MA241 Combinatorics – Marking Sheet 2

Deadline: Monday, 30 October 2006, 3:00.

For this sheet, (B3) and (B6) will be assessed.

(B3). Find a necessary and sufficient condition on the real number $b > 1$ for the equality

$$\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor$$

to hold for all real $x \geq 1$.

Solution. **[15 marks in total.]** We claim that the required condition is $b \in \mathbb{Z}$. (We assume $b > 1$.) **[3 marks.]**

Proof of necessary. Suppose $b \notin \mathbb{Z}$. Put $x := b$. Then

$$\lfloor \log_b x \rfloor = \lfloor \log_b b \rfloor = \lfloor 1 \rfloor = 1$$

but $\lfloor x \rfloor < b$ so $\log_b \lfloor x \rfloor < 1$ so $\lfloor \log_b \lfloor x \rfloor \rfloor \neq 1$. So the studied identity is false, a contradiction, and $b \in \mathbb{Z}$. **[6 marks.]**

Proof of sufficient. Suppose

$$b^k \leq x < b^{k+1} \tag{1}$$

for some $k \in \mathbb{Z}_{\geq 0}$. Then

$$k \leq \log_b x < k+1 \text{ whence } \lfloor \log_b x \rfloor = k \tag{2}$$

Moreover, (1) and (2) are also true for $\lfloor x \rfloor$ instead of x. Therefore

$$\lfloor \log_b x \rfloor = k = \lfloor \log_b \lfloor x \rfloor \rfloor \qquad \textbf{[6 marks].}$$ □

(B6). How many solutions $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ are there of

$$\begin{cases} x_1 = 2005 \\ x_i \leq x_{i+1} < x_i + i^2 \quad \text{for all } i? \end{cases}$$

Solution. **[10 marks.]** This is done by a homogeneous tree. One chooses a solution in $n-1$ steps. In step i one chooses x_{i+1} which can be done i^2 ways. We note that this is independent of what values had been chosen so far, that is, we are dealing with a homogeneous tree. Therefore, the answer is obtained by multiplying these numbers and the result is

$$2^2\, 3^2 \cdots (n-1)^2 = (n-1)!^2.$$ □