MA3D5 Galois Theory – Sheet 1

Deadline: Monday, 26 January 2009, 3:00.

Please put your solutions into the MA3D5 Galois Theory box in front of the Undergraduate Office. Mention your department if it is not mathematics.

(1.1) Let α, β, γ be the roots of the equation $x^3 + px^2 + q = 0$. Find the cubic polynomial equation whose roots are $\alpha^3,\ \beta^3,\ \gamma^3$.

(1.2) Let $T_1,\ \ldots,\ T_n$ be variables. Prove that the polynomial

$$f = \sum_{i<j} T_i^2\, T_j^2$$

is symmetric. Express it explicitly as a polynomial in the elementary symmetric polynomials.

(1.3) Let $\phi_n(x)$ be the cyclotomic polynomial, which is defined to be $\prod(x-\alpha)$ where the product is over the primitive n-th roots of unity.

(a) Prove $\prod_{d|n} \phi_d(x) = x^n - 1$ for all $n \geq 1$. Here, the product is over the positive divisors d of n.

(b) Prove $\phi_n(x) \in \mathbb{Q}(x)$.

(c) Prove $\phi_n(x) \in \mathbb{Q}[x]$.

(1.4) Suppose that $f = x^{n-1} + x^{n-2} + \cdots + 1 \in \mathbb{Q}[x]$ is irreducible ($n \geq 1$). Prove that n is a prime number. (Irreducible means: if $f = gh$ with $g, h \in \mathbb{Q}[x]$ then g or h is invertible in $\mathbb{Q}[x]$, and f is not invertible).

(1.5) Write $\varepsilon := \exp(2\pi i/5)$ for the natural primitive 5th root of 1 as in section 1.5 of the notes; it is a root of the quartic $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the quadratic equation whose two roots are $\varepsilon + \varepsilon^4$ and $\varepsilon^2 + \varepsilon^3$, and hence give radical formulas for $\cos(2\pi/5)$ and $\cos(4\pi/5)$.