

MA106 – Linear Algebra

Assignment 9

March 2011

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems 1, 2, 3 and 5 only must be handed in by **2.00 pm** on **THURSDAY 17 MARCH** (Thursday of the tenth week of term), or they will not be marked.

1. Let $\mathbb{R}[x]_{\leq 2}$ be the vector space of polynomials of degree at most 2 over \mathbb{R} . Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be the bases $1, x, x^2$ and $1, (1-x), (1+x)^2$ of $\mathbb{R}[x]_{\leq 2}$.

(i) Write down the basis change matrices from basis \mathbf{e}'_i to basis \mathbf{e}_i and from basis \mathbf{e}_i to basis \mathbf{e}'_i . [2 marks]

(Note: Use the definition in the lecture notes, which says that the basis change matrix from basis \mathbf{e}_i to basis \mathbf{e}'_i has columns that represent \mathbf{e}_i in terms of the \mathbf{e}'_i .)

(ii) Use (i) to write the polynomial $3 - 4x - 2x^2$ as a linear combination of $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, and then check your answer by direct calculation. [2 marks]

2. Determine whether the following pairs of matrices are equivalent. [2 marks each]

(i) $\begin{pmatrix} 1 & 3 & -2 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & -2 \\ -2 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

(ii) $\begin{pmatrix} 1 & -1 & -2 & 2 \\ -2 & -1 & 1 & -3 \\ -1 & -5 & -4 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & -1 & -2 \\ -1 & 1 & 3 & 1 \end{pmatrix}$.

3. Express the matrix $\begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix}$ as a product of elementary matrices. [5 marks]

4. Find the eigenvalues of each of the following matrices A over the complex numbers \mathbb{C} . For each eigenvalue find one corresponding eigenvector, and then write down a matrix P such that $P^{-1}AP$ is diagonal.

(i) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$, (ii) $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, (iii) $\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$.

5. Find the eigenvalues of the following pairs of matrices, and use them to decide which of the pairs are similar.

(i) $\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}$ and $\begin{pmatrix} 9 & 10 \\ -1 & 1 \end{pmatrix}$; [2 marks]

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 4 & -3 & -2 \\ -4 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 4 & 10 & -1 \end{pmatrix}$; [2 marks]

(iii) $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \end{pmatrix}$. [3 marks]

6. Show that, if λ is an eigenvalue of the matrix A , then λ^2 is an eigenvalue of A^2 . Is the converse true? (Give proof or counterexample.)

7. Let $T : V \rightarrow V$ be a linear map satisfying $T^2 = T$.

(i) Show that $\text{Im}(T)$ and $\text{Ker}(T)$ are complementary subspaces of V . (See Section 4 of the course for the definition of complementary subspaces.)

[*Hint*: Write each $\mathbf{v} \in V$ as $\mathbf{v} = T(\mathbf{v}) + (\mathbf{v} - T(\mathbf{v}))$.]

(ii) Show that a basis of $\text{Im}(T)$ and a basis of $\text{Ker}(T)$ together form a basis of V .

(iii) Deduce that T can be represented by the matrix

$$\begin{pmatrix} I_r & \mathbf{0}_{r,n-r} \\ \mathbf{0}_{n-r,r} & \mathbf{0}_{n-r,n-r} \end{pmatrix}.$$

8. (past exam. question)

(i) State without proof two conditions, each of which implies that an $n \times n$ matrix is similar to a diagonal matrix.

(ii) Find the eigenvalues of the matrix $\begin{pmatrix} 3 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{pmatrix}$.

(iii) For each eigenvalue find the set of all possible eigenvectors.

(iv) Determine whether this matrix is similar to a diagonal matrix, giving reasons for your conclusion.