MA5Q6 Graduate Algebra
Assignment 7
Deadline: Friday 6 December 2013, 12pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics. Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(7.1) Let V be a vector space over a field K with basis $(v_1, \ldots, v_n)$. Then S_n acts on V by linear maps defined by $gv_i = v_{gi}$ for all $g \in S_n$ and $i \in \{1, \ldots, n\}$. Put $X = K(v_1 + \cdots + v_n)$ and

$$Y = \Big\{ \sum_{i=1}^{n} a_i v_i \;\Big|\; a_i \in K \text{ for all } i \text{ and } \sum_{i=1}^{n} a_i = 0 \Big\}.$$

Assume $\mathrm{char}(K) \nmid \#S_n$. Prove that X, Y are simple submodules of V and $V = X \oplus Y$.

(7.2) (Adapted from the 2012 exam MA3E1 Groups and Reps.) We define a group

$$G = \langle x, y, z \mid x^2 = z,\ y^2 = z,\ (xy)^2 = z,\ z^2 = 1 \rangle$$

and consider x, y, z as elements of G. We write $(g_1, \ldots, g_5) = (1, x, y, xy, z)$. All vector spaces and representations are over $\mathbb{C}$.

(a) Prove that there are precisely 4 linear characters of G. Denoting these by $\chi_1, \ldots, \chi_4$, give a table with the values $\chi_i(g_j)$ whenever $1 \leq i \leq 4$, $1 \leq j \leq 5$.

(b) Prove that there exists a unique representation ρ of G such that $\rho(x) = A$, $\rho(y) = B$ where we write

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(c) Prove that ρ is irreducible.

(d) Prove that $\langle z \rangle$ is a normal subgroup of G. Give an explicit presentation of the quotient group $G/\langle z \rangle$.

(e) Prove $\#G = 8$.

(f) Prove that $g_1, \ldots, g_5$ is a maximal set of pairwise non-conjugate elements of G. Give the character table of G.

(7.3) Let $D_8 = \langle (1234), (12)(34) \rangle \subset S_4$ be the dihedral group of order 8.

(a) Find the conjugacy classes of D_8.

(b) Find the character table of D_8.

(7.4) Let G be a group and K a field. Let U, V be left KG-modules and put $W = U \otimes_K V$.

(a) For $g \in G$ prove that there exists a unique K-linear map $L_g \colon W \to W$ such that $L_g(u \otimes v) = (gu) \otimes (gv)$ for all $(u, v) \in U \times V$.

(b) Prove $L_{gh} = L_g L_h$ for all $g, h \in G$. Deduce that W becomes a KG-module by putting $gw = L_g w$ for all $g \in G$, $w \in W$.

(c) Assume that U, V are finite-dimensional. Prove $\chi_W = \chi_U \chi_V$, that is, $\chi_W(g) = \chi_U(g)\, \chi_V(g)$ for all $g \in G$.