THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: June 2008

MA3D50 GALOIS THEORY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Questions 1, 4 and 5 consist of two or three unrelated parts (a), (b), ...

1. (a) (i) State without proof the tower law. **[2]**

(ii) Let L/K be a finite field extension, that is, $[L : K] < \infty$. Prove that L/K is algebraic. **[5]**

(iii) Let $K \subset M$ be fields. Let L be the set of those elements of M that are algebraic over K. Prove that L is a subfield of M. State any results you use. **[5]**

(b) Prove or disprove the following. Let $K \subset L \subset M$ be fields and let $\alpha \in M$ be algebraic over K. Then $[L(\alpha) : L] \leq [K(\alpha) : K]$. **[5]**

(c) Let $\mathbb{F}_q$ be a finite field of q elements. Prove from first principles the following identity in $\mathbb{F}_q[x]$: **[8]**

$$\prod_{\alpha \in \mathbb{F}_q} (x - \alpha) = x^q - x.$$

[But you may use results about finite groups, and that $K[x]$ is a unique factorisation domain for all fields K.]

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2. Let K be a field and let $f \in K[x]$ be a polynomial.

(a) Let $g \in K[x]$ be nonconstant and irreducible. Prove that there exists a field extension $K \subset L$ and an element $\alpha \in L$ such that $g(\alpha) = 0$. **[4]**

(b) Define what a *splitting field* for (K, f) is. **[3]**

(c) Prove that a splitting field for (K, f) exists. [Hint: induction on the degree of f.] **[8]**

(d) Let $\mathbb{C}(t)$ denote the field of rational functions in one variable t. Let $M/\mathbb{C}(t)$ be a splitting field for $f(x) = x^6 - t^2$. Compute $[M : \mathbb{C}(t)]$ and briefly justify. **[4]**

(e) Find splitting fields M/L and L/K such that M/K is not a splitting field, and prove your claims. **[6]**

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3. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $f = x^3 - x^2 - 4x - 1$.

(a) Prove that $f \in \mathbb{Q}[x]$ is irreducible, carefully stating any results you use. **[6]**

(b) Prove that $-(1+\alpha)^{-1}$ is also a root of f. **[6]**

(c) Prove or disprove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois, carefully stating any results you use. **[7]**

(d) Write $\varepsilon = \exp(2\pi i/3)$. Prove that f is irreducible in $\mathbb{Q}(\varepsilon)[x]$. **[6]**

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4. (a) Let $K \subset L$ be fields and let $G = \mathrm{Gal}(L/K)$ be the Galois group. Suppose that G is finite. Let $\mathcal{G}$ be the set of subgroups of G. Let $\mathcal{F}$ be the set of subfields of L containing K. For $F \in \mathcal{F}$, let F^* be the set of $g \in G$ such that $g(x) = x$ for all $x \in F$. For $H \in \mathcal{G}$, let $H^\dagger$ be the set of $x \in L$ such that $g(x) = x$ for all $g \in H$.

(i) Let $F \in \mathcal{F}$. Prove that $F^* \in \mathcal{G}$. **[4]**

(ii) Let $F_1, F_2 \in \mathcal{F}$ and suppose that $F_1 \subset F_2$. Prove that $F_1^* \supset F_2^*$. **[4]**

(iii) State the main theorem of Galois theory. **[2]**

(iv) Prove or disprove the following, carefully stating every result you use. Suppose that $H^{\dagger *} = H$ for all $H \in \mathcal{G}$. Then L/K is normal. **[6]**

(b) Let L/K be an algebraic field extension. Let $\lambda \in L$ be nonzero and such that λ and λ^2 have the same minimum polynomial over K. Prove that λ is a root of unity. **[9]**

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5. (a) Let $t_1, \ldots, t_n$ be variables. Express the polynomial **[6]**

$$S = \sum_{1 \le i < j < k \le n} t_i\, t_j\, t_k\, (t_i + t_j + t_k)$$

in terms of the elementary symmetric polynomials $\sigma_k(t_1, \ldots, t_n)$.

(b) Put $\alpha = 8^{1/4} \in \mathbb{R}$ and $\beta = \alpha + \alpha^2$.

(i) Prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. [Hint: express $\beta(\beta - 2\alpha^2)$ in terms of α.] **[6]**

(ii) Compute $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and prove your result. **[5]**

(c) Let $K \subset L \subset M$ be fields. Let $\alpha \in M$ and let $f \in L[x]$ be a nonzero polynomial all of whose coefficients are algebraic over K, and such that $f(\alpha) = 0$. Prove that α is algebraic over K. You may use without proof that a field extension is finite iff it is algebraic and finitely generated. **[8]**
