MA5Q6 Graduate Algebra
Assignment 5
Deadline: Friday 22 November 2013, 12pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics. Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(5.1) Let A, B be sets and $R \subset A \times B$ a relation. Let P: **Sets** $\to$ **Sets** be the powerset functor. Then $P(A), P(B)$ are ordered by inclusion. We view $P(A)$ and $P(B)$ as categories. Define $f\colon P(A) \to P(B)^{\mathrm{op}}$ and $g\colon P(B)^{\mathrm{op}} \to P(A)$ by

$$f(X) = \{y \in B \mid xRy \text{ for all } x \in X\}$$
$$g(Y) = \{x \in A \mid xRy \text{ for all } y \in Y\}.$$

(a) Prove that (f, g) is an adjoint pair.

(b) Prove $fgf = f$.

(c) Find an adjoint pair (F, G) where $FGF \neq F$.

(5.2) Let $K \subset L$ be fields. Prove $L \otimes_K K[x_1, \ldots, x_n] \cong L[x_1, \ldots, x_n]$.

(5.3) Let R be a commutative ring with one. Let M, N be modules over R. Prove $S(M \oplus N) \cong S(M) \otimes_R S(N)$ as R-algebras.

Hint: You may use that for any R-modules A, B, C there exists an isomorphism $(A \oplus B) \otimes C \to A \otimes C \oplus B \otimes C$ taking $(a + b) \otimes c$ to $a \otimes c + b \otimes c$ for all $(a, b, c) \in A \times B \times C$.

(5.4) Let R be a commutative ring with one. Let M, N be R-modules and $f\colon M \to N$ a homomorphism. Prove that there exists a unique homomorphism of R-algebras $U(f)\colon T(M) \to T(N)$ such that $U(f)|_M = f$. Prove that putting $U(M) = T(M)$ makes U into a functor ***R*-mod** $\to$ **Rings**.

(5.5) Let R be a commutative ring with one. Let M be a free R-module with basis X. Let Y be the image of X in $\wedge(M)$ and choose a total ordering $<$ on Y. Prove that $\wedge^k(M)$ is a free R-module with basis

$$\big(y_1 \cdots y_k \mid y_i \in Y \text{ for all } i,\, y_i < y_{i+1} \text{ for all } i\big).$$