MA3D5 Galois Theory – Sheet 1

Deadline: Monday, 21 January 2008, 3:00.

Solutions to Section B are for handing in, the other sections are not. Section A is easier and is meant as a warming up. Questions in section C are similar to those in section B but not for handing in. Section D is harder.

Please put your solutions into the MA3D5 Galois Theory box in front of the Undergraduate Office. Mention your department if it is not mathematics.

(A1) If $f(x) = a_0\, x^n + a_1\, x^{n-1} + \cdots + a_n$ has roots $\alpha_1, \ldots, \alpha_n$, what polynomial has roots $c\alpha_1, \ldots, c\alpha_n$?

(A2) Let $a, b, c \in \mathbb{C}$. Let K be the radical closure of $\{a, b, c\}$ (that is, the smallest subfield of $\mathbb{C}$ containing a, b, c and such that for all $\alpha \in \mathbb{C}$, $n > 0$, if $\alpha^n \in K$ then $\alpha \in K$). Let L be the radical closure of $\{ab, bc, ca\}$. Prove $K = L$.

(A3) Prove that $x^5 - 3x^3 - 8$ is solvable by radicals, using the results of chapter 1.

(B1) Let α, β, γ be the roots of the equation $x^3 + px^2 + q = 0$. Find the cubic polynomial equation whose roots are α^3, β^3, γ^3.

(B2) Let $\alpha_1, \ldots, \alpha_n$ be variables. Prove that the polynomial

$$f = \sum_{i<j} \alpha_i^2\, \alpha_j^2$$

is symmetric. Express it explicitly as a polynomial in the elementary symmetric polynomials.

(B3) Let $\phi_n(x)$ be the cyclotomic polynomial, which is defined to be $\prod(x - \alpha)$ where the product is over the primitive n-th roots of unity.

(a) Prove $\prod_{d|n} \phi_d(x) = x^n - 1$ for all $n \geq 1$. Here, the product is over the positive divisors d of n.

(b) Prove $\phi_n(x) \in \mathbb{Q}(x)$.

(c) Prove $\phi_n(x) \in \mathbb{Q}[x]$.

(B4) Suppose that $f = x^{n-1} + x^{n-2} + \cdots + 1 \in \mathbb{Q}[x]$ is irreducible ($n \geq 1$). Prove that n is a prime number. (Irreducible means: if $f = gh$ with $g, h \in \mathbb{Q}[x]$ then g or h is invertible in $\mathbb{Q}[x]$, and f is not invertible).

(B5) Let $f \in \mathbb{C}[x_1, x_2, x_3]$ be symmetric. Define $g_1, g_2, g_3 \in \mathbb{C}[a, b, c, d]$ by

$$g_1 = (a + b - c - d)^2, \quad g_2 = (a + c - b - d)^2, \quad g_3 = (a + d - b - c)^2.$$

Prove that $f(g_1, g_2, g_3) \in \mathbb{C}[a, b, c, d]$ is symmetric.

(B6) Write $\varepsilon := \exp(2\pi i/5)$ for the natural primitive 5th root of 1 as in section 1.5 of the notes; it is a root of the quartic $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the quadratic equation whose two roots are $\varepsilon + \varepsilon^4$ and $\varepsilon^2 + \varepsilon^3$, and hence give radical formulas for $\cos(2\pi/5)$ and $\cos(4\pi/5)$.

(C1) Carry out in detail the calculation to prove formulas (1.12) of the notes. Namely, if

$$\left.\begin{array}{rcl} \alpha_1+\alpha_2+\alpha_3+\alpha_4 &=& 0, \\ \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4 &=& r, \\ \alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4 &=& -s, \\ \alpha_1\alpha_2\alpha_3\alpha_4 &=& t, \end{array}\right\}$$

and u, v, w are defined by

$$\left.\begin{array}{rcl} 2\alpha_1 &=& u+v+w, \\ 2\alpha_2 &=& u-v-w, \\ 2\alpha_3 &=& -u+v-w, \\ 2\alpha_4 &=& -u-v+w, \end{array}\right\} \qquad (1)$$

then prove that

$$\left.\begin{array}{rcl} u^2+v^2+w^2 &=& -2r, \\ uvw &=& -s, \\ u^2v^2+u^2w^2+v^2w^2 &=& r^2-4t. \end{array}\right\} \qquad (2)$$

(C2) Let $\Sigma_k = \sum_i \alpha_i^k$ be the power sum. Express Σ_k in terms of the elementary symmetric polynomials if $k = 4, 5$. Do it for $k = 6, 7$ if you know how to use Maple or Mathematica.

(D1) Prove Newton's rule $\sum_{k=0}^{n}(-1)^k\sigma_k\,\Sigma_{n-k} = 0$ where $\Sigma_k = \sum_i \alpha_i^k$ is the power sum.

(D2) Let σ_i be the elementary symmetric functions of $\alpha_1, \dots, \alpha_n$ and τ_i the elementary symmetric functions of $\alpha_1^2, \dots, \alpha_n^2$. Prove:

$$\tau_k = \sum_{i=0}^{2k}(-1)^{k+i}\,\sigma_i\,\sigma_{2k-i}.$$

How to use Maple This is not part of the assignment sheet or of the course, but I recommend doing it. At a unix terminal, type maple; you get a clever logo, and the prompt >. For example, you can calculate $\sum \alpha_i^3$ in terms of elementary symmetric functions (*and get the right answer*!) by the following few lines:

```
> s1:=a+b+c; s2:=a*b+a*c+b*c; s3:=a*b*c;

                 s1 := a + b + c
                 s2 := a b + a c + b c
                 s3 := a b c

> expand(a^3+b^3+c^3-s1^3);

       2       2        2                 2     2       2
  - 3 a b - 3 a c - 3 a b - 6 a b c - 3 a c - 3 b c - 3 b c

> expand(%+3*s1*s2);

                  3 a b c

> evalb(expand(s1^3-3*s1*s2+3*s3) = a^3+b^3+c^3);

                   true
```

Mathematica is very similar.