MA251 - Algebra I

Assignment 3 **Autumn 2016**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your university ID number in the top right-hand corner. Use the problems at the beginning as well as exercises in the lecture notes for a warm up. Solutions to the **FOUR TEST** *problems must be handed in by* **15.00** *on* **MONDAY 14 NOVEMBER** *(Monday of the seventh week of term), or they will not be marked. There will be an award of 5 extra marks for clarity, so do a good job.*

These are practice problems for you to sharpen your teeth on.

P1. Calculate the rank and signature of the quadratic forms corresponding to the matrices

$$\text{(i)} \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1 \\ 1/2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

P2. Call two quadratic forms on the n-dimensional vector space V over field $\mathbb{K}$ *equivalent* if one can be obtained from the other by a change of coordinates. How many equivalence classes are there when (i) $\mathbb{K} = \mathbb{C}$ and $n = 4$; and (ii) $\mathbb{K} = \mathbb{R}$ and $n = 3$.

P3. What is the answer to parts (i) and (ii) of Question P2 for general n?

P4. (i) Show that any 2×2 real orthogonal matrix is equal to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$. (Hence the matrix represents a rotation about the origin or a reflection about a line through the origin in the 2-dimensional plane.)

(ii) Show that a 3×3 real orthogonal matrix A represents either a rotation about a line through the origin, or a reflection about a plane through the origin followed by a rotation (*Hint*: First show that A has an eigenvector $\mathbf{v}$, and change basis to include $\mathbf{v}$.)

P5. Part (ii) of this problem deals with a square root of a matrix. Part (iii) establishes *a polar decomposition* of a matrix A: it is analogous to writing a complex number in the polar form $re^{i\theta}$.

Let A be any invertible $n \times n$ matrix over $\mathbb{R}$.
(i) Show that AA^{T} is symmetric and positive definite.
(ii) Show that there is a symmetric positive definite matrix S with $S^2 = AA^{\mathrm{T}}$.
(iii) Show that there is a symmetric positive definite matrix S and an orthogonal matrix R such that $A = SR$. (*Hint*: same S as in (ii).)

P6. This question outlines a proof of the convergence of the exponential series for any matrix. For $A \in \mathbb{C}^{n,n}$ with entries $(a_{ij})_{1\le i,j\le n}$, let $|A| = \sup_{i,j} |a_{ij}|$.
(i) Show that $|AB| \le n|A||B|$ for any $A, B \in \mathbb{C}^{n,n}$.
(ii) Hence deduce that $|A^k| \le n^{k-1}|A|^k$ for all $k \in \mathbb{N}$.
(iii) Prove that there exists $0 < r < 1$ and $C < \infty$ (depending on A) such that $|A^k|/k! < Cr^k$ for all $k \in \mathbb{N}$, and deduce that for each i, j, the sequence whose nth term is the (i, j) entry of $\sum_{k=0}^{n} A^k/k!$ is convergent.

P7. Find a 2×2 symmetric matrix over $\mathbb{C}$ which is not diagonalisable.

P8. For the following Euclidean vector space (V, ω) and a basis $f_1, \ldots, f_n$, run Gram-Schmidt orthonormalisation process to arrive at an orthonormal basis: $V = \mathbb{R}[X]_{\le 4}$ is the space of real polynomials of degree at most 4,

$$\omega(f(X), g(X)) = \int_{-\infty}^{+\infty} e^{-X^2} f(X)g(X)dX \ ;$$

$f_i = X^{i-1}$, $i = 1, 2, 3, 4, 5$. (*Hint:* You will need the Gaussian integral

$$\int_{-\infty}^{+\infty} X^{2k} e^{-X^2} dX = 2^{-k}(2k-1)!!\sqrt{\pi}$$

where $(2k-1)!!$ is the double factorial defined by $n!! = n \cdot (n-2)!!$ for $n > 0$ odd and $(-1)!! = 1$.)

P9. Find an orthogonal matrix P such that $P^{\mathrm{T}}AP$ is diagonal for

$$\text{(i)}\quad A = \begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}; \qquad \text{(ii)}\quad A = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 3 \end{pmatrix}.$$

The following problems are test problems for you to submit for marking. Write concise but complete solutions only to the questions asked. Additional 5 marks are awarded for clarity.

1. Prove that the 2-dimensional quadratic from $q(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ is positive definite if and only if $\alpha > 0$ and $\beta^2 - 4\alpha\gamma < 0$. [2 marks]

2. Find an orthogonal matrix P such that $P^{\mathrm{T}}AP$ is diagonal for

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$ [3 marks]

3. For the following Euclidean vector spaces (V, ω) and a basis $f_1, \ldots, f_n$, run the Gram-Schmidt orthonormalisation process to arrive at an orthonormal basis.

(i) $V = \mathbb{R}^3$ with the standard dot-product $\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T\mathbf{w}$; $f_1 = (1, 0, 0)^T$, $f_2 = (1, 1, 1)^T$ and $f_3 = (1, -1, 1)^T$. [2 marks]

(ii) $V = \mathbb{R}[X]_{\leq 4}$ is the space of real polynomials of degree at most 4,

$$\omega(f(X), g(X)) = \int_{-1}^{1} f(X)g(X)dX, \qquad f_i = X^{i-1},\ i = 1, 2, 3, 4, 5.$$ [2 marks]

4. Let $\tau \colon W \times V \to \mathbb{K}$ be a bilinear map, where V and W are finite-dimensional vector spaces over a field $\mathbb{K}$. Recall that the dual space is denoted by V^*. For subspaces $X \subset V$, $Y \subset W$ we define

$$X^{\perp} = \{\mathbf{y} \in W \mid \tau(\mathbf{y}, \mathbf{x}) = 0 \text{ for all } \mathbf{x} \in X\},$$
$$Y^{\perp} = \{\mathbf{x} \in V \mid \tau(\mathbf{y}, \mathbf{x}) = 0 \text{ for all } \mathbf{y} \in Y\}.$$

Let U, U_i be subspaces of V.

(i) Prove that $U^{\perp}$ is a subspace of W. [1 marks]

(ii) Prove that $U \subseteq (U^{\perp})^{\perp}$. [1 marks]

(iii) Prove that $U_1 \subseteq U_2$ implies $U_2^{\perp} \subseteq U_1^{\perp}$. [1 marks]

(iv) Show that the map $T_U \colon W \to U^*$ defined by $(T_U\mathbf{w})(\mathbf{u}) = \tau(\mathbf{w}, \mathbf{u})$ for $\mathbf{w} \in W$, $\mathbf{u} \in U$ is a linear map from W to U^*, and that $\ker(T_U) = U^{\perp}$. [1 marks]

(v) Deduce that $\dim(U) + \dim(U^{\perp}) \geq \dim(W)$. [2 marks]

5. Let $A \in K^{n,n}$ be a symmetric matrix over a field K. We define $f(A) = \{t^2 \det(A) \mid t \in K\}$ and $g(A) = \{x^T Ax \mid x \in K^n\}$.

(i) Prove that $f(A) = f(B)$ if A, B are congruent. [1 mark]

(ii) Prove that $g(A) = g(B)$ if A, B are congruent. [1 mark]

(iii) If A is the matrix of a quadratic form q with respect to some basis then we also write $f(q) = f(A)$ and $g(q) = g(A)$. Put $K = \mathbb{Z}_3$, the field of three elements. Using the foregoing, find the congruence classes of the quadratic forms in two variables x, y over K and their f- and g-values. (Two quadratic forms are called congruent if they differ by a change of coordinates.) [3 marks]