# MA241 Combinatorics – Sheet 1

Deadline: Monday, 23 October 2006, 2:00.

Solutions to Section B are for handing in. Section A is easier and is meant as a warming up. Questions in section C are similar to those in section B but not for handing in. Section D is harder and optional.
Please put your solutions into the MA241 Combinatorics box in front of the Undergraduate Office. Mention your department if it is not mathematics.

(A1) (a) Prove the law of exponents for falling powers: $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$.

(b) What is the law of exponents for rising powers? Use it to define $x^{\overline{m}}$ for $m$ a negative integer.

---

(B1) Suppose the rules of the Tower of Hanoi puzzle are changed so that all moves must be to or from the spare peg C. So, for example, to move one disc from A to B you have to take two moves, moving it first from A to C and then from C to B. Let $X_n$ denote the number of moves required to move a pile of $n$ discs from A to B. Find a recurrence relation for $X_n$ but don't solve it.

(B2) We saw during the lectures that $n$ infinite straight lines divide the plane into $n(n+1)/2+1$ regions (provided that no two lines are parallel and no three lines meet at a point). How many of these regions are bounded and how many unbounded?

(B3) Suppose $S_n = S_{n-1} + a_n$ for all integers $n$, and $S_0 = 0$. For positive $n$, express $S_n$ and $S_{-n}$ as a sum of various $a_k$, using the sum sign $\sum$. (For $S_n$ and $S_{-n}$ the answers are required to be different.)

(B4) Use the summation factor method to solve the following, that is, to find a closed formula for $a_n$ and $b_n$ $(n \geq 0)$.

(a) $a_0 = 0,\ 3\,a_n = n\,a_{n-1} - 6 \cdot n!$.

(b) $b_0 = 0,\ b_n = 3\,b_{n-1} + 4^n + 5^{-n}$.

(B5) (a) Differentiate the formula for $\sum_{k=0}^{n-1} x^k$ with respect to $x$ to find a formula for $\sum_{k=0}^{n-1} kx^{k-1}$.

(b) What do you get from the above results by putting $n = \infty$?

(B6) What should be there at the bullets? Draw the pictures involved like the one on page 11 of the lecture notes.

(a) $$\sum_{k=0}^{n} \sum_{\ell=k}^{n} f(k,\ell) = \sum_{\ell=\bullet}^{n} \sum_{k=0}^{\bullet} f(k,\ell).$$

(b) $$\sum_{k=0}^{n} \sum_{\ell=k}^{n} f(k,\ell) = \sum_{m=\bullet}^{\bullet} \sum_{k=\bullet}^{\bullet} f(k,k-m).$$

(c) $\displaystyle\sum_{k=0}^{n}\sum_{\ell=k}^{2k} f(k,\ell) = \sum_{k=\bullet}^{\bullet}\sum_{m=\bullet}^{k} f(k,\bullet).$

(B7) Prove $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2 = 2n\Big(\sum_{i=1}^{n} a_i^2\Big) - 2\Big(\sum_{i=1}^{n} a_i\Big)^2.$

(B8) Show $x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = 1/(x-1)^{\underline{-m}}$.

(B9) Deduce (11) from (9) (pages 13 and 12 of the notes).

(B10) Express $k(k+2)(k+4)$ as a linear combination of rising powers of $k$. Use your result and (10) on page 13 of the notes to compute

$$\sum_{k=1}^{n} k(k+2)(k+4).$$

---

(C1) Evaluate $S_n = \displaystyle\sum_{k=0}^{n}(k+2)(-5)^k$ using the perturbation method.

(C2) We have

$$x^3 = x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} = x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}.$$

Express $x^4$ as a sum of falling and of rising powers. Use your result and (10) on page 14 of the lecture notes to compute $\sum_{1\le k\le n} k^4$.

---

(D1) Express $\dfrac{1}{1} + \dfrac{1}{3} + \cdots + \dfrac{1}{2n-1}$ in terms of harmonic numbers.

(D2) Express $\sum_{k=1}^{n} k\,H_k$ in terms of $n$ and $H_n$.

(D3) Prove *Lagrange's identity* without using induction:

$$\sum_{1\le j\le k\le n}(a_j b_k - a_k b_j)^2 = \Big(\sum_{k=1}^{n} a_k^2\Big)\Big(\sum_{k=1}^{n} b_k^2\Big) - \Big(\sum_{k=1}^{n} a_k b_k\Big)^2,$$

where $a_i$, $b_j$ are real numbers.

(Note that this identity immediately implies *Cauchy's inequality:*

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^2 \le \Big(\sum_{k=1}^{n} a_k^2\Big)\Big(\sum_{k=1}^{n} b_k^2\Big),$$

which says that the absolute value of the scalar product of two vectors is less than or equal to the product of their lengths.)

(D4) Compute

$$\sum_{k\ge 0}\frac{1}{(2k+1)(2k+3)\cdots(2k+2n+1)}.$$

Hint: First find the sum over $\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\}$ and over $\{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \cdots\}$.