MA241 Combinatorics – Marking Sheet 5

Deadline: Monday, 15 January 2007, 3:00.

For this sheet, B3(a) and B6 will be assessed.

(B3).

(a) Let $t \in \mathbb{R}$. Find a closed formula for the exponential generating function for the numbers a_n defined by

$$a_0 = 1, \qquad a_{n+1} = t \sum_{k=0}^{n} \binom{n}{k} a_{n-k} \qquad (n \geq 0).$$

(b) Prove: $a_n = e^{-t} \sum_{k \geq 0} \frac{t^k\, k^n}{k!}$. (Ignore convergence questions.) [Don't mark this part.]

SOLUTION.
(a). **[7 points].** Entirely analogous to the Bell numbers in the printed notes. We have

$$\begin{aligned} A'(z) &= \sum_{n\geq 1} a_n \frac{z^{n-1}}{(n-1)!} = \sum_{n \geq 0} a_{n+1} \frac{z^n}{n!} = t \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} a_k \frac{z^n}{n!} \\ &= t \sum_{n \geq 0} \sum_{k=0}^{n} \frac{a_k}{k!} \frac{1}{(n-k)!} z^n = t \Big(\sum_{k\geq 0} \frac{a_k}{k!} z^k \Big) \Big(\sum_{\ell \geq 0} \frac{z^\ell}{\ell!} \Big) = t\, A(z)\, e^z. \end{aligned}$$

Therefore,

$$\frac{d}{dz} \log A(z) = \frac{A'(z)}{A(z)} = t\, e^z, \quad \log A(z) = t\, e^z + c,$$

and

$$A(z) = e^{t\, e^z + c}$$

for some constant c. Since $e^{t+c} = A(0) = a_0 = 1$ we must have $c = -t$ so

$$A(z) = e^{t\, e^z - t}.$$

(B6). Three faces of a fair dice are labelled a, two are labelled b and one c. Two players X, Y play the following game. They keep tossing the dice until aaa or bb or c occurs (for example, aaa means three a's in a row), then they stop. If aaa or bb occurs then X wins, if c occurs then Y wins.

(a) What are the winning patterns for Y?

(b) Let y_n denote the probability that Y wins after precisely n tosses. Compute

$$Y(z) := \sum_{n \geq 0} y_n \, z^n.$$

(c) What is the probability that Y wins?

SOLUTION. **[18 points in total].**
(a) [**7 points**]. The winning patterns for Y are

$$\left\{ x(y_1 \cdots y_\ell)z \;\middle|\; \begin{array}{l} \ell \geq 0, \;\; x \in \{1, b\}, \\ y_i \in \{ab, aab\}, \\ z \in \{1, a, aa\} \end{array} \right\}$$

[No proof necessary].
(b) [**7 points**]. The probabilities for a, b, c are (respectively) 1/2, 1/3, 1/6. On replacing (a, b, c) by $(z/2, z/3, z/6)$ and "or" by $+$ in a pattern winning for Y and summing over all patterns we get

$$\begin{aligned} Y(z) &= \sum_{n \geq 0} y_n \, z^n = \sum_{\ell \geq 0} (1 + \tfrac{z}{3})(\tfrac{z^2}{6} + \tfrac{z^3}{12})^\ell (1 + \tfrac{z}{2} + \tfrac{z^2}{4}) \tfrac{z}{6} \\ &= \frac{(1 + \frac{z}{3})(1 + \frac{z}{2} + \frac{z^2}{4}) \frac{z}{6}}{1 - \frac{z^2}{6} - \frac{z^3}{12}} = \frac{(3 + z)(4 + 2z + z^2)z}{6(12 - 2z^2 - z^3)}. \end{aligned}$$

(c) [**4 points**]. This is $Y(1) = \dfrac{4 \cdot 7}{6 \cdot 9} = \dfrac{14}{27}$. □