MA5Q6 Graduate Algebra
Assignment 6
Deadline: Friday 29 November 2013, 12pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics. Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(6.1) The 5-lemma. Let

$$\begin{CD}
A @>{p}>> B @>{q}>> C @>{r}>> D @>{s}>> E \\
@V{a}VV @V{b}VV @V{c}VV @V{d}VV @V{e}VV \\
A' @>{p'}>> B' @>{q'}>> C' @>{r'}>> D' @>{s'}>> D'
\end{CD}$$

be a commutative diagram of R-modules with exact rows. Assume that b, d are isomorphisms, a is surjective and e is injective. Prove that c is an isomorphism.

(6.2) Let D be a left R-module. Prove that $\mathrm{Hom}(-, D)\colon {}_R\mathbf{Mod} \to \mathbf{Mod}_R$ is right exact.

(6.3) Let P be an R-module. Prove that P is projective if and only if every short exact sequence of R-modules of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$$

splits.

(6.4) Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

(6.5) Let M be a right R-module and N a left R-module. Prove $\mathrm{Tor}_0(M, N) \cong M \otimes_R N$.

(6.6) Let Q, A be R-modules with Q projective. Prove $\mathrm{Ext}^i(Q, A) = 0$ for all $i > 0$.