MA5Q6 Graduate Algebra
Assignment 3
Deadline: Wednesday 30 October 2013, 12pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics. Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(3.1) Find two distinct functors F: **Groups** $\to$ **Groups** with $F(G) = G$ for all groups G.

(3.2) Let G, H be groups considered as categories (with one object). Let $S, T\colon G \to H$ be functors (group homomorphisms). Prove that there exists a natural transformation $S \to T$ if and only if S, T are conjugate, that is, there exists $h \in H$ such that $Tg = h(Sg)h^{-1}$ for all $g \in G$.

(3.3) Let C, D, E be categories. Prove that if $C \sim D \sim E$ then $C \sim E$.

(3.4) Give a full proof of the diamond lemma, that is, without use of (43).

Hint: For $a \in P$ put $Q(a) = \{b \in A \mid a \twoheadrightarrow b\}$. Let R be the set of elements $a \in A$ such that $Q(a)$ doesn't have a least element.

(3.5) Prove that free groups, as defined by the universal property, are unique in the following sense. Let two free groups $f\colon A \to F$, $g\colon A \to G$ on a set A be given. Then there exists a unique group isomorphism $h\colon F \to G$ such that $g = fh$.

(3.6) Give an example of a concrete category where some free objects don't exist.

(3.7) Let A, B be finite sets with $\#A = m$, $\#B = n$. Prove that the abelianisation of $F(A)$ is isomorphic to $\mathbb{Z}^m$. Deduce that $F(A) \not\cong F(B)$ if $m \neq n$.

(3.8) Describe the coproduct $G * H$ of two groups G, H explicitly. That is, solve its word problem given that it is solvable in G and H. Hint: mimic our explicit construction of the free group.

(3.9) Let $\equiv$ be an equivalence relation on M. Prove that $\equiv$ is a congruence on M if and only if $(a \equiv a',\ b \equiv b') \Rightarrow ab \equiv a'b'$ for all $a, a', b, b' \in M$.