Uniqueness of spherically-symmetric vacuum solutions to Einstein's equations

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We prove that spherically-symmetric vacuum solutions of Einstein's equations with a cosmological constant (Ric $= \Lambda g$, where Λ is not a priori assumed to be constant or independent of t) are necessarily static, have Λ constant, are unique and coincide with the Schwarzschild–de Sitter metric. Relaxing the conditions on Ric we find that assuming that the first off-diagonal term is zero and that the first two diagonal terms are in the correct proportion (again with Λ not assumed constant) corresponds precisely to the static and natural observer field case discussed in [1].

Note Einstein's equations with cosmological constant for a vacuum are usually written Ric $+(\kappa - \frac{1}{2}S)\mathbf{g} = 0$ where Ric is Ricci curvature κ is the cosmological constant and S is the scalar curvature. But this is equivalent to Ric $= \Lambda g$ where $\Lambda = \frac{1}{2}S - \kappa$ is a scalar field, which cannot a priori be assumed constant or independent of t.

Consider the general spherically-symmetric metric

$$ds^2 = -Q\,dt^2 + P\,dr^2 + r^2\,d\Omega^2 \tag{1}$$

where P and Q are positive functions of r and t on a suitable domain. Here t is time, r is radius and $d\Omega^2$, the standard metric on the 2–sphere, is an abbreviation for $d\theta^2 + \sin^2\theta\,d\phi^2$ (or more symmetrically for $\sum_{j=1}^3 dz_j^2$ restricted to $\sum_{j=1}^3 z_j^2 = 1$). The Schwarzschild–de Sitter metric is the case $Q = 1/P = 1 - \Lambda r^2/3 - 2M/r$ with Λ and M constants.

We compute the Ricci curvature using the useful primitive formulae given in [2]. We start by computing R_{tt}, R_{tr} and R_{rr} where $R = \text{Ric}$. From [2, eq (1) p2] we get (after a couple of lines of computation)

$$4R_{tr} = \frac{4\dot{P}}{rP} \tag{2}$$

and from [2, eq (2) p3]

$$\frac{4R_{tt}}{-Q} = -\frac{\dot{P}}{W^2}\dot{W} + \frac{2\ddot{P}}{W} - \frac{4Q'}{rW} - \frac{2Q''}{W} + \frac{Q'W'}{W^2} \tag{3}$$

and

$$\frac{4R_{rr}}{P} = \frac{\dot{P}}{W^2}\dot{W} - \frac{2\ddot{P}}{W} - \frac{4P'}{rP^2} - \frac{2Q''}{W} + \frac{Q'W'}{W^2}. \tag{4}$$

Here we have used dot for diff wrt t and prime for diff wrt r and W is short for PQ. Now assume that Ric_{tr} is zero then from (2) we get $\dot{P} = 0$ and P depends only on r and further the first two terms in (3) and (4) are now zero and these equations reduce to

$$\frac{4R_{tt}}{-Q} = \frac{-4Q'}{rW} - \frac{2Q''}{W} + \frac{Q'W'}{W^2} \tag{5}$$

and

$$\frac{4R_{rr}}{P} = \frac{4P'}{rP^2} - \frac{2Q''}{W} + \frac{Q'W'}{W^2}. \tag{6}$$

Note that only the first terms differ. Now suppose that $\mathrm{Ric}_{ii} = \Lambda g_{ii}$ for $i = r, t$ where Λ is not assumed to be constant. Then the left-hand sides of these equations are the same, and hence

$$\frac{Q'}{W} + \frac{P'}{P^2} = \frac{Q'P + P'Q}{PW} = \frac{W'}{PW} = 0,$$

which is true iff $W' = 0$ ie W depends only on t. If this is the case then $Q(t, r) = W(t)/P(r)$ and we can assume that $W = 1$ by reparametrising t and hence $P = 1/Q$ are functions of r alone and we are in the static case with a natural observer field discussed in [1].

In this case, $\mathrm{Ric} = \Lambda g$ implies that Λ is also a function of r alone and from either equation we read

$$-2Q'/r - Q'' = 2\Lambda. \tag{7}$$

This DE is easily solved. Since $(r^2Q')' = 2rQ' + r^2Q''$ it is equivalent to

$$r^2Q' = -\int 2r^2\Lambda \tag{8}$$

and we have an explicit formula for Q. (In the case Λ is constant we immediately get the Schwarzschild–de Sitter metric up to careful choice of integration constants.)

Now assume that $\mathrm{Ric} = \Lambda g$ for all components. Inspecting [2, eq (1) p2] it is easy to see that all off-diagonal entries are zero (indeed in the general case, only $R_{tr} = R_{rt}$ can be nonzero and we have seen that this is zero in the static case). So the only new content is that $R_{ii} = \Lambda g_{ii}$ for $i = \phi, \theta$. By spherical symmetry the two coordinates give exactly the same information and we examine just the case $i = \phi$.

We use [2, eq (2) p3] again with $\mu = \phi$. Since no other coefficient of g_{ii} involves ϕ the equation reduces to the second term in the summation and further looking at the last term in the equation we see that we must have $\sigma = r$ or θ. We quickly compute the r term to be $-4\sin^2(\theta)(Q + rQ')$ and the θ term reduces to $4\sin^2(\theta)$. Thus $R_{\phi\phi} = \Lambda g_{\phi\phi}$ iff

$$1 - Q - rQ' = r^2\Lambda. \tag{9}$$

Since $(rQ)' = Q + rQ'$ this implies

$$rQ = r - \int r^2\Lambda. \tag{10}$$

We can eliminate $\int r^2\Lambda$ between (8) and (10) to find

$$r^2Q' - 2rQ = -2r + C$$

where C is a constant of intergration. This is readily solved to yield

$$Q = 1 - Kr^2 - \frac{C}{3r}$$

where K is another constant of intergration. Finally substituting for Q in (10) we find that $\Lambda = 3K$ is constant after all and we have the Schwarzschild–de Sitter metric with $3C = 2M$ and the proof is complete.

Note We only need the solution to exist in any open set for the argument above to work. In particular there are no boundary conditions (at a BH or at infinity) assumed.

References

[1] **R S MacKay**, **C Rourke**, *Natural flat observer fields in spherically-symmetric space-times*, J. Phys. A: Math. Theor. 48 (2015) 225204, available at http://msp.warwick.ac.uk/~cpr/paradigm/escape-Jan2015.pdf

[2] **K Z Win**, *Ricci tensor of diagonal metric*, arXiv:gr-qc/9602015