MA5Q6 Graduate Algebra
Assignment 2
Deadline: Tuesday 22 October 2013, 12pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics. Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(2.1) Prove that a morphism in a category has at most one inverse.

(2.2) Prove that **Sets** is not isomorphic to its opposite.

(2.3) Prove that every group is isomorphic to its opposite. Give an example of a monoid that's not isomorphic to its opposite.

(2.4) In an example we saw that ordered sets are essentially the same as certain categories. Do the same for preordered sets. Also for equivalence relations.

(2.5) Let $(P, \leq)$ be an ordered set and $Q \subset P$. A greatest common lower bound or **meet** of Q written $\wedge Q$ is an element $r \in P$ such that for all $s \in P$

$$(s \leq q \quad \text{for all } q \in Q) \iff s \leq r.$$

Likewise, a least common upper bound or **join** of Q written $\vee Q$ is an element $r \in P$ such that for all $s \in P$

$$(q \leq s \quad \text{for all } q \in Q) \iff r \leq s.$$

(a) Prove without using categories that $\wedge Q$ may not exist.

(b) Prove without using categories that if $\wedge Q$ exists then it is unique.

(c) Let C be the category associated with $(P, \leq)$. Then a meet in P is the same as a product in C. Restate this more precisely and prove it.

(d) Let G be a group. Let P be the set of subgroups of G, ordered by inclusion. Prove that $(P, \leq)$ has all joins and meets.

(2.6) Let C, D be categories and $T\colon C \to D$ a functor. Let U, V be isomorphic objects of C. Prove that $T(U)$ and $T(V)$ are isomorphic in D.

(2.7) For a group G let G' be the subgroup of G generated by $\{[a, b] \mid a, b \in G\}$ where $[a, b] = aba^{-1}b^{-1}$.

(a) For a homomorphism of groups $h\colon G \to H$ prove that there exists a unique homomorphism $h'\colon G/G' \to H/H'$ defined by $h'(xG') = h(x)H'$.

(b) Prove that there exists a functor F: **Groups** $\to$ **Ab** which on objects is given by $F(G) = G/G'$ and on morphisms by $F(h) = h'$. It is called the **abelianisation**.

(2.8) The **centre** $Z(G)$ of a group G is the subgroup $\{a \in G \mid ab = ba \text{ for all } b \in G\}$. Prove that there is no functor F: **Groups** $\to$ **Ab** which on objects is $F(G) = Z(G)$. Hint: $S_2 \hookrightarrow S_3 \twoheadrightarrow S_2$.

(2.9) Construct a functor T: **Groups** $\to$ **Rings** that takes a group G to the group algebra $\mathbb{Z}[G]$. Prove your claims.

(2.10) Prove that **Sets** can be embedded into its opposite.