THE UNIVERSITY OF WARWICK

FIRST YEAR EXAMINATION: JUNE 2010

LINEAR ALGEBRA

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) Let V be a vector space over a field K. Explain what it means for vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ to

(i) *span* V; **[2]**

(ii) be *linearly independent;* **[2]**

(iii) form a *basis* of V. **[1]**

b) Prove that if $\mathbf{v_1}, \ldots, \mathbf{v_n}$ span V, then some subsequence of $\mathbf{v_1}, \ldots, \mathbf{v_n}$ forms a basis for V. **[6]**

c) In each of the following examples, decide whether the given vectors are linearly independent, and whether they span the vector space in question. If they do span, find a subset of them which form a basis for the vector space; if they do not, write down a vector which is not a linear combination of the ones given. In all cases the field is $K = \mathbb{R}$.

(i) $V = \mathbb{R}^3$, vectors $(1, 0, -1)$, $(0, -1, 1)$, $(-2, 1, 1)$; **[3]**

(ii) $V = \mathbb{R}^3$, vectors $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(1, 1, 1)$; **[3]**

(iii) $V = \mathbb{R}[x]_{\leq 2}$, vectors $(1 + x)$, $(1 - x)$, x, $(1 + x^2)$; **[4]**

(iv) $V = \mathbb{C}^2$, vectors $(1, 0)$, $(0, 1)$, (i, i). **[4]**

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2. a) Let U and V be vector spaces over a field K. Suppose that $\mathbf{e_1}, \ldots, \mathbf{e_n}$ is a basis for U and that $\mathbf{f_1}, \ldots, \mathbf{f_m}$ is a basis for V. Let $T\colon U \to V$ be a linear transformation. Describe how to write down the $m \times n$ matrix which represents T with respect to the given bases. **[3]**

b) If S and T are two linear transformations from U to V, define the linear transformation $S+T$. If S and T are represented by matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, with respect to the given bases, show that $S+T$ is represented by the matrix $\mathbf{A}+\mathbf{B}$. **[4]**

c) Let M be the function

$$M\colon \operatorname{Hom}_K(U,V) \to K^{m,n}$$

which associates to each linear transformation $U \to V$ its matrix with respect to the given bases. Show that M is a linear transformation. What is the kernel of M? **[6]**

d) For each of the following linear transformations of vector spaces over $\mathbb{R}$, write down the matrix representing the transformation with respect to the given bases.

(i) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+y, x-y)$, basis $(1,0)$, $(1,1)$ for both; **[3]**

(ii) $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x+y+z, x-y)$, bases $(1,0,0)$, $(1,1,0)$, $(1,1,1)$ for $\mathbb{R}^3$ and $(1,0)$, $(0,1)$ for $\mathbb{R}^2$; **[3]**

(iii) $D\colon \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 2}$, $D(f) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - f$, basis $1, x, x^2$ for both; **[3]**

(iv) $I\colon \mathbb{C} \to \mathbb{R}$, $I(z) =$ imaginary part of z, bases $1, i$ for $\mathbb{C}$ and 1 for $\mathbb{R}$. **[3]**

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3. a) Define the *determinant* of an $n \times n$ matrix A over a field K. **[3]**

b) Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 4 \\ 3 & 0 & -1 & 6 \end{pmatrix}$$

over the real numbers $\mathbb{R}$. Using row reduction, find the inverse of $\mathbf{A}$. Show clearly what you are doing at each step. **[12]**

c) Using your calculations from part (b), or otherwise, work out the determinant of $\mathbf{A}$. What is the determinant of $\mathbf{A}^{2010}$? **[6]**

d) If α is any real number, find a solution (involving α) to the equation

$$\mathbf{Ax} = \begin{pmatrix} \alpha \\ 1-\alpha \\ \alpha \\ 0 \end{pmatrix}.$$

[4]

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4. a) Define the *eigenvectors* and *eigenvalues* of an $n \times n$ matrix over a field K. **[3]**

b) Prove that $\lambda \in K$ is an eigenvalue of the $n \times n$ matrix $\mathbf{A}$ if and only if λ is a solution to the characteristic equation **[5]**

$$\det(\mathbf{A} - x\mathbf{I_n}) = 0.$$

c) State what it means for two $n \times n$ matrices to be *similar*, and prove that similar matrices have the same eigenvalues. **[6]**

d) Consider the following two matrices over the real numbers $\mathbb{R}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 2 & 4 \end{pmatrix}.$$

They both have an eigenvalue 1. Describe the eigenvectors corresponding to this eigenvalue for both $\mathbf{A}$ and $\mathbf{B}$. **[8]**

e) Are $\mathbf{A}$ and $\mathbf{B}$ similar? Justify your answer. **[3]**

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5. Determine whether the following statements are true or false. If true, give a brief justification; if false, give a counterexample. (You will get no marks unless you give either a justification or a counterexample.) You will get one bonus mark if you provide full and correct answers to at least 6 parts.

a) Two distinct vectors in $\mathbb{R}^2$ always form a basis for $\mathbb{R}^2$. **[3]**

b) If $\mathbf{A}$ is an $n \times n$ matrix whose entries are all integers, and $\det(\mathbf{A}) = 1$, then $\mathbf{A}^{-1}$ also has all integer entries. **[3]**

c) If $\mathbf{A}$ and $\mathbf{B}$ are invertible $n \times n$ matrices, then $\operatorname{rank}(\mathbf{A} + \mathbf{B}) = \operatorname{rank}(\mathbf{AB})$. **[3]**

d) Every 2×2 matrix is diagonalisable. **[3]**

e) If $\mathbf{A}$ is a 4×4 matrix over $\mathbb{C}$, then $\det(i\mathbf{A}) = \det(\mathbf{A})$. **[3]**

f) An $n \times n$ matrix is singular if and only if it has 0 as an eigenvalue. **[3]**

g) Two similar matrices always have the same eigenvectors. **[3]**

h) If $\mathbf{A}$ is an orthogonal matrix, then $\det(\mathbf{A}) = 1$. **[3]**
