MA4F2 Braid Groups – Sheet 1

Deadline: Wednesday, 26 January 2005, 2:00.

Solutions to Section B are for handing in. Please put your solutions into the MA4F2 Braid Groups box in front of the Undergraduate Office.

(A1) Prove that the following spaces are path-connected: $\mathbb{R}^m$, BS_n, $\mathbb{R}^2 - \mathbb{Q}^2$.

(A2) Prove that BS_2 is homeomorphic to $S^1 \times \mathbb{R}^3$.

(A3) Finish cases 3 and 4 in the proof that the Burau representation is Hermitian.

(B1) Prove the following identities for braids, by applying the Artin relations repeatedly:

(a) $12132 = 23123$

(b) $12^2 32^2 12^2 3 = 32^2 12^2 32^2 1$

(c) $(123)^4 = (321)^4$

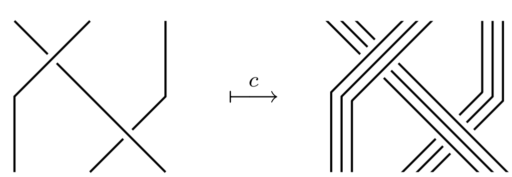
(B2) Let $f, g\colon [a, b] \to BS_n$ be based paths in braid space.

(a) Suppose g is a reparametrisation of f. Sketch the geometric braids of f and g in a typical case.

(b) Suppose f and g are strictly homotopic relative endpoints. Again, sketch the geometric braids of f and g in a typical case.

(B3) Compute the entries of the Burau matrix of $\sigma_1 \cdots \sigma_{n-1} \in B_n$. Hint: Once you have a conjectural answer, try to prove it by induction.

Figure 1: Cabling



(B4) There is a homomorphism called *cabling* $c\colon B_n \to B_{kn}$ which replaces each string by k strings. See figure 1. The k strings always stay neatly in line. Suppose now $k = 2$.

(a) Express $c(\sigma_i)$ in terms of the sigma-generators of B_{2n}.

(b) (Re)prove that c exists in an algebraic way, by using theorem 3.7 and proving that c takes any Artin relation in B_n to one of the consequences of the Artin relations in B_{2n}.

(B5) Let G be a group, $k \in \mathbb{Z}$. Prove that there exists a unique homomorphism $g\colon B_n \to \mathrm{Sym}(G^n)$ such that

$$(x_1,\ \ldots,\ x_n)(g\sigma_i) = (x_1,\ \ldots,\ x_{i-1},\ x_{i+1}^{-1},\ x_{i+1}\,x_i\,x_{i+1},\ x_{i+2},\ \ldots, x_n)$$

for all i.

(B6) (a) Let G be a group and $C \subset G$ a union of conjugacy classes. Show that the B_n-action on G^n preserves C^n.

From now on we put $R = \mathbb{Z}[q, q^{-1}]$,

$$G = \left\{ \begin{pmatrix} q^k & x \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z},\ x \in R \right\}, \qquad C = \left\{ \begin{pmatrix} q & x \\ 0 & 1 \end{pmatrix} : x \in R \right\}.$$

So G is a subgroup of $\mathrm{GL}(2, R)$ and C is a subset of G.

(b) Prove that C is a union of conjugacy classes in G. Deduce that B_n acts on C^n.

(c) On writing

$$\left(\begin{pmatrix} q & x_1 \\ 0 & 1 \end{pmatrix},\ \ldots,\ \begin{pmatrix} q & x_n \\ 0 & 1 \end{pmatrix} \right) D_i = \left(\begin{pmatrix} q & y_1 \\ 0 & 1 \end{pmatrix},\ \ldots,\ \begin{pmatrix} q & y_n \\ 0 & 1 \end{pmatrix} \right),$$

express $(y_1,\ \ldots,\ y_n)$ in terms of $(x_1,\ \ldots,\ x_n)$.

(d) Prove that this recovers the Burau representation.

(e) Devise a diagrammatic notation for the Burau representation and give the example for σ_i.

(B7) Let M be a monoid and let $\approx$ be an equivalence relation on M. Prove that the following are equivalent.

(1) There exists a monoid N and a homomorphism $f\colon M \to N$ such that $x \approx y \Leftrightarrow fx = fy$, for all $x, y \in M$.

(2) For all $w, x, y, z \in M$, if $w \approx x$ and $y \approx z$ then $wy \approx xz$.

(C1) Compute the *determinant* of the Burau Hermitian form, that is, the determinant of the $n \times n$ matrix whose (i, j)-entry is (h_i, h_j).