Test — MA3D5 Galois Theory

Version **1** Monday 23 February 2009

Name: ____________________

Student Number: ____________________

This test covers the material lectured up to and including section 4.2. You can bring any notes you like, but no electronic gadgets.

There are 10 questions. Each question is worth 1 marks. A correct answer is worth 1 mark, an incorrect or empty answer 0. Workings or proofs are neither required nor can give marks.

Every question requires a very short answer: usually less than 10 letters or symbols; often a number. In some cases, there exist many different correct answers.

Give your answers on this sheet and hand it in. You are free to write scratchy notes, but please not on this sheet. No name means no marks. *Good luck!*

1. Put $L = \mathbb{Q}(2^{1/4}) \subset \mathbb{R}$. How many elements does $\mathrm{Gal}(L/\mathbb{Q})$ have?

2. True or false (T/F)? Let A be a UFD and $f, g \in A[X]$. If f divides g and g is primitive then f is primitive.

3. True or false (T/F)? Let $K \subset L$ be fields and $\alpha \in L$. Let $f \in K[X]$ be irreducible, monic and $f(\alpha) = 0$. Then f is the minimum polynomial of α over K.

4. True or false (T/F)? The characteristic of a finite ring is necessarily a prime number.

5. Give an example of a UFD A and a polynomial $f \in A[X]$ which is not Eisenstein though both $f(X-1)$ and $f(X+1)$ are Eisenstein.

6. True or false (T/F)? The polynomial
$$(X^5 - 1)^4 - X^5$$
is solvable by radicals.

7. Find an explicit irreducible polynomial in $\mathbb{F}_2[X]$ of degree 5.

8. True or false (T/F)? Let A be a (commutative) ring and let $f, g \in A[X]$. Then there are $q, r \in A[X]$ such that $f = gq + r$ and $\deg(r) < \deg(g)$.

9. Let σ_k be the elementary symmetric functions in n variables $T_1, \dots, T_n$. Express
$$\sum_{1 \le i < j \le n} (T_i - T_j)^2$$
in terms of the σ_k and n.

10. Find an example of a field K and two extensions $K(\alpha)/K$, $K(\beta)/K$ such that there exists a K-isomorphism $K(\alpha) \to K(\beta)$ although α and β have distinct minimum polynomials over K.

Answers for Version **1**

1. Put $L = \mathbb{Q}(2^{1/4}) \subset \mathbb{R}$. How many elements does $\mathrm{Gal}(L/\mathbb{Q})$ have?

   Answer: 2.

2. True or false (T/F)? Let A be a UFD and $f, g \in A[X]$. If f divides g and g is primitive then f is primitive.

   Answer: true

3. True or false (T/F)? Let $K \subset L$ be fields and $\alpha \in L$. Let $f \in K[X]$ be irreducible, monic and $f(\alpha) = 0$. Then f is the minimum polynomial of α over K.

   Answer: true

4. True or false (T/F)? The characteristic of a finite ring is necessarily a prime number.

   Answer: false

5. Give an example of a UFD A and a polynomial $f \in A[X]$ which is not Eisenstein though both $f(X-1)$ and $f(X+1)$ are Eisenstein.

   Answer: $A = \mathbb{F}_2[Y]$ and $f = X + Y^3$.

6. True or false (T/F)? The polynomial

   $$(X^5 - 1)^4 - X^5$$

   is solvable by radicals.

   Answer: true

7. Find an explicit irreducible polynomial in $\mathbb{F}_2[X]$ of degree 5.

   Answer: The full list is

101001	100101	111101
111011	110111	101111.

   For example the first one is $X^5 + X^3 + 1$.

8. True or false (T/F)? Let A be a (commutative) ring and let $f, g \in A[X]$. Then there are $q, r \in A[X]$ such that $f = gq + r$ and $\deg(r) < \deg(g)$.

   Answer: false

9. Let σ_k be the elementary symmetric functions in n variables $T_1, \ldots, T_n$. Express

   $$\sum_{1 \le i < j \le n} (T_i - T_j)^2$$

   in terms of the σ_k and n.

   Answer: $(n-1)(\sigma_1^2 - 2\sigma_2) - 2\sigma_2$.

10. Find an example of a field K and two extensions $K(\alpha)/K$, $K(\beta)/K$ such that there exists a K-isomorphism $K(\alpha) \to K(\beta)$ although α and β have distinct minimum polynomials over K.

    Answer: $K = \mathbb{Q}$, $\alpha = 0$, $\beta = 1$.