MA241 Combinatorics – Sheet 4

Deadline: Monday, 27 November 2006, 3:00.

Solutions to Section B are for handing in. Please put your solutions into the MA241 Combinatorics box in front of the Undergraduate Office.

(A1) Write down closed formulas for the generating functions of the following sequences: $a_n = 5 \cdot (-3)^n - 7^n$, $b_n = a_n/n!$, $c_n = n\, a_n$, $d_n = a_n/(n+1)$.

(B1) Compute the generating function of $(a_0, a_1, \ldots)$ defined by $a_0 = 0$, $a_1 = 1$ and

$$a_n = -2a_{n-1} + 8a_{n-2} + 4^n \qquad (n \geq 2).$$

(B2) Find the coefficients of the generating function $B(x) = \dfrac{1}{1 + 5x + 7x^2 + 3x^3}$.

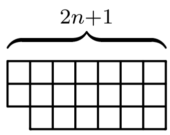
(B3) Compute the generating function of $(c_0, c_1, \ldots)$ defined by $c_0 = 1$ and

$$c_{n-1} = (-1)^n + \sum_{k=0}^{n} c_k \qquad (n \geq 1).$$

(B4) Find recurrence formulas for the numbers p_n and r_n defined as follows (don't solve the recurrences).

(a) p_n is the number of ways to tile a $2 \times n$ rectangle by 3×1 rectangles and 1×1 squares.

(b) r_n is the number of ways to tile a $3 \times 2n$ rectangle by 2×1 rectangles and 2×2 squares. Hint: Also consider s_n defined by

$s_n =$ .

(B5) Let w_n be the number of ways to bracket a string of $2n + 1$ letters ($n \geq 0$) if each pair of brackets contains 3 items, and likewise their are 3 outer items. For example, $w_2 = 3$ because we have three bracketings

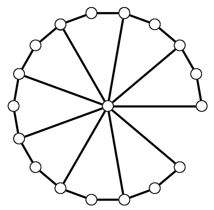
$$(abc)de, \quad a(bcd)e, \quad ab(cde)$$

and $w_3 = 12$ because we have the bracketings

$$\begin{array}{llll} (abc)(def)g, & ((abc)de)fg, & a((bcd)ef)g, & ab((cde)fg), \\ (abc)d(efg), & (a(bcd)e)fg, & a(b(cde)f)g, & ab(c(def)g), \\ a(bcd)(efg), & (ab(cde))fg, & a(bc(def))g, & ab(cd(efg)). \end{array}$$

Find a recurrence formula for the w_n analogous to the recurrence for the Catalan numbers.

(B6) Let K_n denote the graph obtained from the almost-wheel G_n by putting a vertex in the middle of each boundary edge. Find a recurrence formula for u_n, the number of spanning trees of K_n.



The graph K_9.

(B7) Compute the generating function of $(d_0, d_1, \dots)$ defined by $d_0 = 1/2$, $d_n = 0$ for $n < 0$ and

$$nd_n = \sum_{k=0}^{n} d_{k-2}(-1)^{n-k} \qquad (n \geq 1).$$

(B8) Give another proof of the Inversion Formula (Theorem 7 on page 30) using generating functions and binomial convolution (page 63). Hint: Consider the exponential generating functions of the sequences $f(n)$ and $g(n)$ and $(-1)^n f(n)$ and $(-1)^n g(n)$.

(C1) (a) Find a closed formula for $f(x) = \sum_{n\geq 0} H_n x^n$.

(b) Find a closed formula for $g(x) = \sum_{n\geq 0} H_n \frac{x^{n+1}}{n+1}$.

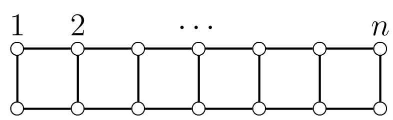
(C2) Let $k, \ell \geq 0$ and $p \geq k + \ell$. Prove

$$\sum_{m=k}^{p-\ell} \binom{m}{k}\binom{p-m}{\ell} = \binom{p+2}{k+\ell+2}.$$

Hint: Apply Proposition 16 on page 51 to $n = k+1$ and $n = \ell + 1$ and multiply. Apply a third time and compare coefficients.

(C3) Compute $\sum_{k\geq 1} \frac{x^{k+2}}{k(k+2)}$. Hint: Partial fractions.

(C4) Let t_n denote the number of spanning trees of the ladder graph of $2n$ vertices. Find a closed expression for the generating function of these numbers.



The ladder graph of $2n$ vertices.

(D1) Give a closed formula for $\sum_{n\geq 0} \binom{2n+1}{n} z^n$.