MA241 Combinatorics – Sheet 2

Deadline: Monday, 30 October 2006, 2:00.

Solutions to Section B are for handing in. Please put your solutions into the MA241 Combinatorics box in front of the Undergraduate Office.

(B1) Let $n \geq 0$. Express $T_n = \sum_{0 \leq k \leq n} 5^{\lceil \sqrt{k} \rceil}$ in terms of n and $a := \lceil \sqrt{n} \rceil$.

(B2) Show that if $n \in \mathbb{Z}$ then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$. Which integers n satisfy $\lfloor n/2 \rfloor + \lfloor n/3 \rfloor + \lceil n/6 \rceil = n$?

(B3) Find a necessary and sufficient condition on the real number $b > 1$ for the equality

$$\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor$$

to hold for all real $x \geq 1$.

(B4) (a) All women in a village want to get married. How many ways are there to achieve this if there are k women and ℓ men in the village?

(b) Same question if a man can have more than one wife but not conversely.

(B5) (a) How many ways are there to divide p indistinguishable cakes among q people?

(b) Same question if nobody gets more than one cake.

(B6) How many solutions $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ are there of

$$\begin{cases} x_1 = 2005 \\ x_i \leq x_{i+1} < x_i + i^2 \quad \text{for all } i? \end{cases}$$

(B7) Let $n \geq 1$ be an integer. Prove:

(a). $\displaystyle\sum_k \binom{n}{k} = 2^n;$ (b). $\displaystyle\sum_k \binom{n}{2k} = 2^{n-1}.$

(B8) (a) Prove

$$\sum_{k,\ell \in \mathbb{Z}} \binom{a}{k} \binom{b}{\ell} \binom{c}{n-k-\ell} = \binom{a+b+c}{n} \qquad (1)$$

where $a, b, c \in \mathbb{R}$, $n \in \mathbb{Z}_{\geq 0}$ by mimicking our combinatorial proof of Vandermonde convolution.

(b) Give another proof of (1) by using Vandermonde convolution twice.

(B9) Give a closed formula for

$$T = \sum_{k=0}^{m} (n+1-k) \binom{m}{k} \binom{n}{k}^{-1} \qquad (n \geq m \geq 0 \text{ integers}).$$

(C1) Express $\sum_{0\le k\le n} \lfloor \sqrt{k}/2 \rfloor$ in terms of n and $a := \lfloor \sqrt{n}/2 \rfloor$.

(C2) How many of the numbers 2^k, for $0 \le k \le n$, have leading digit 1 in decimal notation?

(C3) In this exercise you will prove the binomial theorem using differentation. Define

$$f_n(x,y) = (x+y)^n - \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

You will prove $f_n = 0$ by induction on n. If it is known for $n-1$, prove that the *partial derivatives* vanish:

$$\frac{\partial}{\partial x} f_n = 0 = \frac{\partial}{\partial y} f_n \qquad (2)$$

(the partial differentation $\partial/\partial x$ means to differentiate with respect to x, viewing y constant, and vice versa). Use (without proving it) that (2) implies that f_n is constant.

(C4) (a) Let $n \ge 0$. Prove:

$$\left(\sum_{k=0}^{n} \binom{n}{k} (-x)^k\right)\left(\sum_{k=0}^{n} \binom{n}{k} x^k\right)\left(\sum_{k=0}^{n} \binom{n}{k} x^{2k}\right) = \sum_{k=1}^{n} \binom{n}{k} (-x^4)^k.$$

(b) What identity do you obtain by extracting the coefficients of x^{4t} ($t \in \mathbb{Z}_{\ge 0}$) on both sides of the identity in (a)? A proof is not necessary.

(D1) Evaluate the sum $S_n = \sum_{k\ge 1} \lfloor n/2^k + 1/2 \rfloor$ for integers $n \ge 0$.

(D2) Prove

$$\sum_k \binom{n}{k} \binom{n}{2m-k} (-1)^k = \binom{n}{m} (-1)^m$$

for nonnegative integers m, n.