MA241 Combinatorics – Marking Sheet 3

Deadline: Monday, 13 November 2006, 3:00.

For this sheet, (B4) and (B8) are marked.

(B4). Let $0 \le \ell \le n$. Prove:

$$\begin{bmatrix} n+1 \\ \ell+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{\ell}.$$

Hint: Apply part 4 of Theorem 12 to x and $x+1$ and use the binomial theorem.

SOLUTION. **[15 marks].** We have

$$\begin{aligned}
\sum_k \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k &= x^{\overline{n+1}} && \text{by Theorem 12 part 4} \\
&= x(x+1)^{\overline{n}} \\
&= x \sum_k \begin{bmatrix} n \\ k \end{bmatrix} (x+1)^k && \text{by Theorem 12 part 4} \\
&= x \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \sum_\ell \binom{k}{\ell} x^\ell && \text{by the Binomial Theorem} \\
&= \sum_\ell x^{\ell+1} \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{\ell}.
\end{aligned}$$

Taking the coefficients of $x^{\ell+1}$ on both sides gives the desired result. □

(B8). Every integer $n > 0$ can uniquely be written in the *Fibonary expansion*

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$

where $k_i > k_{i+1} + 1$ and $k_r > 1$.

(a) Prove that the Fibonary expansion of $9F_n$ (with $n \ge 6$) is

$$9F_n = F_{n+4} + F_{n+1} + F_{n-2} + F_{n-4}.$$

(b) Find without proof the Fibonary expansion of $4F_n + F_{n-1}$ ($n \ge 1$).

SOLUTION. (a). **[4 marks].** One proves that

$$9F_n = F_{n+4} + F_{n+1} + F_{n-2} + F_{n-4}$$

is true (say, for $n \ge 4$) by induction. It is then immediate that the right hand side is indeed the Fibonary expansion provided $n \ge 6$.

(b). **[6 marks in total, of which 4 for the case $n \ge 4$].**

$$4F_n + F_{n-1} = \begin{cases} F_4 + F_2 & n = 1, \\ F_5 & n = 2, \\ F_6 + F_2 & n = 3, \\ F_{n+3} + F_{n-2} & n \ge 4. \end{cases}$$

□