MA251 - Algebra I

Assignment 1 **Autumn 2017**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your university ID number in the top right-hand corner. Use the problems at the beginning as well as exercises in the lecture notes for a warm up. Solutions to the **FOUR TEST** *problems must be handed in by* **15.00** *on* **MONDAY 23 OCTOBER** *(Monday of the fourth week of term), or they will not be marked. There will be an award of 5 extra marks for clarity, so do a good job.*

These are practice problems for you to sharpen your teeth on.

P1. Find the eigenvalues of each of the following matrices A over the complex numbers $\mathbb{C}$. For each eigenvalue find one corresponding eigenvector, and then write down a matrix P such that $P^{-1}AP$ is diagonal.

(i) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, (iii) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, (iv) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

P2. Find the eigenvalues of the following pairs of matrices, and use them to decide which of the pairs are similar.

(i) $\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$;

(ii) $\begin{pmatrix} 8 & 3 & -6 \\ 2 & 1 & 0 \\ 10 & 4 & -7 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 4 & 10 & -1 \end{pmatrix}$;

(iii) (harder) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix}$.

P3. Say whether the following statements are true or false. If true, briefly state why. If false, give a counterexample. In all parts, $A = (\alpha_{i,j})$ is an $n \times n$ matrix over a field $\mathbb{K}$ containing $\frac{1}{2}$, the trace of A is the sum of diagonal elements: $\mathrm{tr}(A) = \sum_{i=1}^{n} \alpha_{i,i}$.
(i) If A is skew-symmetric, that is $A^T = -A$, then $\mathrm{tr}(A) = 0$.
(ii) If n is odd and A is skew-symmetric, then $\det(A) = 0$.
(iii) If $A^3 = I_n$, then $\det(A) = 1$.
(iv) If $A^2 = A$, then A is singular.
(v) If $n = 2$, then $A^2 - \mathrm{tr}(A)A + \det(A)I_2 = A^2 - \mathrm{tr}(A)A - \frac{1}{2}\mathrm{tr}(A^2)I_2 + \frac{1}{2}\mathrm{tr}(A)^2 I_2 = 0$.

P4. Compute the following determinants. You may want to use elementary row and/or column operations to reduce the matrix to a simpler form first.

(i) $\begin{vmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}$; (ii) $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & 1 \end{vmatrix}$; (iii) $\begin{vmatrix} 1 & 2 & 1 & -2 & -1 \\ 2 & 0 & 1 & 0 & 1 \\ -3 & 0 & 8 & 0 & -1 \\ 3 & 0 & 0 & -2 & 1 \\ -3 & -5 & -3 & 1 & 2 \end{vmatrix}$;

(iv) $\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix}$ where θ is some real number.

P5. Write down a matrix in JCF with minimal polynomial $(x+1)(x-2)^3(x+3)^2$ and characteristic polynomial $(-1-x)^2(2-x)^4(-3-x)^3$.

P6. Let $T\colon V \to V$ be a linear map, where $\dim(V) = n$, and suppose that $T^n = 0$ and that there exists a vector $v \in V$ with $T^{n-1}(v) \neq 0_V$. Prove that the vectors v, $T(v)$, $T^2(v), \ldots,$ $T^{n-1}(v)$ are linearly independent and that the nullity of T is 1.

P7. For the following matrices A, find minimal polynomials, matrices B, and invertible matrices P, such that $P^{-1}AP = B$ is in Jordan Canonical Form.

$$(i)\ \begin{pmatrix} 4 & 1 \\ -1 & 6 \end{pmatrix} \quad (ii)\ \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix} \quad (iii)\ \begin{pmatrix} 10 & -4 & -8 \\ 8 & -2 & -8 \\ 8 & -4 & -6 \end{pmatrix} \quad (iv)\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}$$

P8. Let A, B be square matrices and recall the block sum $A \oplus B$. Prove that $A \oplus B$ and $B \oplus A$ are similar. Generalise to more blocks. Hint: Translate this into the language of linear maps rather than matrices.

The following problems are test problems for you to submit for marking. Write concise but complete solutions only to the questions asked. Additional 5 marks are awarded for clarity.

1. Let $V = \mathbb{R}_{\leq 2}[x]$ be the vector space of polynomials of degree at most 2 over $\mathbb{R}$. Consider the following bases of V:

$$E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (1, x, x^2), \qquad E' = (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (1 + x,\ 1 + x^2,\ (1 + x)^2).$$

(i) Write down the change of base matrices for E and E' and vice versa. [1 mark]

(ii) Using the matrices found in (i), write the polynomial $2 + x - x^2$ as a linear combination of $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. [1 mark]

(iii) Write the linear map $T\colon V \to V$ given by $(Tf)(x) = x\, f'(x)$ in the basis E. [1 mark]

(iv) Using the matrices found in (i), write T in the basis E'. [1 mark]

2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$, where $n \geq 2$, $\mathbb{K}$ is a field. You will prove that

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \ldots & \alpha_2^{n-1} \\ & \ldots & \ldots & \ldots & \ldots \\ 1 & \alpha_n & \alpha_n^2 & \ldots & \alpha_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i).$$

(i) Show it for $n = 2$. [1 mark]

(ii) Show it for $n = 3$. [1 mark]

(iii) Use induction to show it for general n. *Hint:* Use elementary column operations to reduce the first row to $(1, 0, 0 \ldots)$ and such that the $(n-1) \times (n-1)$ in the bottom right corner is the matrix associated to $(\alpha_2, \ldots, \alpha_n)$. [3 marks]

3. For the following matrices A over $\mathbb{C}$, find minimal polynomials, matrices B, and invertible matrices P, such that $P^{-1}AP = B$ is in Jordan Canonical Form.

$$(i)\ \begin{pmatrix} -22 & 9 \\ -49 & 20 \end{pmatrix} \qquad (ii)\ \begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & -5 \\ 0 & 0 & 2 \end{pmatrix} \qquad (iii)\ \begin{pmatrix} -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

[2 marks each]

4. Let $T\colon V \to V$ be a linear map for which $T^k = 0$ (the zero map) for some $k > 0$. (Linear maps with this property are called nilpotent.) Assume $V \neq 0$. Prove that 0 is an eigenvalue of T, and that it is the only eigenvalue of T. [5 marks]