

MA106 – Linear Algebra

Assignment 4

February 2011

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **2, 4, 5** and **6** only must be handed in by **2.00 pm** on **THURSDAY 10 FEBRUARY** (Thursday of the fifth week of term), or they will not be marked.

1. For the following pairs of matrices A, B , determine whether any of $A + B$, AB and BA are defined and calculate those that are defined.

$$1. A = \begin{pmatrix} -1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}; \quad 2. A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -3 \\ 1 & 2 \end{pmatrix};$$

$$3. A = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -3 & 2 \end{pmatrix}; \quad 4. A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Are the following identities satisfied by all $n \times n$ matrices A and B , where $I = I_n$ is the $n \times n$ identity matrix? Give either a proof or counterexample. [2 marks each]

- (i) $(A - B)(A + B) = A^2 - B^2$;
- (ii) $(A - I)(A^2 + A + I) = A^3 - I$.

3. Find the matrices representing the following linear maps $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$, with respect to the standard bases of $\mathbb{R}^3, \mathbb{R}^4$:

- (i) $T(\alpha, \beta, \gamma) = (\alpha + \beta + \gamma, 2\beta - \gamma, \alpha - \beta, \alpha - \beta)$;
- (ii) $T(\alpha, \beta, \gamma) = (0, \alpha + \gamma, \gamma, 2\alpha - 3\gamma)$.

4. Find the matrices representing the following geometrical transformations $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, with respect to the standard basis.

- (i) T rotates each point through an angle θ about the x -axis in the direction that moves the point $(0, 0, 1)$ towards the point $(0, -1, 0)$. [2 marks]
- (ii) T reflects each point in the (x, z) -plane. [1 marks]

5. (Effect of using a different basis.) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear map defined by $T(\alpha, \beta) = (\alpha + \beta, \alpha - \beta)$. Write down the matrix of T , firstly using the standard basis of \mathbb{R}^2 and secondly using the basis $(-1, 1), (0, -2)$ of \mathbb{R}^2 .

[1+2 marks]

6. Let $S : U \rightarrow V$ and $T : V \rightarrow W$ be linear maps, where U, V and W are vector spaces over the same field K . Prove the following:

- (i) $\text{Rank}(TS) \leq \text{Rank}(T)$; [2 marks]
- (ii) $\text{Rank}(TS) \leq \text{Rank}(S)$; [3 marks]
- (iii) If $U = V$ and S is nonsingular then $\text{Rank}(TS) = \text{Rank}(T)$; [2 marks]
- (iv) If $V = W$ and T is nonsingular then $\text{Rank}(TS) = \text{Rank}(S)$. [3 marks]

[This question is quite difficult, but important. (i) and (iii) are easier than (ii) and (iv). As a hint for (ii), note that if $\text{Rank}(S) = r$, then $\text{Im}(S)$ is spanned by r vectors.]

7. Let $T : V \rightarrow W$ be a linear map, where $\dim(V)$ and $\dim(W)$ may be assumed to be finite. Prove:

- (i) T is injective if and only if there is a linear map $S : W \rightarrow V$ with $ST = I_V$;
- (ii) T is surjective if and only if there is a linear map $S : W \rightarrow V$ with $TS = I_W$.