Test — MA3D5 Galois Theory

Name: ______________________

Version 1 Monday 18 February 2007

Student Number: ______________________

This test covers the material lectured up to and including section 3.1. You can bring any other notes you like, but no calculators.

Each question is worth 1 point. A correct answer is worth 1 point, an incorrect or empty answer 0. Workings or proofs are neither required nor can give points.

A question with a list a–d is a multiple choice question; you should find the unique correct statement in the list.

The open questions require a very short answer: usually less than 10 letters or symbols; often a number. In some cases, there exist many different correct answers.

You can probably give all answers on this sheet. If not, for example because the sheet turns into a mess, please continue on an empty sheet. No name means no points. *Good luck!*

1. Give an explicit example of an irreducible polynomial in $\mathbb{Q}[x]$ of degree 13 as well as the name or number of the theorem that almost immediately implies that it is irreducible.

2. True or false (T/F)? Let K be a field. Let $f \in K[x]$ be a nonconstant polynomial. Then $K[x]/(f)$ is a field.

3. True or false (T/F)? Let K be a field. Let $f \in K[x]$ be a nonconstant polynomial. Then there exists a field extension L/K and an $\alpha \in L$ such that $f(\alpha) = 0$.

4. True or false (T/F)? Let $k \subset K \subset L$ be fields. Let $x_1, \dots, x_m \in L$ be k-independent and let $y_1, \dots, y_m \in K$ be k-independent. Then $\{x_i y_j\}_{i,j}$ are k-independent.

5. Let I be the ideal in $\mathbb{Z}$ generated by $\{9, 12\}$.
   a. I is neither maximal nor prime.
   b. I is prime but not maximal.
   c. I is prime and maximal.
   d. I is maximal but not prime.

6. Compute the cyclotomic polynomial $\phi_{12}(x)$ explicitly.

7. True or false (T/F)? Every ring of characteristic 9 has a subfield.

8. Find an explicit monic polynomial in $\mathbb{Z}[x]$ of degree 4 with a root at $\sqrt{2} + \sqrt{3} + \sqrt{6}$. Write it in standard form $x^4 + ax^3 + bx^2 + cx + d$.

9. Find the remainder of division of the polynomial x^7 by $x^5 - x^4 - 1$.

10. Define $g_1, g_2, g_3 \in \mathbb{Z}[t_1, t_2, t_3]$ by

$$g_1 = t_1^2 - t_2^2, \quad g_2 = t_2^2 - t_3^2, \quad g_3 = t_3^2 - t_1^2.$$

Consider the following statements.

(1): If $f \in \mathbb{Z}[x_1, x_2, x_3]$ is symmetric then so is $f(g_1, g_2, g_3)$.

(2): If $f \in \mathbb{Z}[x_1, x_2, x_3]$ is symmetric then so is $f(g_1^2, g_2^2, g_3^2)$.

a. (1) is false, (2) is true.
b. Both are false.
c. (1) is true, (2) is false.
d. Both are true.

11. True or false (T/F)? Let $K \subset L \subset M$ be fields and let $\alpha \in M$ be algebraic over K. Then $\min_L(\alpha)$ divides $\min_K(\alpha)$ in $L[x]$. Here, $\min_K(\alpha)$ denotes the minimum polynomial of α over K.

12. Find an example of fields $K \subset L \subset M$ such that $1 < [M : L] < \infty$ but every element of M, algebraic over K, is in K.

13. Give an example of a surjective ring homomorphism $R \to S$ such that S is a field but R is not an integral domain.

14. Let $t_1, \ldots, t_n$ be variables. Express $\sum_{i=1}^{n} t_i^{-1}$ explicitly in terms of the elementary symmetric polynomials in the t_i.

15. (See the proof of theorem 1.5). Find the leading monomial in the polynomial

$$\prod_{1 \le i < j \le 4} (t_i + t_j).$$

16. True or false (T/F)? Let $K \subset L$ be fields and $\alpha \in L$. Then $K[\alpha] = K(\alpha)$.

Answers for Version **1**

1. Give an explicit example of an irreducible polynomial in $\mathbb{Q}[x]$ of degree 13 as well as the name or number of the theorem that almost immediately implies that it is irreducible.

   Answer: $x^{13} - 2$, Eisenstein=2.36 or 2.37.

2. True or false (T/F)? Let K be a field. Let $f \in K[x]$ be a nonconstant polynomial. Then $K[x]/(f)$ is a field.

   Answer: False.

3. True or false (T/F)? Let K be a field. Let $f \in K[x]$ be a nonconstant polynomial. Then there exists a field extension L/K and an $\alpha \in L$ such that $f(\alpha) = 0$.

   Answer: True.

4. True or false (T/F)? Let $k \subset K \subset L$ be fields. Let $x_1, \ldots, x_m \in L$ be k-independent and let $y_1, \ldots, y_m \in K$ be k-independent. Then $\{x_i y_j\}_{i,j}$ are k-independent.

   Answer: False.

5. Let I be the ideal in $\mathbb{Z}$ generated by $\{9, 12\}$.

   a. I is neither maximal nor prime.
   b. I is prime but not maximal.
   c. I is prime and maximal.
   d. I is maximal but not prime.

   Answer: c.

6. Compute the cyclotomic polynomial $\phi_{12}(x)$ explicitly.

   Answer: $x^4 - x^2 + 1$.

7. True or false (T/F)? Every ring of characteristic 9 has a subfield.

   Answer: False.

8. Find an explicit monic polynomial in $\mathbb{Z}[x]$ of degree 4 with a root at $\sqrt{2} + \sqrt{3} + \sqrt{6}$. Write it in standard form $x^4 + ax^3 + bx^2 + cx + d$.

   Answer: $x^4 - 22x^2 - 48x - 23$.

9. Find the remainder of division of the polynomial x^7 by $x^5 - x^4 - 1$.

   Answer: $x^4 + x^2 + x + 1$.

10. Define $g_1, g_2, g_3 \in \mathbb{Z}[t_1, t_2, t_3]$ by

    $$g_1 = t_1^2 - t_2^2, \quad g_2 = t_2^2 - t_3^2, \quad g_3 = t_3^2 - t_1^2.$$

    Consider the following statements.

    (1): If $f \in \mathbb{Z}[x_1, x_2, x_3]$ is symmetric then so is $f(g_1, g_2, g_3)$.

    (2): If $f \in \mathbb{Z}[x_1, x_2, x_3]$ is symmetric then so is $f(g_1^2, g_2^2, g_3^2)$.

    a. (1) is false, (2) is true.
    b. Both are false.
    c. (1) is true, (2) is false.
    d. Both are true.

    Answer: a.

11. True or false (T/F)? Let $K \subset L \subset M$ be fields and let $\alpha \in M$ be algebraic over K. Then $\min_L(\alpha)$ divides $\min_K(\alpha)$ in $L[x]$. Here, $\min_K(\alpha)$ denotes the minimum polynomial of α over K.

    Answer: True.

12. Find an example of fields $K \subset L \subset M$ such that $1 < [M : L] < \infty$ but every element of M, algebraic over K, is in K.

    Answer: $M = \mathbb{C}(t)$, $L = \mathbb{C}(t^2)$, $K = \mathbb{C}$.

13. Give an example of a surjective ring homomorphism $R \to S$ such that S is a field but R is not an integral domain.

    Answer: $\mathbb{Z}/4 \to \mathbb{Z}/2$.

14. Let $t_1, \ldots, t_n$ be variables. Express $\sum_{i=1}^n t_i^{-1}$ explicitly in terms of the elementary symmetric polynomials in the t_i.

    Answer: σ_{n-1}/σ_n.

15. (See the proof of theorem 1.5). Find the leading monomial in the polynomial

    $$\prod_{1 \le i < j \le 4} (t_i + t_j).$$

    Answer: $t_1^3 t_2^2 t_3$.

16. True or false (T/F)? Let $K \subset L$ be fields and $\alpha \in L$. Then $K[\alpha] = K(\alpha)$.

    Answer: False.