MA5Q6 Graduate Algebra
Assignment 4
Deadline: Wednesday 13 November 2013, 12pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics. Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(4.1) Apply the methods of section 3.5 to the monoid

$$\langle a, b, c \mid abc = ac,\ cba = ba \rangle.$$

That is, find a complete rewriting system.

(4.2) Using only the universal property, prove that $A \otimes_R B$ is generated as $\mathbb{Z}$-module by $\{a \otimes b \mid (a, b) \in A \times B\}$.

(4.3) Let $m, n \geq 1$ and put $d = \gcd(m, n)$. Prove $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.

(4.4) Let A be an (S, R)-bimodule and B an (R, T)-bimodule. Let $f\colon A \times B \to A \otimes_R B$ be their tensor product (here we ignore S, T). Make $A \otimes_R B$ into an (S, T)-bimodule as in proposition 69(b). Prove that f is an extended tensor product as well.

(4.5) Let V, V', W, W' be finite-dimensional vector spaces over a field K with bases $\{v_i \mid i\}$, $\{v'_i \mid i\}$, $\{w_i \mid i\}$, $\{w'_i \mid i\}$. Let $p\colon V \to V'$ (respectively, $q\colon W \to W'$) be linear maps with matrices A (respectively, B). Find the matrix of the linear map $p \otimes q\colon V \otimes W \to V' \otimes W'$ with respect to the bases $\{v_i \otimes w_j \mid i, j\}$ and $\{v'_i \otimes w'_j \mid i, j\}$.

(4.6) Let P, Q be ordered sets. Recall that we can regard them as categories.

(a) What is a functor $P \to Q$? That is, describe it in the language of ordered sets.

(b) What does it mean for two functors $F\colon P \to Q$ and $G\colon Q \to P$ to be adjoint?

(4.7) Show that there is no functor G: **Sets** $\to$ **Groups** which is *right* adjoint to the forgetful functor F: **Groups** $\to$ **Sets**. You may use that left adjoints preserve coproducts.

(4.8) Let F: **Top** $\to$ **Sets** be the forgetful functor that takes a topological space to the underlying set. Prove that F has a left adjoint and a right adjoint (which may be distinct).