MA5Q6 Graduate Algebra
Assignment 1
Deadline: Tuesday 16 October 2013, 12pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics. Hand in all questions to Carole Fisher's office.

You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult. This first homework is largely warm-up, and so I can see the level of the class (so also let me know if you found this easy, or difficult).

(1.1) Let M be a monoid and G the set of invertible elements of M. Prove that G, equipped with the same multiplication as M, is a group.

(1.2) Let G be a monoid. Assume that for all $a \in G$ there exists b such that $ab = 1$. (We say that b is a right inverse to a). Prove that G is a group.

(1.3) How many semigroups of two elements are there, up to isomorphism? How many monoids? How many groups?

(1.4) Let $G = \mathrm{Sym}(\mathbb{Z}/7)$, so $G \cong S_7$. Define $f, g \in G$ by $f(x) = x + 1$ and $g(x) = 2x$. Let H be the subgroup of G generated by f, g. How many elements does H have?

(1.5) Let G be a group. Let $\mathrm{Aut}(G)$ be the set of automorphisms of G, that is, isomorphisms $G \to G$.

(a) Prove that $\mathrm{Aut}(G)$ equipped with composition is a group.

(b) Let C_n denote a cyclic group of order n. Let p be a prime number. Prove $\mathrm{Aut}(C_p) \cong \mathrm{Aut}(C_{p-1})$.

(1.6) Recall that the **characteristic** of a field K is the smallest $n \geq 1$ for which 1 added to itself n times is zero, or zero if there is no such n. Show that the characteristic of a field is either zero or a prime number p.

(1.7) Prove that the set of **quaternions**

$$\mathbb{H} := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \right\}$$

is a subring of $M(2, \mathbb{C})$. Prove that it is a division ring.

(1.8) Let $(P, \leq)$ be an ordered set (some people say partially ordered set). A **chain** in P is a subset inheriting a total order from $(P, \leq)$. An element $a \in P$ is said to be **maximal** if $a < b$ for no $b \in P$.

Zorn's lemma states the following. Let $(P, \leq)$ be an ordered set. Suppose that for every chain $C \subset P$ there exists $a \in P$ such that $c \leq a$ for all $c \in C$. Then P has a maximal element.

Use Zorn's lemma to prove that every vector space V (not necessarily spanned by finitely many elements) over a field K has a basis.