THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: never

MOCK EXAM MA4F20 BRAID GROUPS 2004–2005

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. (a) Give a topological space whose fundamental group is the braid group B_n, without using any definition of B_n. You don't need to define the topology on the space or to prove your result. **[3]**

(b) In this part of the question, you will prove the following.

> **Diamond Lemma.** Let $>_0$ be an acyclic relation on a set L. Let $\geq$ denote its reflexive-transitive closure. Suppose that L has no infinite descending chains, that is, there are no $x_0,\ x_1,\ \ldots \in L$ with $x_0 > x_1 > \cdots$. Suppose also that for all $a, b, c \in L$, if $a >_0 b$ and $a >_0 c$ then there exists $d \in L$ such that $b \geq d$ and $c \geq d$. If L has a greatest element, then it has a least element.

For $x \in L$, let $L(x) := \{y \in L \mid x \geq y\}$. Let L_0 be the set of elements $x \in L$ such that $L(x)$ has no least element. For $x \in L - L_0$ let $M(x)$ be the least element of $L(x)$. In parts (a), (b), (c), suppose L_0 is non-empty.

(i) Prove that L_0 has a minimal element a (that is, there is no $b \in L_0$ such that $a > b$). **[2]**

(ii) Let a be as before. Suppose $a >_0 b_i$ and $a \neq b_i$ $(i = 1, 2)$. Prove that $b_i \notin L_0$ and $M(b_1) = M(b_2)$. **[2]**

(iii) Deduce a contradiction. **[2]**

(iv) Finish the proof. **[2]**

(c) For $x \in \Sigma_n$, let $N(x)$ denote the set of reflections separating 1 from x. Let $x \in \Sigma_n$ and $1 \leq i < j \leq n$. Prove that $(ij) \in N(x^{-1}) \Leftrightarrow xi > xj$. **[7]**

(d) We consider the classical Garside structure on B_4. We write i for σ_i. Apply the word reversing process to the word $2^{-1}1^{-1}1^{-1}2^{-1}32221$. Use your answer to give a word for the join $2112 \vee 32221$. **[5+2]**

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2. (a) Consider the monoid presentation (S, R) where $S = \mathbb{Z}_{\geq 0}$ and

$$R = \{(a, b) = (a-1, b+2) \mid a, b \in \mathbb{Z}_{\geq 0},\ a > 0\}.$$

(i) Prove that (S, R) is a well-founded rewriting system. **[2]**

(ii) Prove that it is confluent. **[3]**

(iii) Find all R-minimal words. **[3]**

(b) Compute the classical greedy form of the braid **[6]**

$$r\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix} r\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}$$

(where rx is the classical simple braid).

(c) Let $S \subset \Sigma_n$ be the set of fundamental reflections and write $s_i := (i, i+1) \in S$. Define $t\colon S \to B_n$ by $t(s_i) := \sigma_i$. Let $(x_1, \ldots, x_k) \in S^*$ be a minimal expression for $x \in \Sigma_n$ (that is, $x_1 \cdots x_k = x$ with k minimal). Without using pictures, prove that $(tx_1) \cdots (tx_k)$ depends only on x. **[11]**

Hint: Induction on k. You may use that $(\Sigma_n, \leq)$ is a lattice and that

$$s_i \vee s_j = \begin{cases} s_i\, s_j\, s_i & |i-j| = 1 \\ s_i\, s_j & |i-j| > 1. \end{cases}$$

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3. (a) Let $a, b \in \mathbb{C}$ and put **[5]**

$$A_1 := \begin{pmatrix} 1 & 1 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that there exists a unique homomorphism $f\colon B_3 \to \mathrm{GL}(3, \mathbb{C})$ such that $f(\sigma_1) = A_1$, $f(\sigma_2) = A_2$.

(b) Let F denote the free group on S. Prove that every element of F can uniquely be written as a reduced word in S (that is, an element of $(S \cup S^{-1})^*$ not containing xx^{-1} or $x^{-1}x$ as subword for any $x \in S$). **[5]**

(c) In this part of the question, proofs are not necessary. We define the group E_{n+1} by

$$E_{n+1} = \{g \in B_{n+1} \mid (\pi g)(n+1) = n+1\}$$

where π is the usual homomorphism $B_{n+1} \to \Sigma_{n+1}$.

(i) Give the definitions of two homomorphisms $u\colon E_{n+1} \to B_n$ and $v\colon B_n \to E_{n+1}$ such that $uv = \mathrm{id}_{B_n}$. **[3]**

(ii) We write $M_n := \ker u$. What split short exact sequence do we get from (i)? Which of the involved groups are X, Y, Z if the following should be correct? The group X acts on Y, yielding a semi-direct product Z. **[4]**

(iii) Give a presentation for M_n and draw pictures of your generators for M_n. **[3]**

(d) For $x \in \Sigma_n$, let $N(x)$ denote the number of reflections separating 1 from x, and $\ell(x) = |N(x)|$. Prove that for all $x, y \in \Sigma_n$ we have **[5]**

$$\ell(xy) = \ell(x) + \ell(y) \quad \Leftrightarrow \quad N(x) \subset N(xy).$$

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4. (a) Give a monoid which is not isomorphic to a submonoid of any group and prove your result. **[3]**

(b) Define the following BKL elements in B_4: $a = a_{12}$, $b = a_{23}$, $c = a_{34}$, $d = a_{13}$. Put $A = \{a, b, c, d\}$, let B_4^+ be the submonoid of B_4 generated by A and let $\leq$ be the ordering on B_4^+ defined by $x \leq z \Leftrightarrow \exists y \in B_4^+\colon xy = z$. Prove that $(B_4^+, \leq)$ is not a lattice. Deduce that B_4^+ is not a Garside monoid, not even if one replaces A with another set of generators. **[6]**

(c) Find a minimal expression for **[7]**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 1 & 7 & 2 & 4 & 5 \end{pmatrix} \in \Sigma_7.$$

(d) Let $S \subset \Sigma_n$ denote the set of fundamental reflections. For $x \in \Sigma_n$, let $r(x)$ be the simple braid associated with $x \in \Sigma_n$ and $N(x)$ the set of reflections separating 1 from x. For $I \subset S$ we write $w_I = \vee I$ and $\Delta_I = r(w_I) = \vee(rI)$. You may use the following without proving it.

$$\text{Let } I \subset S. \text{ Then } N(w_I) \cap S = I. \tag{1}$$

$$\text{Let } s \in S,\ x \in \Sigma_n. \text{ Then } s \in N(x) \Leftrightarrow s \leq x. \tag{2}$$

Let $x \in \Sigma_n$ and put $J := S \cap N(x)$.

(i) Prove that $w_I \leq x \Leftrightarrow I \subset J$, for all $I \subset S$. **[5]**

(ii) Let $\mu\colon B_n^+ \to \mathbb{Z}$ denote the Möbius function on the classical positive braid monoid. Let $x \in \Omega$. Prove **[4]**

$$\mu(x) = \begin{cases} (-1)^{\#I} & \text{if } x = \Delta_I \text{ for some } I \\ 0 & \text{otherwise.} \end{cases}$$

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5. (a) Let G be a group, $k \in \mathbb{Z}$. Prove that there exists a unique homomorphism $g\colon B_n \to \mathrm{Sym}(G^n)$ such that **[4]**

$$(x_1, \ \ldots, \ x_n)(g\sigma_i) = (x_1, \ \ldots, \ x_{i-1}, \ x_{i+1}^{-1}, \ x_{i+1}\, x_i\, x_{i+1}, \ x_{i+2}, \ \ldots, x_n)$$

for all i.

(b) Let M denote the monoid given by the presentation

$$\big(a, b, c \;\big|\; abbc = bcab, \ bcca = cabc, \ caab = abca\big)$$

associated to the complement f on $A = \{a, b, c\}$.

Your proofs in (i), (ii) and (iii) should not depend on a drawing of the Dehornoy graph.

(i) Prove that f is normed and coherent. **[3]**

(ii) Use the word reversing process to compute $\Delta := a \vee b \vee c$. **[5]**

(iii) Prove that Δ is a Garside element. **[4]**

(iv) Draw the Dehornoy graph Γ. **[3]**

(v) Let $\ell\colon M \to \mathbb{Z}$ denote the homomorphism defined by $\ell(a) = \ell(b) = \ell(c) = 1$. Compute **[3]**

$$\sum_{x \in M} t^{\ell(x)} \in \mathbb{Z}[[t]]$$

and show your working.

(vi) Compute the greedy form for $cbcca$ and show your working. **[3]**
