THE UNIVERSITY OF WARWICK

MSc EXAMINATION: April 2013

Graduate Algebra

Time Allowed: **3 hours**

Candidates should attempt **all 4** questions.

Read carefully the instructions on the answer book and make sure that the numbers required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

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1. a) Let M, N be R-modules, where R is a commutative ring with unit. Show that

$$M \otimes_R N \cong N \otimes_R M.$$

[8]

b) Let K be a finite field of order 256. Describe $\mathbb{Z}/5\mathbb{Z} \otimes_{\mathbb{Z}} K[x, y]$. You must fully justify your answer. **[5]**

c) Describe a basis for $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$ as a vector space over $\mathbb{R}$. You do not need to justify your answer. **[4]**

d) Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map given by the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

so $\phi(\mathbf{e}_1) = \mathbf{e}_1 + 3\mathbf{e}_2$. Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a linear map given by the matrix

$$\begin{pmatrix} 5 & 6 & 7 \end{pmatrix},$$

so $\psi(a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3) = 5a + 6b + 7c$. Write down the matrix for the map $\phi \otimes \psi : \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3 \to \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}$ using your basis for $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$ from the previous part, and *this* convention for matrices of linear transformations. **[8]**

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2. a) Show that if P_1, P_2 are projective R-modules then $P_1 \oplus P_2$ is a projective R-module. **[5]**

b) Compute $\mathrm{Tor}_4^{\mathbb{Z}}(\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/5\mathbb{Z})$ **[8]**

c) Let X be a set of size five. Set $Ob(\mathcal{C}) = X$, and set $\hom(x, y) = \emptyset$ for $x, y \in X$ with $x \neq y$, and $\hom(x, x) = \{x\}$ for all $x \in X$. Show that $\mathcal{C}$ is a category. **[5]**

d) Do there exist products in the category $\mathcal{C}$? You must fully justify your answer. **[7]**

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3. a) Let $R = \mathbb{C}[\mathbb{Z}/5\mathbb{Z}]$ be the group algebra of the cyclic group of order five. Show that R is not an integral domain by giving two elements $a, b \in R$ with $ab = 0$. **[7]**

b) (i) Let F_2 be the free group with two generators a, b. Is F_2 generated by the group elements aba^{-1} and bab^{-1}? **[4]**

(ii) Let G be the group $\langle a, b | aba^{-1}, bab^{-1} \rangle$. What is the cardinality of G? **[4]**

c) Explain why $\mathbb{R}$ is not a normal extension of $\mathbb{Q}(\sqrt{2})$. Justify any claims you make in your answer. **[5]**

d) Show that if L/K is a separable normal field extension of degree 13 then there is no field L' with $K \subsetneq L' \subsetneq L$. **[5]**

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4. For each question, indicate if the statement is true or false, and *justify* your answer. If the statement is false this means giving an explicit counterexample and explaining why it is a counterexample. If the statement is true this means giving a proof. No credit will be given for a correct answer without justification.

a) The map that takes a group to its centre gives a functor from the category of groups to the category of abelian groups. **[5]**

b) If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, and F is a functor from abelian groups to abelian groups, then
$0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence. **[5]**

c) The group ring $\mathbb{C}[S_4]$ has a subring isomorphic to $M_2(\mathbb{C})$. (Recall that in our definition of subring it does not need to contain 1). **[5]**

d) $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is isomorphic to $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ as abelian groups. **[5]**

e) The group ring $\mathbb{Z}/2\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ is semisimple. **[5]**
