MA241 Combinatorics – Marking Sheet 1

Deadline: Monday, 23 October 2006, 2:00.

For this sheet, (B4)(b) and (B10) will be assessed.

(B4)(b). Use the summation factor method to solve the following, that is, to find a closed formula for b_n $(n \geq 0)$.
(b) $b_0 = 0,\ b_n = 3\,b_{n-1} + 4^n + 5^{-n}$.

SOLUTION. **[10 marks].** Write $b_n = 3^n\,c_n$. Then the recursion becomes

$$c_n = c_{n-1} + (\tfrac{4}{3})^n + 15^{-n}.$$

Therefore

$$c_n = \sum_{k=1}^{n} \Big((\tfrac{4}{3})^n + 15^{-n} \Big) = \frac{(\frac{4}{3})^{n+1} - \frac{4}{3}}{\frac{4}{3} - 1} + \frac{(\frac{1}{15})^{n+1} - \frac{1}{15}}{\frac{1}{15} - 1}$$

$$= \frac{(\frac{4}{3})^n - 1}{1 - \frac{3}{4}} + \frac{(\frac{1}{15})^n - 1}{1 - 15}$$

and

$$b_n = 3^n\,c_n = \frac{4^n - 3^n}{1 - \frac{3}{4}} + \frac{(\frac{1}{5})^n - 3^n}{1 - 15}$$

$$= 4(4^n - 3^n) + \frac{5^{-n} - 3^n}{-14} = 4^{n+1} - \frac{1}{14}5^{-n} - \frac{55}{14}3^n. \qquad \square$$

(B10). Express $k(k+2)(k+4)$ as a linear combination of rising powers of k. Use your result and (10) on page 13 of the notes to compute

$$\sum_{k=1}^{n} k(k+2)(k+4).$$

SOLUTION. **[15 marks].** There are constants a, b such that

$$k(k+2)(k+4) = k(k+1)(k+2) + a\,k(k+1) + b\,k$$

for all k. By putting $k = -1$ we find $(-1)(1)(3) = -b$, that is, $b = 3$. By looking at the coefficient of k^2 we find $6 = 3 + a$, that is, $a = 3$. Using (11) on page 13 about rising powers rather than what was wrongly suggested in the exercise about falling powers, we find

$$\sum_{k=0}^{n} k(k+2)(k+4) = \sum_{k=0}^{n} (k^{\overline{3}} + 3\,k^{\overline{2}} + 3\,k^{\overline{1}}) = \frac{n^{\overline{4}}}{4} + 3\,\frac{n^{\overline{3}}}{3} + 3\,\frac{n^{\overline{2}}}{2}$$

$$= \frac{n(n+1)}{4}\Big((n+2)(n+3) + 4(n+2) + 6\Big)$$

$$= \frac{n(n+1)}{4}(n^2 + 9n + 20) = \frac{n(n+1)(n+4)(n+5)}{4}.$$

(The last factorisation is not required). $\square$