MA241 Combinatorics – Marking Sheet 1

Deadline: Wednesday, 26 January 2005, 2:00.

For this sheet, B1(b), B2, B6(abc) and B7 will be assessed.

Marks are in the margin. (Each assessment has 25 points in total.)

Correct answers always get full marks even if they are different from mine. In general, partially correct answers get partial marks to marker's discretion, if not indicated.

(B1) Prove the following identities for braids, by applying the Artin relations repeatedly:

(b) $12^2 32^2 12^2 3 = 32^2 12^2 32^2 1$

Solution.

[3] (b). We have

$$12\underline{232}\overline{212}23 = 1\underline{232}31\underline{212}3 = \underline{13}2\underline{3311}2\underline{13} = 3121133231$$

and then the same way back with 1 and 3 swapped. □

(B2) Let $f, g\colon [a, b] \to BS_n$ be based paths in braid space.

(a) Suppose g is a reparametrisation of f. Sketch the geometric braids of f and g in a typical case.

(b) Suppose f and g are strictly homotopic relative endpoints. Again, sketch the geometric braids of f and g in a typical case.

[2+2] Solution. Left to marker's discretion. □

(B6)

(a) Let G be a group and $C \subset G$ a union of conjugacy classes. Show that the B_n-action on G^n preserves C^n.

From now on we put $R = \mathbb{Z}[q, q^{-1}]$,

$$G = \left\{ \begin{pmatrix} q^k & x \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z},\ x \in R \right\}, \qquad C = \left\{ \begin{pmatrix} q & x \\ 0 & 1 \end{pmatrix} : x \in R \right\}.$$

So G is a subgroup of $\mathrm{GL}(2, R)$ and C is a subset of G.

(b) Prove that C is a union of conjugacy classes in G. Deduce that B_n acts on C^n.

(c) On writing

$$\left(\begin{pmatrix} q & x_1 \\ 0 & 1 \end{pmatrix}, \ \ldots, \ \begin{pmatrix} q & x_n \\ 0 & 1 \end{pmatrix}\right) D_i = \left(\begin{pmatrix} q & y_1 \\ 0 & 1 \end{pmatrix}, \ \ldots, \ \begin{pmatrix} q & y_n \\ 0 & 1 \end{pmatrix}\right),$$

express $(y_1, \ \ldots, \ y_n)$ in terms of $(x_1, \ \ldots, \ x_n)$.

SOLUTION.

[3] (a). Let $(x_1, \ldots, x_n)\sigma_i = (y_1, \ldots, y_n)$. Suppose that all x_k are in C. For $k \notin \{i, i+1\}$ we have $y_k = x_k \in C$. Moreover

$$y_i = x_{i+1} \in C, \qquad y_{i+1} = x_{i+1}^{-1} \, x_i \, x_{i+1} \in C$$

because C is a closed under conjugation.

We have shown that σ_i preserves C^n. But the σ_i generate B_n and therefore, B_n preserves C^n.

[3] (b). We have

$$\begin{pmatrix} q^k & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^k & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} q^{k+1} & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^{-k} & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & * \\ 0 & 1 \end{pmatrix}$$

where $*$ stands for anything (and different things each time). So C is a union of conjugacy classes in G.

[3] (c). Let us first consider the case $n = 2$, $i = 1$ (which is essentially all cases as we will later state.) We have

$$\begin{aligned} \left(\begin{pmatrix} q & x_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} q & x_2 \\ 0 & 1 \end{pmatrix}\right) D_1 &= \left(\begin{pmatrix} q & x_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} q & x_2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} q & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q & x_2 \\ 0 & 1 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} q & x_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} q^{-1} & -q^{-1}x_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^2 & qx_2 + x_1 \\ 0 & 1 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} q & x_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} q & x_2 + q^{-1}x_1 - q^{-1}x_2 \\ 0 & 1 \end{pmatrix}\right) \end{aligned}$$

so

$$(y_1, y_2) = (x_2, q^{-1}x_1 + (1 - q^{-1})x_2).$$

In general we have

$$\begin{aligned} y_j &= x_j & (j \notin \{i, i+1\}) \\ y_i &= x_{i+1} \\ y_{i+1} &= q^{-1}x_i + (1 - q^{-1})x_{i+1} \end{aligned}$$

□

(B7) Let M be a monoid and let $\approx$ be an equivalence relation on M. Prove that the following are equivalent.

(1) There exists a monoid N and a homomorphism $f \colon M \to N$ such that $x \approx y \Leftrightarrow fx = fy$, for all $x, y \in M$.

(2) For all $w, x, y, z \in M$, if $w \approx x$ and $y \approx z$ then $wy \approx xz$.

SOLUTION.

[1] **(1) implies (2).** If $w \approx x$ and $y \approx z$ then $fw = fx$ and $fy = fz$ then $f(wy) = (fw)(fy) = (fx)(fz) = f(xz)$ then $wy \approx xz$. This proves (1) implies (2).

(2) implies (1).

[1] **Defining N as a set.** $N = M/{\approx}$ (which makes sense because $\approx$ is an equivalence relation.)

[2] **Defining multiplication on N.** $[w][y] := [wy]$ where $[w]$ is the equivalence class of w. We need to prove that this is well-defined. Suppose $[w] = [x]$ and $[y] = [z]$. Then $w \approx x$ and $y \approx z$ so $wy \approx xz$ by (a) so $[wy] = [xz]$. This proves well-definedness of multiplication in N.

[2] **Proof that N is a monoid.** Associativity of multiplication: $([x][y])[z] = [xy][z] = [(xy)z] = [x(yz)] = [x][yz] = [x]([y][z])$. On writing 1 for the identity in M, we prove that $[1]$ is an identity for N: $[1][x] = [1x] = [x] = [x1] = [x][1]$. So N has an identity.

[2] **The homomorphism.** Define $f\colon M \to N$ by $fx := [x]$. We claim that f is a homomorphism. We have $f(xy) = [xy] = [x][y] = (fx)(fy)$. Moreover $f1 = [1]$ so f takes identity to identity. This proves that f is a homomorphism.

[1] **Proof of (1).** We have $fx = fy \Leftrightarrow [x] = [y] \Leftrightarrow x \approx y$ by construction. $\square$