THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: June 2005

MA4F2 BRAID GROUPS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

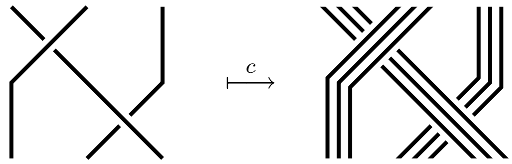
Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Figure 1: Cabling



1. (a) There is a homomorphism $c\colon B_n \to B_{nk}$ called cabling which replaces each string by k strings. See figure 1. Suppose $k = 3$. Express $c(\sigma_i)$ in terms of the sigma generators of B_{nk}. A proof is not necessary. **[4]**

(b) In this part of the question, you will prove the following.

> **Diamond Lemma.** Let $>_0$ be an acyclic relation on a set L. Let $\geq$ denote its reflexive-transitive closure. Suppose that L has no infinite descending chains, that is, there are no $x_0,\ x_1,\ \ldots \in L$ with $x_0 > x_1 > \cdots$. Suppose also that for all $a, b, c \in L$, if $a >_0 b$ and $a >_0 c$ then there exists $d \in L$ such that $b \geq d$ and $c \geq d$. If L has a greatest element, then it has a least element.

For $x \in L$, let $L(x) := \{y \in L \mid x \geq y\}$. Let L_0 be the set of elements $x \in L$ such that $L(x)$ has no least element. For $x \in L - L_0$ let $M(x)$ be the least element of $L(x)$. In parts (a), (b), (c), suppose L_0 is non-empty.

(i) Prove that L_0 has a minimal element a (that is, there is no $b \in L_0$ such that $a > b$). **[2]**

(ii) Let a be as before. Suppose $a >_0 b_i$ and $a \neq b_i$ $(i = 1, 2)$. Prove that $b_i \notin L_0$ and $M(b_1) = M(b_2)$. **[2]**

(iii) Deduce a contradiction. **[2]**

(iv) Finish the proof. **[2]**

(c) Consider the complement f on the alphabet $A = \{a, b, c\}$ associated with the presentation

$$(a, b, c \mid ab = ca,\ aac = baa,\ cbc = bcb).$$

You may assume without proving it that f is normed and coherent.

(i) Draw the Dehornoy graph without proof. **[3]**

(ii) Prove that $\Delta := a \vee b \vee c$ exists and give an expression for it. **[4]**

(iii) Prove that Δ is not a Garside element. **[6]**

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2. (a) Let A be a set. Give the definition of *reflexive-transitive closure* of a relation $R \subset A \times A$. **[2]**

(b) (i) Prove that $(a, b, c \mid b^3c \to ba, \quad c^2a \to ab)$ is a well-founded rewriting system. **[2]**

(ii) Prove that it is not confluent. **[3]**

(iii) Show how one can make it into a complete rewriting system by adding just one rewriting rule, without changing the monoid presented by it. **[6]**

(c) Compute the classical greedy form of the braid **[6]**

$$r\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} r\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} r\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

(where rx is the classical simple braid).

(d) Let $A, B \subset \mathrm{Ref}$ be sets of reflections in Σ_n such that **[6]**

$$\big[(ij) \in T \text{ and } (jk) \in T\big] \quad \Rightarrow \quad (ik) \in T \qquad (1 \le i < j < k \le n), \qquad (1)$$

$$\big[(ij) \notin T \text{ and } (jk) \notin T\big] \quad \Rightarrow \quad (ik) \notin T \qquad (1 \le i < j < k \le n) \qquad (2)$$

for $T = A, B$. Let C be the least subset of Ref containing $A \cup B$ and satisfying (1). Prove that C satisfies (2).

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3. (a) Let M be a semigroup [1] and $\approx$ an equivalence relation on M. Prove that the following are equivalent. **[6]**

(1) There exists a semigroup N and a homomorphism $f\colon M \to N$ such that $fx = fy \Leftrightarrow x \approx y$ for all $x, y \in M$.

(2) For all $w, x, y, z \in M$, if $w \approx x$ and $y \approx z$ then $wy \approx xz$.

(b) Let G denote the group presented by **[6]**

$$\left(\sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\,\sigma_j\,\sigma_i = \sigma_j\,\sigma_i\,\sigma_j & j = i+1 \\ \sigma_i\,\sigma_j = \sigma_j\,\sigma_i & |i-j| > 1 \end{array} \right)$$

and H by

$$\left(\begin{array}{l} a_{ij} = a_{ji} \\ (\,1 \le i < j \le n) \end{array} \;\middle|\; \begin{array}{l} a_{ij}\,a_{jk} = a_{jk}\,a_{ki} = a_{ki}\,a_{ij} \;\; (i < j < k) \\ a_{ij}\,a_{k\ell} = a_{k\ell}\,a_{ij} \;\; (i < j < k < \ell \text{ or } j < k < \ell < i) \end{array} \right).$$

Prove that there is a unique homomorphism $f\colon G \to H$ such that $f(\sigma_i) = a_{i,i+1}$. Formulate theorems which you use in your proof.

(c) We write i for $s_i \in S \subset \Sigma_n$. Is 234312434 a minimal expression? **[6]**

(d) We consider the classical Garside structure on B_4. We write i for σ_i. Apply the word reversing process to the word $1^{-1}3^{-1}2^{-1}2^{-1}1233$. Use your answer to give a word for the join $2231 \vee 1233$. **[5+2]**

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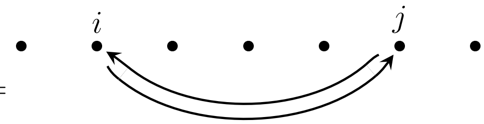
[1] Recall that a *semigroup* is a pair $(G, *)$ of a set G and an associative binary operation $*$ on G. This is not the same as a monoid. Recall also that for G, H semigroups, a homomorphism $f\colon G \to H$ is a map satisfying $f(xy) = (fx)(fy)$ for all $x, y \in G$.

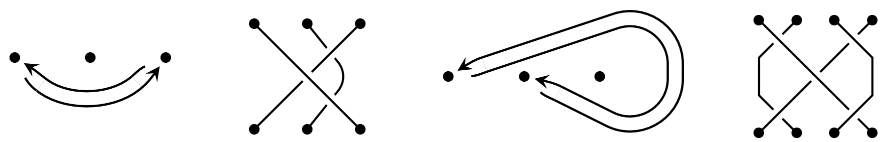
4. (a) (i)

$a_{ij} =$ 

Let a_{ij} denote the BKL generators of the braid group B_n (see the figure). Write $p_{ij} = a_{ij}^2$. Express $a_{ij}\, p_{kn}\, a_{ij}^{-1}$ ($1 \le i < j < n$ and $1 \le k < n$) in terms of the p_{tn} ($1 \le t < n$), treating some cases separately. Proofs are not needed. **[7]**

(ii) Use (i) to express $p_{ij}\, p_{in}\, p_{ij}^{-1}$ in terms of all p_{tn} if $i < j < n$. **[3]**

(iii) In the figure below you can find, as an example, one braid x in two diagrammatic notations, called horizontal and vertical. Moreover, a braid y is given in horizontal notation and a braid z in vertical. Sketch y in vertical notation. Sketch z in horizontal notation. **[4]**



x horizontally $\quad$ x vertically $\quad$ y horizontally $\quad$ z vertically

(b) Let $S \subset \Sigma_n$ denote the set of fundamental reflections. For $x \in \Sigma_n$, let $r(x)$ be the simple braid associated with $x \in \Sigma_n$ and $N(x)$ the set of reflections separating 1 from x. For $I \subset S$ we write $w_I = \vee I$ and $\Delta_I = r(w_I) = \vee(rI)$. You may use the following without proving it.

$$\text{Let } I \subset S. \text{ Then } N(w_I) \cap S = I. \tag{3}$$

$$\text{Let } s \in S,\, x \in \Sigma_n. \text{ Then } s \in N(x) \Leftrightarrow s \le x. \tag{4}$$

Let $x \in \Sigma_n$ and put $J := S \cap N(x)$.

(i) Prove that $w_I \le x \Leftrightarrow I \subset J$, for all $I \subset S$. **[5]**

(ii) Let $\mu\colon B_n^+ \to \mathbb{Z}$ denote the Möbius function on the classical positive braid monoid. Let $x \in \Omega$. Prove **[4]**

$$\mu(x) = \begin{cases} (-1)^{\#I} & \text{if } x = \Delta_I \text{ for some } I \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Give without proof two polynomials $f, g \in \mathbb{Z}[t]$ such that **[2]**

$$\frac{f}{g} = \sum_{x \in B_4^+} t^{\ell(x)}$$

where $\ell\colon B_n^+ \to \mathbb{Z}$ is the homomorphism defined by $\ell(\sigma_i) = 1$.

5. (a) Let G be a group, $k \in \mathbb{Z}$. Prove that there exists a unique homomorphism $f\colon B_n \to \mathrm{Sym}(G^n)$ such that **[4]**

$$(x_1,\ \ldots,\ x_n)(f\sigma_i) = (x_1,\ \ldots,\ x_{i-1},\ x_{i+1},\ x_{i+1}^{-k}\, x_i\, x_{i+1}^{k},\ x_{i+2},\ \ldots, x_n)$$

for all i and all $x_j \in G$.

(b) We define a complement f on $A = \{a, b, c\}$ by

$$f(a,b) = ca \qquad f(b,c) = ab \qquad f(c,a) = bc$$
$$f(b,a) = cb \qquad f(c,b) = ac \qquad f(a,c) = ba$$

and we let M denote the monoid presented by the associated presentation

$$\big(a, b, c \mid aba = cbc, \quad bcb = aca, \quad cac = bab\big).$$

Your proofs should not depend on a drawing of the Dehornoy graph. From (iv) on proofs are not necessary.

(i) Prove that f is normed and coherent. **[4]**

(ii) Use the word reversing process to compute $\Delta := a \vee b \vee c$. **[5]**

(iii) Prove that Δ is a Garside element. **[4]**

(iv) Draw the Dehornoy graph Γ. **[3]**

(v) Label each vertex x of Γ with the value of the Möbius function $\mu(x)$. **[3]**

(vi) What has μ to do with $\sum_{x \in M} x$? **[2]**

END