MA106 – Linear Algebra

Assignment 3 **January 2011**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1, 2, 3** *and* **5** *only must be handed in by* **2.00 pm** *on* **THURSDAY 3 FEBRUARY** *(Thursday of the fourth week of term), or they will not be marked.*

1. Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? Just answer yes or no in each case. [1 mark for each part]

(i) $\{(\alpha, \beta, \gamma) \mid \beta^3 = \alpha^3\}$; (ii) $\{(\alpha, \beta, \gamma) \mid \gamma^2 \geq 0\}$;

(iii) $\{(\alpha, \beta, \gamma) \mid \alpha + \beta = \gamma\}$; (iv) $\{(\alpha, \beta, \gamma) \mid \alpha + \beta = 7\gamma + 1\}$.

2. Find the dimensions of the subspaces of K^n that are spanned by the following sequences of vectors, and in each case find a subsequence that is a basis of this subspace. [2 marks for each part]

(i) $(1,1,0,0)$, $(0,1,0,1)$, $(1,0,0,-1)$, $(0,0,1,0)$, $(1,0,1,-1)$, $(0,2,1,2)$, where $K = \mathbb{R}$;

(ii) $(1,1,0,0)$, $(0,1,0,1)$, $(1,0,0,1)$, $(0,1,1,0)$, $(1,0,1,0)$, $(1,1,1,1)$, where K is the finite field $\{0,1\}$ with 2 elements;

(iii) $(-1,3,3,-1)$, $(3,-9,-9,3)$, $(1,-3,-3,1)$, $(-2,-3,-3,-2)$, $(2,-3,-3,2)$, $K = \mathbb{R}$;

3. (i) Let W be a subspace of a vector space V. Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are linearly independent vectors in W which do not span W. Show that there exists $\mathbf{w}_{m+1} \in W$ such that $\mathbf{w}_1, \ldots, \mathbf{w}_{m+1}$ are linearly independent. [2 marks]
(ii) Deduce that if W is a subspace of V and V has finite dimension, then W has finite dimension, and $\dim(W) \leq \dim(V)$. [2 marks]
(iii) Let W_1 and W_2 be subspaces of a finite-dimensional vector space V with $W_1 \subseteq W_2$, and suppose that $\dim(W_1) = \dim(W_2)$. Prove that $W_1 = W_2$. [3 marks]

4. Let W be a subspace of a vector space V of finite dimension. Show that W has a complementary subspace in V (i.e. there is a subspace X of V such that $V = W + X$ and $W \cap X = \{\mathbf{0}\}$).

5. Find the ranks and nullities of the following linear maps $T : U \to V$, and find bases of the kernel and image of T in each case.
(i) $U = \mathbb{R}^4$, $V = \mathbb{R}^4$, $T(\alpha, \beta, \gamma, \delta) = (\alpha + \gamma, \gamma + \delta, \alpha + \beta, \beta + \delta)$; [2 marks]
(ii) $U = \mathbb{R}[x]_{\leq 5}$, $V = \mathbb{R}[x]_{\leq 5}$ (polynomials of degree at most 5 over $\mathbb{R}$), $T(f) = f''''$ (fourth derivative of $f \in U$). [1 mark]

6. Describe the following linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ geometrically. In all cases (x, y) is a vector in $\mathbb{R}^2$, and we write $T(x, y)$ rather than $T((x, y))$.

(i) $T(x, y) = (-2x, -2y)$; (ii) $T(x, y) = (0, y)$;

(iii) $T(x, y) = (y, x)$; (iv) $T(x, y) = (x/\sqrt{2} + y/\sqrt{2}, -x/\sqrt{2} + y/\sqrt{2})$.

7. Suppose that the linear map $T : U \to V$ is a bijection. So T has an inverse map $T^{-1} : V \to U$. Prove that T^{-1} is a linear map.