MA241 Combinatorics – Sheet 3

Deadline: Monday, 13 November 2006, 3:00.

Solutions to Section B are for handing in. Please put your solutions into the MA241 Combinatorics box in front of the Undergraduate Office.

(A1) In the lectures we unfolded the basic recurrence formula

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$$

one way, to obtain

$$\left\{ {n \atop k} \right\} = \Big(\sum_{m=0}^{\ell-1} (k-m) \left\{ {n-m-1 \atop k-m} \right\} \Big) + \left\{ {n-\ell \atop k-\ell} \right\}.$$

Now unfold the other way, keeping $\left\{ {n-1 \atop k-1} \right\}$ but continuing with $k\left\{ {n-1 \atop k} \right\}$ and repeating any number of times. A proof is not necessary.

(A2) By using the identity $\left[{n \atop k} \right] = (n-1)\left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right]$ in the case $k = 2$, prove by induction that

$$\left[{n+1 \atop 2} \right] = n!\, H_n \qquad \text{for all } n \geq 0.$$

(B1) Use the inversion theorem to prove

$$\left\{ {n \atop m} \right\} = \sum_k \binom{n}{k} \left\{ {k+1 \atop m+1} \right\} (-1)^{n-k} \qquad (n, m \geq 0).$$

You may use, without proving it, that

$$\left\{ {n+1 \atop m+1} \right\} = \sum_k \binom{n}{k} \left\{ {k \atop m} \right\} \qquad (n, m \geq 0).$$

(B2) Let $n \geq 0$. Prove

$$\sum \binom{2n}{2k, 2\ell, 2m} = \frac{9^n + 3}{4},$$

where the sum is over all triples $(k, \ell, m) \in (\mathbb{Z}_{\geq 0})^3$ with $k + \ell + m = n$. Hint: Use the trinomial theorem to expand

$$\sum_{x,y \in \{-1,1\}} (1 + x + y)^{2n}.$$

(B3) Put $X_m = \{1, \ldots, m\}$. How many surjective maps from X_m to X_n are there?

(B4) Let $0 \le \ell \le n$. Prove:

$$\left[{n+1 \atop \ell+1}\right] = \sum_k \left[{n \atop k}\right]\binom{k}{\ell}.$$

Hint: Apply part 4 of Theorem 12 to x and $x+1$ and use the binomial theorem.

(B5) Prove $\displaystyle\sum_{k=0}^{n} k^m = \sum_\ell \left\{{m \atop \ell}\right\}\frac{(n+1)^{\underline{\ell+1}}}{\ell+1}$. Hint: (10) on page 13 of the notes.

(B6) Suppose we balance n cards over the edge of a table, all cards being of length 2, but of different weights. What is the largest possible overhang if card k (counted from the top) has weight $1/k$? Does the overhang tend to ∞ as $n \to \infty$?

(B7) Prove Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for the Fibonacci numbers F_n. Use induction on n, but you must prove it for negative as well as for positive n.

(B8) Every integer $n > 0$ can uniquely be written in the *Fibonary expansion*

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$

where $k_i > k_{i+1} + 1$ and $k_r > 1$.

(a) Prove that the Fibonary expansion of $9F_n$ (with $n \ge 6$) is

$$9F_n = F_{n+4} + F_{n+1} + F_{n-2} + F_{n-4}.$$

(b) Find without proof the Fibonary expansion of $4F_n + F_{n-1}$ $(n \ge 1)$.

(B9) Let $A(z) = \sum_n a_n z^n$, $B(z) = \sum_n b_n z^n$, $C(z) = \sum_n c_n z^n$, and assume $0 = a_n = b_n = c_n$ for negative n. Express C in A and B in each of the following cases.

(a) $c_n = a_{2n}$.

(b) $c_n = a_{\lfloor n/2 \rfloor - 1}$.

(c) $c_n = 4^n(n+3)\,a_n + b_n$.

(d) $c_n = \displaystyle\sum_{k=0}^{n} 5^n\, a_k\, b_{n-k}$.

(e) $c_n = \displaystyle\sum_{k=1}^{n} \frac{2^k\, a_{n-k}}{k}$.

(f) $c_n = \displaystyle\sum_{2k+\ell \le n} a_k\, b_\ell$.

(C1) Let $0 \le \ell \le n$. Prove:

$$\sum_k (-1)^{k-\ell}\left\{{n \atop k}\right\}\left[{k \atop \ell}\right] = [n = \ell].$$

Hint: Use parts 1 and 3 of Theorem 12.

(C2) Prove $\displaystyle\left[{n \atop k}\right] = \sum_{m=0}^{n-1}\left[{n-1-m \atop k-1}\right](n-1)^{\underline{m}}$ for nonnegative integers k, n.