MA241 Combinatorics – Sheet 5

Deadline: Monday, 15 January 2007, 3:00.

Solutions to Section B are for handing in. Please put your solutions into the MA241 Combinatorics box in front of the Undergraduate Office.

(A1) Use binomial convolution to prove $\sum_{j=0}^{m} \binom{m+1}{j} B_j = [m = 0]$.

(A2) Compute $\sum_{k,\ell \geq 0} k \binom{k+\ell}{k} x^k \, y^\ell$.

(B1) Prove $B_n = 0$ if $n > 2$ is odd. Hint: let $f(z)$ denote the exponential generating function of the Bernoulli numbers and consider $f(z) - f(-z)$.

(B2) (a) On writing

$$f(z) = \frac{z}{e^z - 1},$$

prove that $z\, f' + (z-1)\, f + f^2 = 0$.

(b) Use (a) to find a closed expression (involving Bernoulli numbers) for

$$\sum_{k=0}^{n} \binom{n}{k} B_k \, B_{n-k}.$$

(B3) (a) Let $t \in \mathbb{R}$. Find a closed formula for the exponential generating function for the numbers a_n defined by

$$a_0 = 1, \qquad a_{n+1} = t \sum_{k=0}^{n} \binom{n}{k} a_{n-k} \qquad (n \geq 0).$$

(b) Prove: $a_n = e^{-t} \sum_{k \geq 0} \frac{t^k \, k^n}{k!}$. (Ignore convergence questions.)

(B4) Prove

$$\sum_{k,\ell \geq 0} \binom{k+\ell}{k} \frac{x^k \, y^\ell}{\ell!} = \frac{e^{y(1-x)^{-1}}}{1-x}.$$

Hint: Proposition 16 on page 51.

(B5) We have a coin which shows heads with probability p and tails with probability $1 - p = q$. We throw the coin n times. What is the probability that we get heads an even number of times?

(B6) Three faces of a fair dice are labelled a, two are labelled b and one c. Two players X, Y play the following game. They keep tossing the dice until aaa or bb or c occurs (for example, aaa means three a's in a row), then they stop. If aaa or bb occurs then X wins, if c occurs then Y wins.

(a) What are the winning patterns for Y?

(b) Let y_n denote the probability that Y wins after precisely n tosses. Compute

$$Y(z) := \sum_{n \geq 0} y_n \, z^n.$$

(c) What is the probability that Y wins?

(B7) We have a coin which shows heads with probability p and tails with probability $1 - p = q$. Fix $n \geq 1$. We keep tossing the coin until h^n occurs (n times heads in a row), then we stop. Let F denote the RV which counts the number of tosses.

(a) Compute $F(z)$ $(:= G_F(z))$. Hint: Treat the patterns starting with t separately from those starting with h.

(b) Compute the expected number of tosses $E(F)$.

(B8) Make a list of the 17 most important tricks in the printed notes, and count how many times they are used in the printed notes. Two examples are: changing order of summation (5 times), unfolding (3 times).

(C1) Let $e(n)$ denote the number of partitions of X_n such that no part has only one element. Compute the exponential generating function $E(z)$ for $e(n)$. Hint: Express the Bell number $b(n)$ as a linear combination of $e(0), \ \ldots, \ e(n)$.

(D1) (a) Prove

$$\sum_{k \geq 0} \frac{t^k \, k^n}{k!} = e^t \sum_{m \geq 0} \left\{ {n \atop m} \right\} t^m.$$

Hint: Show that the coefficients of t^k on both sides are equal.

(b) Use (a) and exercise (B3) to compute $\displaystyle\sum_{m,n \geq 0} \left\{ {n \atop m} \right\} \frac{t^m \, x^n}{n!}$.

(D2) Prove

$$\sum_{n_1, \ldots, n_k \geq 0} \min(n_1, \ldots, n_k) \, x_1^{n_1} \cdots x_k^{n_k} = \frac{x_1 \cdots x_k}{(1 - x_1) \cdots (1 - x_k)(1 - x_1 \cdots x_k)}.$$