

MA106 – Linear Algebra

Assignment 5

February 2011

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems 1, 3 and 4 only must be handed in by **2.00 pm** on **THURSDAY 17 FEBRUARY** (Thursday of the sixth week of term), or they will not be marked.

1. Let U and V be vector spaces of dimensions n and m over K , and let $\text{Hom}_K(U, V)$ be the vector space over K of all linear maps from U to V . Find the dimension and describe a basis of $\text{Hom}_K(U, V)$. (You may find it helpful to use the correspondence with $m \times n$ matrices over K .) [4 marks]

2. Reduce the following matrices to row-reduced form by using elementary row operations, and hence find their ranks.

(i) $\begin{pmatrix} 1 & 2 & -6 \\ -3 & -6 & 18 \\ 2 & 4 & -12 \end{pmatrix};$ (ii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix};$

(iii) $\begin{pmatrix} 0 & 1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 2 & 0 \\ 2 & -1 & 2 & -2 & 1 \end{pmatrix}.$

3. Write down the augmented matrices for the following systems of simultaneous linear equations (with coefficients in \mathbb{R}), reduce the matrices to row-reduced form, and then write down the set of all solutions of the equations in the form $\underline{\mathbf{x}} + \text{Nullspace}(A)$ where A is the matrix for the original system; for example, one correct solution to (iii) is

$$(-1, 1, 0) + \{\alpha(2, -1, 0) + \beta(-3, 0, 1) \mid \alpha, \beta \in \mathbb{R}\},$$

but you should choose a different particular solution $\underline{\mathbf{x}}$ in your solution to (iii)!

[2 marks each]

(i) $5x - y = 4, \quad x + y = 1, \quad 3x - 3y = 2;$

(ii) $-y + 2z = 0, \quad x - y - z = -2, \quad -x - y + 5z = 2, \quad -2x - y + 8z = 4;$

(iii) $x + 2y + 3z = 1, \quad 2x + 4y + 6z = 2;$

(iv) $x - y + z = 0, \quad x + y - z = -1, \quad x + 5y - 5z = 3.$

4. Let $T : V \rightarrow V$ be a linear map, where V is a finite-dimensional vector space. Then T^2 is defined to be the composite TT of T with itself, and similarly $T^{i+1} = TT^i$ for all $i \geq 1$. Suppose that $\text{Rank}(T) = \text{Rank}(T^2)$.

(i) Prove that $\text{Im}(T) = \text{Im}(T^2)$. [2 marks]

(ii) For $i \geq 1$, let $U_i : \text{Im}(T) \rightarrow \text{Im}(T)$ be defined as the restriction of T^i to the subspace $\text{Im}(T)$ of V . Show that U_i is nonsingular for all i . [3 marks]

(iii) Deduce that $\text{Rank}(T) = \text{Rank}(T^i)$ for all $i \geq 1$. [3 marks]

5. Show that there is only one possible row reduced form that can be obtained from a given matrix A by performing elementary row operations.