MA4F2 Braid Groups – Sheet 2

Deadline: Wednesday, 9 February 2005, 2:00.

Solutions to Section B are for handing in. Please put your solutions into the MA4F2 Braid Groups box in front of the Undergraduate Office.

Needless to say, you may use previous parts of questions even if you haven't proved them.

(A1) Let R be a relation on a set A. Prove that the following are equivalent.

(1) R is contained in an ordering on A.

(2) $x_0 R x_1 R \cdots R x_n = x_0 \Rightarrow x_i = x_0$ for all i.

(3) The reflexive-transitive closure of R is an ordering.

(A2) Formulate and *prove the group analogues to 5.15 and 5.17.

(A3) Let G be a group generated by $S \subset G$. Let $g, x, y \in G$. Prove $d_S(x, y) = d_S(gx, gy)$ and $d_S(x, y) = \ell_S(x^{-1}y)$.

(A4) Suppose $n \geq 3$. Prove that the Cayley graph $\Gamma(\Sigma_n, S)$ (viewed as a graph (V, E, ∂)) has an automorphism which does not come from left multiplication by an element $h \in \Sigma_n$.

(B1) In this exercise you will prove the diamond lemma.

Retain the notation and assumptions of the diamond lemma. For $x \in L$, let $L(x) := \{y \in L \mid x \geq y\}$. Let L_0 be the set of elements $x \in L$ such that $L(x)$ has no least element. For $x \in L - L_0$ let $M(x)$ be the least element of $L(x)$. In parts (a), (b), (c), suppose L_0 is non-empty.

(a) Prove that L_0 has a minimal element a (that is, there is no $b \in L_0$ such that $a > b$).

(b) Suppose $a >_0 b_i$ and $a \neq b_i$ $(i = 1, 2)$. Prove $b_i \notin L_0$ and $M(b_1) = M(b_2)$.

(c) Deduce a contradiction.

(d) Finish the proof.

(B2) Let B_3^+ be the monoid presented by $(S_1, R_1) = (1, 2 \mid 121 = 212)$. (Warning: $\varnothing$ is the identity, 1 is not.) Let B_3 be the group presented by the same presentation, regarded as a group presentation. From the general theory of presentations it follows that there is a natural homomorphism $f\colon B_3^+ \to B_3$. One of the things you prove in this exercise is that f is injective.

(a) Prove that B_3^+ is presented [1] by

$$(S_2, R_2) := \left(1, 2, \Delta \,\middle|\, \begin{array}{ll} 121 \to \Delta, & 212 \to \Delta, \\ 1\Delta \to \Delta 2, & 2\Delta \to \Delta 1, \end{array}\right)$$

(b) Construct a map $g\colon S_2^* \to \mathbb{Z}_{\geq 0}$ such that $g(x) > g(y)$ if $x \xrightarrow[R]{} y$. You don't need to prove that g has this property. Deduce that (S_2, R_2) is a well-founded rewriting system for B_3^+.

[1] Recall our convention $xRy \Leftrightarrow (x, y) \in R \Leftrightarrow (x = y) \in R \Leftrightarrow (x \to y) \in R$, four notations for the same thing.

(c) Prove that (S_2, R_2) is a complete rewriting system for B_3^+.

(d) Prove that you don't get a complete rewriting system for B_3^+ if you remove the last rewriting rule $2\Delta \to \Delta 1$.

(e) Compute the R_2-minimal form for 22121122121222121.

(f) Prove that

$$(S_3, R_3) := \left(1, 2, \Delta, \delta \;\middle|\; \begin{array}{cc} 121 \to \Delta, & 212 \to \Delta, \\ 1\Delta \to \Delta 2, & 2\Delta \to \Delta 1, \\ 1\delta \to \delta 2, & 2\delta \to \delta 1, \\ \Delta\delta \to \varnothing, & \delta\Delta \to \varnothing \end{array} \right)$$

is a monoid presentation for B_3.

(g) From now on you may assume without proving it that (S_3, R_3) is a well-founded rewriting system for B_3. Prove that (S_3, R_3) is a complete rewriting system for B_3.

(h) Prove that every R_2-minimal word in $\{1, 2, \Delta\}^*$ is also R_3-minimal. Deduce that f is injective.

(i) There is a homomorphism (called length) $\ell \colon B_3^+ \to \mathbb{Z}$, $\ell(1) = \ell(2) = 1$. (You don't need to prove this.) Clearly, the rewriting system (S_2, R_2) preserves the length, that is, for all $(x, y) \in R$ one has $\ell(x) = \ell(y)$. Use this to compute the formal power series

$$\sum_{x \in B_3^+} t^{\ell(x)}.$$

(B3) Draw the Cayley graph for the presentation $G = \langle x, y \mid x^3,\ y^2,\ (xy)^3 \rangle$. Make your drawing planar, that is, without crossings. Choose a vertex a and label each vertex b with the distance $d_S(a, b)$. You needn't prove that your drawing is correct. Prove that G is isomorphic to the alternating group A_4.

(B4) Make a planar drawing of the Cayley graph for the presentation $\langle x, y \mid x^5,\ y^2, (xy)^3 \rangle$. (It has 60 vertices so you need to draw somewhat small to make it fit on one page.) A proof is not necessary.

(B5) Prove that there exists $w_0 \in \Sigma_n$ such that $N(w_0) = \mathrm{Ref}$. Give a formula for $w_0(i)$ for all $i \in I_n$. Show that $x \leq w_0$ for all $x \in \Sigma_n$.

(B6) Let $x \in \Sigma_n$ and $1 \leq i < j \leq n$. Prove that $(ij) \in N(x^{-1}) \Leftrightarrow xi > xj$.

(B7) Compute the length $\ell(x)$ for $x = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 9 & 6 & 8 & 3 & 2 & 5 & 1 \end{smallmatrix}\right) \in \Sigma_9$.

(B8) (a) Let $r \in \mathrm{Ref}$, $g, a, b \in \Sigma_n$. Prove that r separates a from b if and only if grg^{-1} separates ga from gb.

(b) Let $x, y \in \Sigma_n$ be such that $N(x) \subset N(xy)$. Prove that $N(xy)$ is the disjoint union of $N(x)$ and $x\,N(y)\,x^{-1}$.

(C1) Let p be a prime number and let $\mathbb{F}_p$ or $\mathbb{Z}_p$ be the field of p elements. Prove that $\mathrm{SL}(2, \mathbb{F}_p)$ is presented by $\langle x, y \mid xyx = yxy,\ (xy)^6,\ x^p \rangle$.

(C2) Let $r \in \mathrm{Ref}$ and $x, y \in \Sigma_n$. Prove that r separates x from y if and only if $d(x, y) > d(x, ry)$.