MA251 - Algebra I

Assignment 2 **Autumn 2017**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your university ID number in the top right-hand corner. Use the problems at the beginning as well as exercises in the lecture notes for a warm up. Solutions to the **FOUR TEST** *problems must be handed in by* **15.00** *on* **MONDAY 30 OCTOBER** *(Monday of the fifth week of term), or they will not be marked. There will be an award of 5 extra marks for clarity, so do a good job.*

These are practice problems for you to sharpen your teeth on.

P1. Show that a 2×2-matrix A with $A^3 = 0$ also satisfies $A^2 = 0$.

P2. Let A be an 8×8-matrix A over $\mathbb{R}$, and suppose that $c_A(z) = (1-z)^8$ and $\mu_A(z) = (z-1)^4$. Write down the possible JNFs for A. How would you decide which was the correct JNF?

P3. Is it true that for all $n \times n$-matrices A, B over $\mathbb{C}$, the matrices AB and BA are similar? Give a proof or counterexample.

P4. Let $q \colon V \to K$ be a quadratic form. Prove that, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

$$q(\mathbf{u}+\mathbf{v}+\mathbf{w}) - q(\mathbf{u}+\mathbf{v}) - q(\mathbf{v}+\mathbf{w}) - q(\mathbf{u}+\mathbf{w}) + q(\mathbf{u}) + q(\mathbf{v}) + q(\mathbf{w}) = \mathbf{0}.$$

P5. Write down the symmetric matrices corresponding to the quadratic forms
(i) $3x^2 - 7xy + 11y^2$; (ii) $xy + yz + xz$; (iii) $w^2 - xy + z^2$.

P6. A bilinear form $\tau \colon V \times V \to K$ is called *alternating* or *anti-symmetric* if $\tau(\mathbf{u}, \mathbf{v}) = -\tau(\mathbf{v}, \mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in V$.

(i) Show that the form τ is alternating if and only if $\tau(\mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{v} \in V$;

(ii) Show that any bilinear form on V is equal to the sum of a symmetric form and an alternating form.

P7. An $n \times n$ matrix is called *orthogonal* if $A^{\mathrm{T}} A = I_n$. Let A be an orthogonal matrix over $\mathbb{R}$.

(i) Prove $\det(A) = \pm 1$.

(ii) Prove that, if λ is an eigenvalue of A with $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$.
(*Hint*: Transpose the equation $A\mathbf{v} = \lambda \mathbf{v}$.)

P8. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, with real coefficients.

(i) Using either JNF or Lagrange's interpolation, compute A^n explicitly for a natural number n.

(ii) Find a polynomial $f(Z)$ of degree less than 2 such that $e^{tA} = f(A)$ and compute e^{tA} explicitly.

(iii) Solve the system of differential equations

$$\begin{cases} x'(t) &= 2x(t) + y(t) \\ y'(t) &= x(t) + 2y(t) \end{cases}$$

with initial condition $x(0) = 1$ and $y(0) = -1$.

P9. (Lagrange interpolation). Let $\alpha_1, \ldots, \alpha_\ell \in K$ be distinct. Let $k_1, \ldots, k_\ell \geq 1$ and put $n = k_1 + \cdots + k_\ell$. Let $\beta_{i,j} \in K$ be given whenever $1 \leq i \leq \ell$ and $0 \leq j < k_i$.

Prove that there exists a unique polynomial $h \in K[x]$ of degree $< n$ such that $h^{(j)}(\alpha_i) = \beta_{i,j}$ for all i, j ($h^{(j)}$ is the j th derivative).

P10. Let $D: \mathbb{R}[X] \to \mathbb{R}[X]$ be the unique linear map such that $D(X^n) = n\, X^{n-1}$ for all n (differentiation). For $t \in \mathbb{R}$ we define $e^{tD}(f) = \sum_{n \geq 0} t^n\, D^n(f)/n!$. Prove that this is a finite sum and that it equals $f(X+t)$.

The following problems are test problems for you to submit for marking. Write concise but complete solutions only to the questions asked. Additional 5 marks are awarded for clarity.

1. Let $A = \begin{pmatrix} -1 & -3 \\ 3 & 5 \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

(i) Using either JNF or Lagrange's interpolation, compute A^n explicitly for a natural number n. [2 marks]

(ii) Find a polynomial $f(Z)$ of degree less than 2 such that $e^{tA} = f(A)$ and compute e^{tA} explicitly. [2 marks]

(iii) Solve the system of differential equations

$$\begin{cases} x'(t) &= -x(t) &- &3y(t) \\ y'(t) &= 3x(t) &+ &5y(t) \end{cases},$$

with initial condition $x(0) = 1$ and $y(0) = 1$. [1 mark]

2. Let A be a near-Jordan block, that is, a matrix obtained from a Jordan block by possibly changing the first column. Prove that no two Jordan blocks in any Jordan canonical form for A have the same eigenvalue. [4 marks]

3. (i) Write down the symmetric matrix A corresponding to the quadratic form $q(\mathbf{v}) = xy + yz + 2xz$ in the 4 variables x, y, z. [1 mark]

(ii) Find a change of coordinates to put q in the form $\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2$. [2 marks]

(iii) Write down the corresponding change of basis matrix P, and verify that $P^{\mathrm{T}}AP$ is diagonal. [2 marks]

4. Let V be a vector space over a field $\mathbb{K}$. Let $\mathbf{e}_i$, $i \in I$ be a (possibly infinite) basis for V, that is, every element of V can uniquely be written as $\sum_J a_j\, \mathbf{e}_j$ where $J \subset I$ is a finite subset and $a_j \in K \setminus \{0\}$ for all $j \in J$. *The dual vector space* V^* is defined as the set of all linear maps $V \to \mathbb{K}$. While elements of V are called vectors, elements of V^* should be called *covectors*. Given the basis of V as above, we define a covector $\mathbf{e}^i \in V^*$, $i \in I$ by $\mathbf{e}^i(\mathbf{e}_j) = \delta_{ij}$ (1 if $i = j$, 0 otherwise).

(i) Prove that the covectors $\mathbf{e}^i$, $i \in I$ are linearly independent. [1 marks]

(ii) Prove that there exists unique $T \in V^*$ such that $T(\mathbf{e}_i) = 1$ for all i. [1 marks]

(iii) Assuming that V is finite-dimensional, prove that the covectors $\mathbf{e}^i$ form a basis and write T explicitly as a linear combination of the covectors $\mathbf{e}^i$. [1 marks]

(iv) Assuming that V is infinite-dimensional, prove that T is not a linear combination of $\mathbf{e}^i$. (It follows that the vectors $\{\mathbf{e}^i\}$ do not form a basis.) [1 marks]

(v) Assume that V is finite-dimensional. From what we proved it follows that both V and V^* are vector spaces of the same dimension. Consider the linear bijection $\phi: V \to V^*$ defined by $\phi(\mathbf{e}_i) = \mathbf{e}^i$. Show that this bijection depends on the original choice of basis. (*Hint: Consider 2 different bases in a one dimensional vector space and compute bijections explicitly*) [2 marks]