THE UNIVERSITY OF WARWICK

SECOND YEAR EXAMINATION: 2004-2005

Mock exam – MA241 COMBINATORICS

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. (a) Which values should be put at the dots to make it correct? **[5]**

$$\sum_{k=1}^{n}\sum_{\ell=1}^{k} f(k,\ell) = \sum_{\ell=\bullet}^{\bullet}\sum_{m=\bullet}^{\bullet} f(\ell+m,\ell).$$

A proof is not necessary.

(b) Find a closed formula for the generating function of the numbers a_n defined by $a_0 = 1$ and **[10]**

$$3a_n = \sum_{k=0}^{n-1} 2^{n-k}\, a_k \qquad (n \geq 1).$$

(c) Let $n \geq 0$. Express $T_n = \sum_{0\leq k\leq n} 5^{\lfloor\sqrt{k}\rfloor}$ as a rational function of n and $a := \lfloor\sqrt{n}\rfloor$.

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2. (a) Let a, b be real numbers with $a < b$. Finish the following table, which gives a formula for the number of integers in four kinds of real intervals, in terms of the floor and ceiling. One entry is given as an example. A proof is not necessary. **[5]**

	X	$\lvert X\rvert$
1	$\{n \in \mathbb{Z} \mid a \leq n \leq b\}$	?
2	$\{n \in \mathbb{Z} \mid a \leq n < b\}$	?
3	$\{n \in \mathbb{Z} \mid a < n \leq b\}$	$\lfloor b\rfloor - \lfloor a\rfloor$
4	$\{n \in \mathbb{Z} \mid a < n < b\}$	?

(b) Let $X(m)$ be any set of m elements. For $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}$ with $n_1 + \cdots + n_k = m$, we define **[10]**

$$A(n_1, \ldots, n_k) = \left\{ (Y_1, \ldots, Y_k) \;\middle|\; \begin{array}{l} Y_i \subset X(m) \\ Y_1 \cup \cdots \cup Y_k = X(m) \\ |Y_i| = n_i \end{array} \right\}.$$

Prove that the cardinality of $A(n_1, \ldots, n_k)$ is

$$\frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!}.$$

You may assume, without proving it, that it is true for $k = 2$. Your proof does not need to be extremely formal.

(c) We define B_n, E_n $(n \geq 0)$ by **[10]**

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} \frac{B_n x^n}{n!}, \qquad \frac{1}{e^x + 1} = \sum_{n \geq 0} \frac{E_n x^n}{n!}.$$

Prove:

$$2^{n-1} B_n = \sum_k \binom{n}{k} B_k E_{n-k}.$$

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3. (a) The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Each natural number $n \geq 0$ has a unique Fibonary expansion

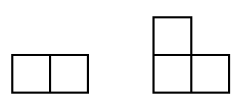
$$n = F_{k_1} + \cdots + F_{k_r},$$

where $k_i > k_{i+1} + 1$ and $k_r > 1$.

(i) Prove that the Fibonary expansion of $7F_n$ is $F_{n+4} + F_{n-4}$ if $n \geq 6$. **[7]**

(ii) Find without proof the Fibonary expansion of $3F_n$ if $n \geq 4$. **[6]**

(b) Let p_n denote the number of ways to tile a $2 \times n$ rectangle by dominoes and hooks. Express p_n in terms of $p_{n-1}, p_{n-2}, \ldots, p_{n-5}$. **[12]**



domino hook

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4. (a) Prove **[5]**

$$\binom{r}{k} = (-1)^k \binom{k-1-r}{k}, \qquad r \in \mathbb{R},\ k \in \mathbb{Z}_{\geq 0}.$$

(b) Find a closed formula for the coefficients t_n if **[10]**

$$\sum_{n\geq 0} t_n x^n = \frac{x^3}{(1+2x)(1+3x)}.$$

(c) Prove: **[10]**

$$\left\{ {n \atop m} \right\} = \sum_k \binom{n}{k} \left\{ {k+1 \atop m+1} \right\} (-1)^{n-k}.$$

You may use, without proving it, that

$$\left\{ {n+1 \atop m+1} \right\} = \sum_k \binom{n}{k} \left\{ {k \atop m} \right\},$$

and that

$$\sum_k (-1)^k \binom{n}{k} = 0 \quad \text{if } n > 0.$$

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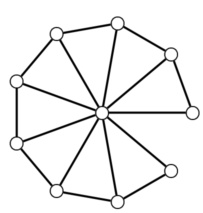
5. (a) Give a precise definition of $\left\{ {n \atop k} \right\}$. **[3]**

(b) We keep tossing a fair coin until ht^2h (head-tail-tail-head) occurs, then we stop. Let F denote the RV which counts the numbers of tosses.

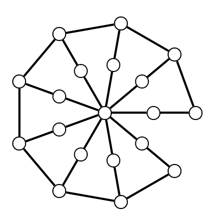
(i) Compute $F(z)$ $(:= G_F(z))$. **[8]**

(ii) Compute the expected number of tosses $E(F)$. **[4]**

(c) Let K_n be the graph obtained from the almost-wheel G_n by adding a vertex in the middle of each edge that contains the central vertex, and let s_n be the number of spanning trees for K_n. Find a recurrence formula for the s_n. **[10]**



The graph G_9



The graph K_9
