Trunks and Classifying Spaces

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SUMMARY

Trunks are objects loosely analogous to categories. Like a category, a trunk has vertices and edges (analogous to objects and morphisms), but instead of composition (which can be regarded as given by preferred triangles of morphisms) it has preferred squares of edges. A trunk has a natural cubical nerve, analogous to the simplicial nerve of a category. The classifying space of the trunk is the realisation of this nerve. Trunks are important in the theory of racks [FR]. A rack X gives rise to a trunk $\mathcal{T}(X)$ which has a single vertex and the set X as set of edges. The **rack space** BX of X is the realisation of the nerve $N\mathcal{T}(X)$ of $\mathcal{T}(X)$. The connection between the nerve of a trunk and the usual (cubical) nerve of a category determines in particular a natural map $BX \to BAs(X)$ where $BAs(X)$ is the classifying space of the associated group of X. There is an extension to give a classifying space for an augmented rack, which has a natural map to the loop space of the Brown–Higgins classifying space of the associated crossed module [FR section 2 and BH2].
The theory can be used to define invariants of knots and links since any invariant of the rack space of the fundamental rack of a knot or link is ipso facto an invariant of the knot or link.

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Introduction

In this paper we introduce **trunks**. Trunks are objects loosely analogous to categories. Like a category, a trunk has vertices and edges (analogous to objects and morphisms in a category), but instead of composition (which can be regarded as given by preferred triangles of morphisms) it has *preferred squares* of edges. The associativity axiom for a category can be defined by considering tetrahedra in the category. The analogue for a trunk, which is optional, uses cubes. There is also a concept of Trunk–**map** between trunks, analogous to functors between categories.

However our main interest in trunks arises from the fact that a trunk has a natural cubical **nerve**, analogous to the usual simplicial nerve of a category. The realisation of this nerve (as a topological space) plays the role of classifying space for the trunk. There are, as we shall see in this paper, strong connections between this space and other well known classifying spaces.

The terminology "trunk" is intended to suggest the word "truncated" because one way to think of a trunk is as a particular type of cubical set truncated at dimension 2. Note that in earlier versions of this paper, we have used the word "species" instead of trunk. This has been abandoned because of the potential confusion with existing usage.

We were lead to formulate the concept of a trunk by our work on racks. We remind the reader that **rack** (originally spelled wrack) is the first of the (many) terms used to describe a set with a binary operation such that right multiplication is an automorphism, see [CW, B, J, K, Mv] and in particular the historical notes in the introduction to [FR]. Racks are important because a codimension two embedding with transverse orientation (a **link**) has a **fundamental rack** and this fundamental rack is a classifying invariant for irreducible links of 1–manifolds in 3–manifolds [FR]. A rack X gives rise to a trunk $\mathcal{T}(X)$. The transition from rack to trunk is analogous to the transition from group to category. The trunk $\mathcal{T}(X)$ has a single vertex and the set X is the set of edges. The preferred squares are of the following type, where the rack operation is denoted by a^b.

$$\begin{tikzcd} * \arrow[r, "a^b"] & * \\ * \arrow[u, "b"] \arrow[r, "a"'] \arrow[ur, phantom, "\circlearrowleft"] & * \arrow[u, "b"'] \end{tikzcd}$$

The **rack space** BX of X is the realisation of the nerve $N\mathcal{T}(X)$ of $\mathcal{T}(X)$. The 1–cells of BX can be identified with elements of X and the 2–cells are determined by the preferred squares.

The preferred square given above can be pictured as part of a link diagram with arcs labelled by a, b and a^b.

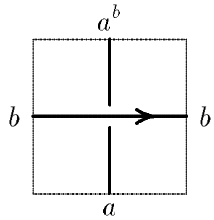


Diagram of a typical 2–cell of the rack space

Thus, given a map of a surface M to BX, we can use transversality to construct a link diagram in M labelled by elements of X. Similar considerations apply to higher dimensional manifolds, and it follows that BX is the classifying space for (cobordism classes of) link diagrams labelled by X. Moreover there is a corresponding classifying bundle ξ/BX which is a mock bundle [BRS chapter 2] embedded in $BX \times \mathbb{R}$. This is the first of an infinite family of bundles (defined for any cubical set) which we call **James bundles**. This theory can be used to define many new invariants of knots and links. Any invariant of the rack space of the fundamental rack of a knot or link is *ipso facto* an invariant of the knot or link.

In this paper we shall concentrate on the formal properties of these spaces and shall defer consideration of the James bundles, invariants, the geometric properties of BX and related topics in stable homotopy theory to later papers.

The connection between the nerve of a trunk and the usual (cubical) nerve of a category determines in particular a natural map $BX \to BAs(X)$ where $BAs(X)$ is the classifying space of the associated group of X. Moreover there is an extension to give a classifying space for an augmented rack, which has a natural map to the loop space of the Brown–Higgins classifying space of the associated crossed module [FR section 2 and BH2].

This paper is organised as follows. Section 1 contains the definitions and examples of trunks that we shall need and the connections between trunks and categories, and section 2 contains some basic material on cubes and cubical sets. In section 3 we define the nerve of a trunk and its realisation. As particular examples we have the rack space and the augmented rack space. The connection with the classifying space for groups follows quickly from the definition. Finally in section 4 we give some miscellaneous results on the algebraic topology of the various classifying spaces and the connection with the Brown–Higgins classifying space.

1. Trunks

In this section we consider the notion of a **trunk**. This is a mathematical object loosely analogous to a category. A trunk is a directed graph, in other words it has vertices and directed edges in analogy to the objects and morphisms of a category, but instead of composition (which can be regarded as given by preferred *triangles* of morphisms) it has preferred *squares* of edges.

Definition An **oriented square** in a directed graph is a sequence of edges a, b, c, d (repetitions allowed) which can be represented in two ways:

$$\begin{array}{ccc} C & \xrightarrow{c} & D \\ \uparrow b & \circlearrowleft & \uparrow d \\ A & \xrightarrow{a} & B \end{array} \qquad \textbf{or} \qquad \begin{array}{ccc} B & \xrightarrow{d} & D \\ \uparrow a & \circlearrowright & \uparrow c \\ A & \xrightarrow{b} & C \end{array}$$

The orientation symbols $\circlearrowleft$ and $\circlearrowright$ serve to recover the sequence from the representation.

We shall sometimes refer to the edge a as the **base** of the square.

Definition A **trunk** consists of a directed graph Γ, together with a collection of oriented squares in Γ called **preferred squares.**

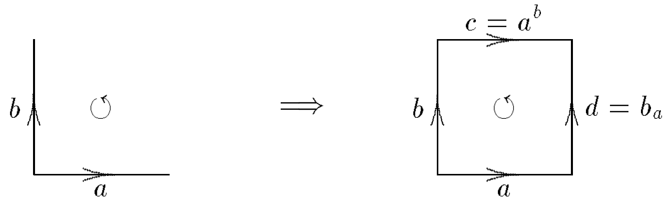
We will define three types of trunk by adding to the basic definition. The following definition is the most important for our purposes.

Definition A **corner trunk** is a trunk which satisfies the following two **corner axioms**.

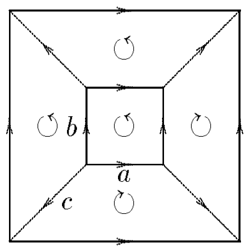
C1 Given edges $a\colon A \to B$ and $b\colon A \to C$ then there are unique edges $c\colon C \to D$ and $d\colon B \to D$ such that the following square with base a

$$\begin{array}{ccc} C & \xrightarrow{c} & D \\ \uparrow b & \circlearrowleft & \uparrow d \\ A & \xrightarrow{a} & B \end{array}$$

is preferred. Notice that the other edges are determined by two partially defined binary operations which we write $c = a^b$ and $d = b_a$.



C2 In the following diagram, if the three squares centre, south and west are given and preferred, then the diagram can be completed as shown so that the squares; outside, east and north are preferred. The outside square has the anticlockwise orientation.



Given C1, an equivalent statement is that the edges a, b, c determine the entire diagram of preferred squares.

Remarks

(1) In a corner trunk the *corner* of a square or cube (i.e. the edges a, b or a, b, c) has a unique completion to form the final square or cube, and this is why we have chosen this terminology.

(2) The two corner axioms are loosely analogous to the composition and associativity laws of a category. The analogy can be sharpened by replacing the corner, which comprises co-original edges (morphisms), by consecutive edges, as in the category axioms. This leads to an alternative theory of corner trunk which, in the key example of racks (see 1.3 example 3 below), coincides with the theory given here. We have decided not to pursue this alternative approach. This is because the formulae for the boundary maps in the nerve (see section 3) are considerably simpler in the co-original approach.

1.1 Lemma *Whenever the relevant operations are defined, the binary operations of a corner trunk satisfy the following axioms.*

(a) $a^{bc_b} = a^{cb^c}$

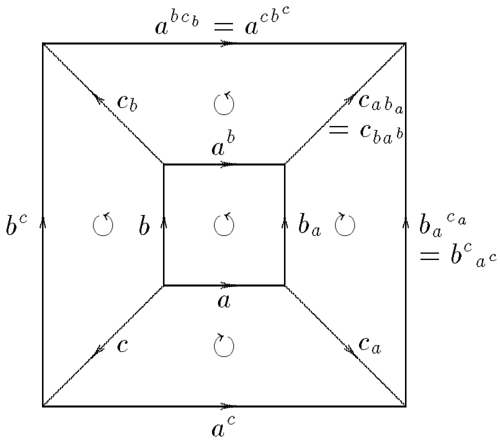
(b) $a_{bc_b} = a_{cb^c}$

(c) $a_b{}^{c_b} = a^c{}_{b^c}$

Remark We follow and extend the convention for bracketing which is understood for ordinary exponentials (of numbers). Thus $a^{bc} = (a^b)^c$ and $a^{b^c} = a^{(b^c)}$, and similar equations hold for the other action a_b. For mixed compositions of actions, the bracketing is determined by the position of the symbols. Thus $a^b{}_c = (a^b)_c$ and $a^{b_c} = a^{(b_c)}$. As an example, the first rule (a) in lemma 1.1, with brackets reads

$$(a^b)^{(c_b)} = (a^c)^{(b^c)}.$$

Proof We check these equalities by looking at the coincidental labelling on the following diagram.



□

Definition A **trunk with identities** or **pointed trunk** is a trunk together with a chosen edge $A \xrightarrow{e_A} A$, called an **identity loop** for each vertex A, and with the following squares preferred for each edge $A \xrightarrow{a} B$.

$$
\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\uparrow e_A & \circlearrowleft & \uparrow e_B \\
A & \xrightarrow{a} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
B & \xrightarrow{e_B} & B \\
\uparrow a & \circlearrowleft & \uparrow a \\
A & \xrightarrow{e_A} & A
\end{array}
$$

Any trunk S can be **pointed** to yield the trunk S^+, which is obtained from S by adjoining one new "identity" loop for each vertex of S, and allowing the above new types of preferred square.

Finally we have

Definition A **trunk with composition** is a trunk in which Γ is a category with composition of morphisms, written $a \cdot b$, and satisfying the following laws.

Horizontal Composition: Preferred squares a, b, c, d and a', b', c', d' with $d = b'$ may be composed horizontally to form the preferred square $a \cdot a', b, c \cdot c', d'$ with base $a \cdot a'$.

Vertical Composition: Preferred squares a, b, c, d and a', b', c', d' with $c = a'$ may be composed vertically to form the preferred square $a, b \cdot b', c', d \cdot d'$ with base a.

$$
\begin{tikzcd}
\cdot \arrow[r, "c"] & \cdot \arrow[r, "c'"] & \cdot \\
\cdot \arrow[u, "b"] \arrow[r, "a"'] \arrow[ur, phantom, "\circlearrowleft"] & \cdot \arrow[u] \arrow[r, "a'"'] \arrow[ur, phantom, "\circlearrowleft"] & \cdot \arrow[u, "d'"']
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\cdot \arrow[r, "c \cdot c'"] & \cdot \\
\cdot \arrow[u, "b"] \arrow[r, "a \cdot a'"'] \arrow[ur, phantom, "\circlearrowleft"] & \cdot \arrow[u, "d'"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdot \arrow[r, "c'"] & \cdot \\
\cdot \arrow[u, "b'"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \cdot \arrow[u, "d'"'] \\
\cdot \arrow[u, "b"] \arrow[r, "a"'] \arrow[ur, phantom, "\circlearrowleft"] & \cdot \arrow[u, "d"']
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\cdot \arrow[r, "c'"] & \cdot \\
\cdot \arrow[u, "b \cdot b'"] \arrow[r, "a"'] \arrow[ur, phantom, "\circlearrowleft"] & \cdot \arrow[u, "d \cdot d'"']
\end{tikzcd}
$$

1.2 Lemma *With the notation above a corner trunk with composition satisfies the following.*

(1) $(a \cdot a')^b = a^b \cdot {a'}^{b_a}$

(2) $(b \cdot b')_a = b_a \cdot {b'}_{a^b}$

(3) $a^{b \cdot b'} = a^{bb'}$

(4) $b_{a \cdot a'} = b_{aa'}$

$\square$

1.3 Examples of trunks

1. Any category C determines a directed graph $\Gamma(C)$ with vertices the objects of C and edges the morphisms of C. Let $\mathrm{Trunk}(C)$ be the trunk with underlying graph $\Gamma(C)$, whose preferred squares are the commutative squares in C. This trunk has identities and compositions.

2. Let C be a skeletal category (so all isomorphisms are identities) with pushouts. Then C determines a corner trunk with underlying graph $\Gamma(C)$ and preferred squares the pushout squares in C. This trunk also has identities and compositions.

3. **The Rack Trunk**

This is the key example. A **rack** is a set X with a binary operation written a^b such that $a \mapsto a^b$ is a bijection for all $b \in X$ and such that the **rack identity**

$$a^{bc} = a^{cb^c}$$

holds for all $a, b, c \in X$.

For example if $a, b \in G$ are two members of a group then conjugation, given by $a^b := b^{-1}ab$, defines a rack structure on G. If $a, b \in \mathbb{Z}$ are two integers then $a^b := a + 1$ defines a rack structure on $\mathbb{Z}$ which does not come from conjugation in a group. For more examples of racks see [FR].

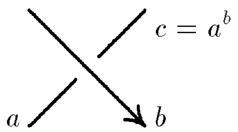
A rack X defines a trunk $\mathcal{T}(X)$ with a single vertex and with X the set of edges. The preferred squares are of the following type, one for each pair $a, b \in X$.

$$\begin{tikzcd} * \arrow[r, "a^b"] & * \\ * \arrow[u, "b"'] \arrow[r, "a"'] \arrow[ur, phantom, "\circlearrowleft"] & * \arrow[u, "b"'] \end{tikzcd}$$

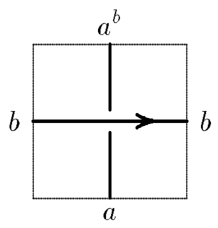
This is a corner trunk with the rack operation a^b being the rack operation and the other operation b_a being trivial.

Remarks

(1) A link diagram determines a rack (the fundamental rack) by labelling the arcs by generators and reading a relation at each crossing as follows.



For full details, and for the alternative topological definition of the fundamental rack, which applies to arbitrary codimension two links, see [FR]. Notice that the preferred square, above, can be correlated with this crossing as follows.



This connection between the rack trunk and link diagrams will be explored in a later paper.

(2) If one of the operations in a corner trunk is trivial (for example if $a_b = a$ for all a), then the three laws of lemma 1.1 reduce to the rack identity for the other operation. Thus these laws define a natural algebraic object which might be called a **birack**. This object, like the rack itself, has an application to links. A link diagram has a fundamental birack obtained by labelling arcs, and using both operations at crossovers. One operation determines the change of label on the understring and the other on the overstring.

Rack actions

We shall now generalise the rack trunk in a variety of ways. We need the idea of an **action** of a rack on a set. Let X be a rack and Y a set. We say that X **acts** on Y if for each $a \in X$ we have a bijection

$$\begin{array}{rcl} \phi_a\colon Y & \longrightarrow & Y \\ y & \longmapsto & y^a \end{array}$$

such that the following **rack action identity** holds

$$y^{ab} = y^{ba^b} \quad \text{for all} \quad y \in Y, \quad a, b \in X.$$

We also say that Y is an X–**set**.

1.4 Further examples of trunks

1. The Action Rack Trunk

An X–set Y gives rise to a trunk $\mathcal{T}_Y(X)$ by taking Y as the set of vertices and edges of the form $y \xrightarrow{a} y^a$, for $y \in Y$, $a \in X$. the preferred squares are then of the following type.

$$\begin{array}{ccc} y^b & \xrightarrow{a^b} & y^{ab} = y^{ba^b} \\ \uparrow b & \circlearrowleft & \uparrow b \\ y & \xrightarrow{a} & y^a \end{array}$$

Notice that $\mathcal{T}_Y(X)$ is a corner trunk. The binary operations in $\mathcal{T}_Y(X)$ can be read from the above diagram. In particular if we write (y, a) for the edge $y \xrightarrow{a} y^a$ then the operations are given by the formulae

$$(y, a)^{(y,b)} = (y^b, a^b), \qquad (y, a)_{(y,b)} = (y^a, b).$$

The trunk $\mathcal{T}_Y(X)$ covers the trunk $\mathcal{T}(X)$ in a way which will be made precise in section 3, see theorem 3.7.

We now give some important special cases.

2. The Extended Rack Trunk

Here $X = Y$ and the action is just the action of the rack on itself. Thus we have a trunk $\mathcal{T}_X(X)$. The extended rack trunk is again a corner trunk with binary operations which can be read from the above diagram. In particular if we write (a, b) for the edge $a \xrightarrow{b} a^b$ then the operations are given by the formulae

$$(a, b)^{(a,c)} = (a^c, b^c), \qquad (a, b)_{(a,c)} = (a^c, b).$$

3. The Augmented Rack Trunk

Let G be a group and let X be a G–set with G action written $x \cdot g$.†

Recall from [FR] that an **augmented rack** comprises a group G, a G–set X and a function $\partial : X \to G$ satisfying the **augmentation identity**

$$\partial(a \cdot g) = g^{-1} \partial a\, g \quad \text{for all} \quad a \in X, \quad g \in G.$$

Then X becomes a rack by defining a^b to be $a \cdot \partial b$. For more information on augmented racks see [FR].*

If $\partial : X \to G$ is an augmented rack then G can be regarded as an X–set by defining g^a to be $(\partial a)^{-1} g$. Thus again we have a special case of the action rack trunk, a trunk $\mathcal{T}_G(X)$, with vertices the set G, and edges $g \xrightarrow{a} g^a$, for $a \in X$.

4. There are two similar trunks associated to an augmented rack with group G which are given by redefining g^a to be $g \partial a$ and $(\partial a)^{-1} g \partial a$ respectively.

Pointing the Rack Trunk

The rack trunk (1.3 example 3) can be pointed by replacing X by a pointed rack:

An **identity** in a rack is an element $e \in X$ such that $e^a = e$ and $a^e = a$ for all $a \in X$. Identities in racks are not unique. A **pointed rack** is a rack X with a chosen identity.

Let X be any rack, then X can be **pointed** to form the pointed rack X^+, by adding an external identity as follows. As a set $X^+ = X \amalg \{e\}$, and the rack operation is extended so that $e^e = e$, $e^a = e$ and $a^e = a$ for all $a \in X$. It is easy to check that X^+ is a rack — indeed X^+ is just the disjoint union of X with the trivial rack on one element.‡ Conversely a pointed rack can be unpointed by deleting the identity, and it is easy to see that pointing gives a one to one correspondence between unpointed and pointed racks.

† In this paper all group actions will be right actions and written in this way except briefly at the end of section 3 where we consider left actions.

* Note that what we have called an augmented rack (in analogy with Joyce's terminology augmented quandle [J]) is called a crossed G–set by Freyd and Yetter [FY].

‡ The **disjoint union** $X \amalg Y$ of two racks is defined in Breiskorn [B], the operations of X on X and Y on Y are the usual ones, whilst X and Y act trivially on each other.

Let X be a pointed rack and $e \in X$ be its identity element. Then the trunk $\mathcal{T}(X)$ of 1.3 example 3 is pointed by taking the edge $0 \xrightarrow{e} 0$ as identity at the (unique) vertex. Pointing *either* the rack *or* the corresponding trunk yields the same result:

1.5 Lemma *If X is an unpointed rack then there is an isomorphism of pointed trunks*

$$\mathcal{T}(X^+) \cong (\mathcal{T}(X))^+.$$

□

Remark The action rack trunk can also be pointed in a similar way. This involves the concepts of an **action of a pointed rack**, which is a rack action which satisfies the further property that the identity acts trivially. If X is a pointed rack then the trunk $\mathcal{T}_X(X)$ of 1.4 example 2 is pointed by taking $a \xrightarrow{e} a$ as the identity at the vertex a. However in this case pointing the rack is not equivalent to pointing the trunk, because $\mathcal{T}_{X^+}(X^+)$ has a vertex e not present in $(\mathcal{T}_X(X))^+$. In fact $\mathcal{T}_{X^+}(X^+)$ is the disjoint union of $(\mathcal{T}_X(X))^+$ with a copy of $\mathcal{T}(X^+)$.

For pointed versions of 1.4 examples 3 and 4 we need the notion of an **augmented rack with identity** or **pointed augmented rack** which is an augmented rack $\partial : X \to G$ such that

(1) X is pointed

(2) $\partial(e_X) = e_G$ and

(3) $e_X \cdot g = e_X$ for all $g \in G$.

Any augmented rack can be **pointed** to give a pointed augmented rack by pointing X and using the axioms above to extend the action of G. If $\partial : X \to G$ is pointed then the trunk $\mathcal{T}_G(X)$ of 1.4 example 3 is pointed by taking $g \xrightarrow{e} g$ as the identity at the vertex g. The two trunks in example 4 are pointed in exactly the same way. The analogue of lemma 1.5 holds in these three examples:

$$\mathcal{T}_G(X^+) \cong (\mathcal{T}_G(X))^+.$$

Trunk–**maps and actions by** Trunk–**maps**

Definition A Trunk–**map** between trunks is a map which takes vertices to vertices, directed edges to directed edges and preferred squares to preferred squares. Trunk–maps between trunks are analogous to functors between categories. There is a category of trunks and Trunk–maps. If S and S' are trunks then the set of Trunk–maps from S to S' is written $\mathrm{Hom}(S, S')$.

A rack X **acts** on a trunk S if, for all $x \in X$, there is a Trunk–map $(.)^x\colon S \to S$ such that the analogue of the rack identity holds. More precisely suppose that we use the notation

$$(A \xrightarrow{c} B)^x := (A^x \xrightarrow{c^x} B^x),$$

then $\alpha^{xy} = \alpha^{yx^y}$ for all $x, y \in X$ where α can be either an edge or a vertex of S.

Similarly a group G **acts** on S if for each $g \in G$ there is a Trunk–map $.g\colon S \to S$, such that, if we write $(A \xrightarrow{c} B).g := (A \cdot g \xrightarrow{c \cdot g} B \cdot g)$, then $\alpha \cdot 1 = \alpha$ and $\alpha \cdot (gh) = (\alpha \cdot g) \cdot h$ for all $g, h \in G$.

1.6 Examples

1. A functor $\mathcal{F}\colon C \to D$ between categories gives a Trunk–map Trunk$(\mathcal{F})$: Trunk$(C) \to$ Trunk(D). In other words Trunk is a functor from categories to trunks.

2. Let X be a rack, then there is a Trunk–map

$$p_X\colon\ \mathcal{T}_X(X) \longrightarrow \mathcal{T}(X)$$

defined on edges by mapping $x \xrightarrow{a} x^a$ to $0 \xrightarrow{a} 0$.

3. The last example generalises. Let Y, Z be X–sets and $q\colon Y \to Z$ an X–map. Then q induces a Trunk–map

$$q_\sharp\colon \mathcal{T}_Y(X) \longrightarrow \mathcal{T}_Z(X)$$

defined by mapping the edge $y \xrightarrow{a} y^a$ to the edge $q(y) \xrightarrow{a} q(y^a) = q(y)^a$. Thus $\mathcal{T}_{-}(X)$ is a functor from the category of X–sets to the category of trunks and Trunk–maps.

4. **The action of the rack on the rack trunk**

Let X be a rack and Y an X–set, then each element $x \in X$ determines a Trunk–map $(.)^x\colon \mathcal{T}_Y(X) \longrightarrow \mathcal{T}_Y(X)$ by

$$\begin{array}{ccc} y^a & & y^{ax} = y^{xa^x} \\ a\uparrow & \overset{(.)^x}{\longmapsto} & \uparrow a^x \\ y & & y^x \end{array}$$

and note that the notation α^x for the action of X on all three of Y, X and S is consistent.

The Trunk–map $q_\sharp$ of example 3 is **equivariant** in the sense that it commutes with this X–action.

5. **The edge action of the group on the augmented rack trunk**

Suppose that $\partial\colon X \to G$ is an augmented rack, then as we saw in 1.4 example 3, G becomes an X–set with action $g^x := (\partial x)^{-1} g$. Hence by the last example there is an X–action on the augmented rack trunk $\mathcal{T}_G(X)$. This action is in fact compatible with an action of G on $\mathcal{T}_G(X)$:

$$\begin{array}{ccc} g^a & & h^{-1}g^a \\ a\uparrow & \overset{.h}{\longmapsto} & \uparrow a\cdot h \\ g & & h^{-1}g \end{array}$$

(Compatible means that $\alpha^x = \alpha \cdot \partial x$.)

Note that the right hand edge in the diagram *is* an edge because

$$\begin{aligned} h^{-1}g^a &= h^{-1}(\partial a)^{-1}g \\ &= h^{-1}(\partial a)^{-1}hh^{-1}g \\ &= (\partial(a\cdot h))^{-1}h^{-1}g \quad \text{by the augmentation identity} \\ &= (h^{-1}g)^{(a\cdot h)}. \end{aligned}$$

We call this action the **edge action** of G on $\mathcal{T}_G(X)$, to distinguish it from the following action.

6. The vertex action of the group on the augmented rack trunk

There is another action of G on $\mathcal{T}_G(X)$ given by right multiplication on vertices which we call the **vertex action**:

$$\begin{tikzcd} g^a = (\partial a)^{-1}g & & g^a h = (\partial a)^{-1}gh \\ g \arrow[u, "a"] & \overset{h_\sharp}{\longmapsto} & gh \arrow[u, "a"'] \end{tikzcd}$$

The vertex action is **over** $\mathcal{T}(X)$ i.e. we have a commutative triangle of Trunk–maps:

$$\begin{tikzcd} \mathcal{T}_G(X) \arrow[rr, "h_\sharp"] \arrow[dr, "p_G"'] & & \mathcal{T}_G(X) \arrow[dl, "p_G"] \\ & \mathcal{T}(X) & \end{tikzcd}$$

This diagram suggests that p_G is in some sense the projection of a principal G–bundle. This will be made precise in section 3, theorem 3.7.

Trunk–maps and categories

A trunk S determines an associated category $\mathrm{Cat}(S)$. The objects of $\mathrm{Cat}(S)$ are just the vertices of S. The morphisms of $\mathrm{Cat}(S)$ are generated by the edges of S with the relations which follow from insisting that preferred squares commute. Trunk–maps become functors. Thus Cat is a functor from the category of trunks to the category of categories.

We have already seen in example 1 above that there is a functor Trunk from categories to trunks.

1.7 Lemma *The functors* Trunk *on categories and* Cat *on trunks are adjoint. In particular, given a trunk S and a category C, then the set of* Trunk*–maps* $\mathrm{Hom}(S, \mathrm{Trunk}(C))$ *is in natural bijective correspondence with the set of functors* $\mathrm{Fun}(\mathrm{Cat}(S), C)$. □

Pointed versions

Given pointed trunks S and T, a **pointed** Trunk–**map** is a Trunk–map which carries identity loops to identity loops. If S is unpointed but T is pointed then any Trunk–map $S \to T$ extends uniquely to a pointed Trunk–map $S^+ \to T$. If S is pointed then $\mathrm{Cat}^+(S)$ is constructed in a similar way to $\mathrm{Cat}(S)$ identifying identity loops in S with identities in the category. If S is unpointed, we can naturally identify $\mathrm{Cat}^+(S^+)$ with $\mathrm{Cat}(S)$. Note that $\mathrm{Trunk}(C)$ is always pointed. Trunk and Cat^+ are also adjoint in the following sense:

1.8 Lemma *Given a pointed trunk S and a category C then the set of pointed* Trunk*–maps* $\mathrm{Hom}^+(S, \mathrm{Trunk}(C))$ *is in natural bijective correspondence with the set of functors* $\mathrm{Fun}(\mathrm{Cat}^+(S), C)$. □

2. Cubes and Cubical Sets

In this section we outline the theory of cubical sets that we need for the application to trunks. To begin with we shall consider cubical sets *without* degeneracies and we shall call these □–sets or square sets to distinguish from the usual notion of cubical sets. We start with a definition of □–sets and □–maps.

A **□–set** $X = \{X_n\}$ consists of a collection of sets X_n, one for each for each natural number $n \geq 0$, and functions $\partial_i^\epsilon : X_n \to X_{n-1}$, $1 \leq i \leq n$, $\epsilon \in \{0, 1\}$, satisfying the following *face relations*:

$$\partial_{j-1}^\eta \partial_i^\epsilon = \partial_i^\epsilon \partial_j^\eta \quad 1 \leq i < j \leq n \qquad \epsilon, \eta \in \{0, 1\}.$$

There is a notion of **□–map** $f = \{f_n\} : X \to Y$ between □–sets, namely for each n an $f_n : X_n \to Y_n$ and commuting relations with the face maps, $f_{n-1}\partial_i^\epsilon = \partial_i^\epsilon f_n$.

We shall give presently alternative definitions for □–sets and maps based on cubes. First we need to establish some terminology.

Cubes

Define the n–**cube** I^n to be the subset $[0, 1]^n$ of $\mathbb{R}^n$. A **vertex** of I^n is defined to be a point of the form $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ where $\epsilon_i = 0$ or 1 and $i \in \{1, 2, \ldots, n\}$. It is often convenient to regard a vertex as a subset of $\{1, 2, \ldots, n\}$ as follows: $A \subset \{1, 2, \ldots, n\}$ corresponds to the vertex with $\epsilon_i = 1$ if and only if $i \in A$. We shall use both subset and coordinate notation interchangeably.

A p–**face** of I^n is a subset defined by setting $n - p$ coordinates equal to either 0 or 1. In particular there are $2n$ $(n-1)$–faces determined by setting $x_i = \epsilon$ where

$i \in \{1, 2, \ldots, n\}$, and $\epsilon \in \{0, 1\}$. The 1–faces are called **edges** and have two vertices given in subset notation by A and $A \cup \{i\}$ where $i \notin A$. Faces can be given a canonical orientation. In particular edges are oriented $A \to A \cup \{i\}$ and 2–faces are oriented

$$\begin{array}{ccc} A \cup \{j\} & \to & A \cup \{i\} \cup \{j\} \\ \uparrow & \circlearrowleft & \uparrow \\ A & \to & A \cup \{i\} \end{array}$$

where $i < j$.

Thus I^n can be regarded as a *trunk* by taking its vertices and edges as the vertices and edges of the trunk and 2–faces as preferred squares, oriented as above. It can also be regarded as a *category*, namely the category generated by this trunk. Thus the cube can be regarded in three alternative ways: as a *topological space* (as usual), as a *trunk* or as a *category*. The trunk and category versions carry all the formal structure of the cube and are the appropriate versions of the cube to be used in the context of trunks and categories respectively. We shall not normally need to distinguish between the three versions, and will use the notation I^n for all three. The context will make clear which version is being used. However, if we do need to be more precise, we shall use the notation I^n_{Trunk} for the trunk and $I^n_{\mathrm{Cat}} = \mathrm{Cat}(I^n_{\mathrm{Trunk}})$ for the category versions.

The category version has an elegant description in terms of subsets of $\{1, \ldots, n\} as follows$.

2.1 Lemma *The category I^n_{Cat} can be naturally identified with the category whose objects are the subsets of $\{1, \ldots, n\}$ and whose morphisms are the inclusions.*

Proof The objects of I^n_{Cat} are the vertices of I^n which (in subset notation) are the subsets of $\{1, \ldots, n\}$. The edges of I^n_{Trun} are $A \to A \cup \{i\}$, i.e. particular inclusions of subsets. So we have to show that any inclusion can be expressed as a composition of edges (which is obvious) and that any two such expressions are related by preferred squares. But if $f: A \to B$ is an inclusion of subsets and if $|B| = |A| + r$ then there is a correspondence between expressions $f = f_r \ldots f_1$, where the f_i are edges, and the symmetric group S_r — each edge introduces a new element of $B \setminus A$. We can now pass from one expression to another by a sequence of relations (preferred squares) just as one can get from one element of S_r to another by a sequence of transpositions. $\square$

The category $\square$

Let $p \leq n$ and let J be a p–face of I^n. Then there is a canonical isometric identification $\gamma: J \to I^p$ defined by forgetting the fixed coordinates. If $\mathrm{inc}: J \to I^n$ is the inclusion then we call the composition $Y = \mathrm{inc} \circ \gamma^{-1}: I^p \to I^n$ a **face map**.

Definition The category $\square$ is the category whose objects are the n–cubes I^n for $n = 0, 1, \ldots, n$ and whose morphisms are the face maps.

The face maps defined by the $(n-1)$–faces are denoted $\delta_i^\epsilon: I^{n-1} \to I^n$, and given by:

$$\delta_i^\epsilon(x_1, x_2, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \epsilon, x_i, \ldots, x_{n-1}).$$

Note that the effect of δ_i^ϵ on the vertex A (in subset notation) is to add one to numbers $\geq i$, and to insert i if $\epsilon = 1$.

The following relations hold:

$$\delta_i^\epsilon \delta_{j-1}^\omega = \delta_j^\omega \delta_i^\epsilon \quad 1 \leq i < j \leq n \quad \text{and} \quad \epsilon, \omega \in \{0,1\}. \tag{$*$}$$

Any face map Y can be uniquely written $\delta_{i_k}^{\epsilon_k} \cdots \delta_{i_1}^{\epsilon_1}$ where each $\epsilon_j \in \{0,1\}$ and we assume either $i_1 < \cdots < i_k$, or $i_k \leq \cdots \leq i_1$.

Notice that the vertex $A = (A_1, \ldots, A_n)$ in coordinate notation can be identified with the morphism $\delta_n^{A_n} \ldots \delta_1^{A_1}: I^0 \to I^n$ in $\square$, and that morphisms in $\square$ preserve edges. An edge $A \to A \cup \{i\}$ in I^n can be identified with the morphism

$$\delta_n^{A_n} \ldots \delta_{i+1}^{A_{i+1}} \delta_{i-1}^{A_{i-1}} \ldots \delta_1^{A_1}: I^1 \to I^n.$$

A face map carries (oriented) edges and squares of I^p to (oriented) edges and squares of I^n and hence defines both a Trunk–map $I^p_{\text{Trunk}} \to I^n_{\text{Trunk}}$, and a functor $I^p_{\text{Cat}} \to I^n_{\text{Cat}}$. Thus there are two isomorphic copies of $\square$ given by replacing I^n by I^n_{Trunk} and I^n_{Cat} respectively. We shall not need to distinguish between $\square$ and these two copies. However note that the trunk copy has a particularly simple definition as follows.

2.2 Lemma *The category $\square$ can be naturally identified with the category which has for* objects I^n_{Trunk}, $n = 1, 2, \ldots$ *and* morphisms *all* Trunk*–maps $I^p \to I^n$ for $p \leq n$.*

Proof We have seen that a face map determines a Trunk–map. We need to check the converse statement, that any Trunk–map corresponds to a face map. But a Trunk–map is determined by its effect on the **original** edges $\emptyset \to \{i\}$ for each i and the preferred squares that any two of these form imply that their images are mutually perpendicular. The result now follows easily. $\square$

$\square$–Sets and their Realisations

We now give the second equivalent definitions of $\square$–sets and maps.

A $\square$**–set** is a functor $X: \square^{op} \to Sets$ where $\square^{op}$ is the opposite category of $\square$ and $Sets$ denotes the category of sets.

A $\square$**–map** between $\square$–sets is a natural transformation.

We write X_n for $X(I^n)$, λ^* for $X(\lambda)$ and in particular ∂_i^ϵ for $X(\delta_i^\epsilon) = (\delta_i^\epsilon)^*$.

The equivalence of the two definitions of $\square$–sets and maps follows from the formulae for the composition of face maps given in equation $(*)$ above.

The **realisation** $\| X \|$ of a $\square$–set X is given by making the identifications $(\lambda^* x, t) \sim (x, \lambda t)$ in the disjoint union $\coprod X^n \times I^n$, where I^n has its topological meaning.

Notice that $\| X \|$ is a CW–complex with one n–cell for each element of X^n.

Degeneracies

We construct a larger category $\square^+$ which contains $\square$ as a subcategory by introducing **degeneracies**. Let $\eta_i: I^n \to I^{n-1}$, where $1 \le i \le n$, be defined by collapsing the i-th coordinate. In other words

$$\eta_i(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Thus the effect of η_i on the vertex A (in subset notation) is to subtract one for numbers greater than i, and to delete i if $i \in A$. A **degeneracy** is a composition of η_i's for varying i.

The category $\square^+$ has objects I^n and morphisms generated by face maps and degeneracies, and therefore by the δ_i^ϵ and η_i. The following relations hold.

(1) $\quad \delta_i^\epsilon \delta_{j-1}^\omega = \delta_j^\omega \delta_i^\epsilon \quad 1 \le i < j \le n \quad$ and $\quad \epsilon, \omega \in \{0, 1\} \quad$ as before, and

(2) $\quad \eta^{j-1}\eta^i = \eta^i \eta^j \qquad i < j$

(3) $$\eta^j \delta_i^\epsilon = \begin{cases} \delta_i^\epsilon \eta^{j-i} & \text{if } i < j \\ \text{identity} & \text{if } i = j \\ \delta_{i-1}^\epsilon \eta^j & \text{if } i > j \end{cases}$$

By (2) any degeneracy can be written uniquely in the form $\eta^{j_s} \ldots \eta^{j_1}$ where $j_s < \ldots < j_1$. By (3) any morphism in $\square^+$ can be written uniquely as a composition $\lambda\mu$ where λ is a face map (a composition of δ_i^ϵ's) and μ is a degeneracy. Thus a morphism in $\square^+$ is a projection of I^p onto a face of I^n.

We can extend the definitions given above to get cubical sets with degeneracies, $\square^+$–sets, and their realisation (with degeneracies collapsed) $|X|$, by simply allowing λ to be a morphism of $\square^+$ in the definitions of $\square$–set and realisation. If X is a cubical set with degeneracies then we use the notation s_i for $X(\eta_i)$.

Remark Note carefully the difference in notation (and concept) of $\| X \|$ and $|X|$. The former is the realisation *without* degeneracies and the latter *with* degeneracies. For a specific example consider the singular cubical complex of a space X. This is defined by: $S(X)_n = \{f: I^n \to X\}$ the set of continuous maps.

$\lambda^*(f) = f \circ \lambda$ where λ is a morphism in $\square$ or $\square^+$ in the two contexts.

Here $\| S(X) \|$ is quite different from $|S(X)|$, for example if X is a point then $|S(X)|$ is also a point, whilst $\| S(X) \|$ has the homotopy type of $\Omega(S^2)$, the loop space on the 2–sphere, by the James' construction [Ja].

To distinguish, we shall use $S^+(X)$ for the singular complex *with* degeneracies and $S(X)$ *without*. This is consistent with the use of $(\ )^+$ in other contexts.

We have also used the notation $(\ )^+$ for both *pointed* trunks racks etc. (in the last section) and for the versions of $\square$ and realisations *with degeneracies*. This is deliberate. As we shall see in the next section, the effect of pointing is essentially the same as introducing degeneracies.

There are again isomorphic copies of $\square^+$ obtained by interpreting I^n as I^n_{Cat} or the pointed version of I^n_{Trunk}, which we shall denote I^{n+}, and there is a pointed/degeneracy version of lemma 2.2 which has a similar proof.

2.3 Lemma *The category $\square^+$ is naturally isomorphic to the category which has for* objects I^{n+}, $n = 1, 2, \ldots$ *and* morphisms *all (pointed)* Trunk*–maps* $I^{p+} \to I^{n+}$. $\square$

Kan complexes and Kan fibrations

There is a theory of Kan complexes and Kan fibrations for $\square$–sets.

Notice first that the cube I^n can be regarded as a $\square$–set in its own right with the set of p–cubes $(I^n)_p$ being the set of face maps $I^p \to I^n$ and with λ^* given by composition. The realisation of I^n is then I^n (as topological space) and the corresponding CW–complex has the usual cell structure.

Let c_n denote the unique n–cube of I^n and let $B^n_{i,\epsilon}$ be the subcomplex of I^n comprising all the proper faces of c_n other than $\partial^\epsilon_i c_n$. A $\square$–map $b: B^n_{i,\epsilon} \to X$ is called an n–**box** in X, and an extension $f: I^n \to X$ is called a **filler** of the box b. If every n–box has a filler then we say X is **Kan in dimension** n. A **box** in X is an n–box for some n. If every box has a filler we say X is a **Kan complex**.

Kan [Ka] showed that Kan complexes with degeneracies have a good homotopy theory and the homotopy groups have a simple combinatortial description. Here degeneracies are necessary as an example of Hintze [H] shows.

Let C^{n-1} be the subcomplex of I^n of all proper faces. A **closed** n–**box** in X is then a map $c: C \to X$ and c has a **filler** if c extends to I^n .

A **Kan fibration** $f: X \to Y$ is a $\square$–map such that given any box $b: B^n_{i,\epsilon} \to X$ and any filler $c: I^n \to Y$ for $f \circ b$ there is a filler $\hat{c}$ for b so that $c = f \circ \hat{c}$. There is a good theory of Kan fibrations for cubical sets with degeneracies and in particular a Kan fibration has a long exact sequence of homotopy groups, see [H].

3. The nerve of a trunk and the rack space

In this section we describe how a $\square$–set (the nerve) can be associated to any trunk.

Let S be a trunk. The **nerve** of S, denoted NS, is the $\square$–set given as follows.

The set, NS_n, of n–cubes of NS is defined to be $\mathrm{Hom}(I^n, S)$, i.e. the set of Trunk–maps from the cube trunk to S.

Face maps are given by $\lambda^*(f) = f \circ \lambda$ where $\lambda: I^p \to I^n$ is a face map (Trunk–map) and $f: I^n \to S$ is an n–cube (Trunk–map).

3.1 Examples

1. **The Rack Space**

Let X be a rack and let $\mathcal{T}(X)$ be the corner trunk associated to X described in 1.3 example 3. We shall write BX for the realisation of the nerve, $\| N\mathcal{T}(X) \|$. This is the **rack space**.

2. **The Action Rack Space**

Let X be a rack and Y an X–set. Let $\mathcal{T}_Y(X)$ be the corner trunk described in 1.4 example 1. We shall write $B_Y X$ for the realisation of the nerve $\| N\mathcal{T}_Y(X) \|$. This is the **action** rack space. A particular example is the **extended rack space** $B_X X$. The Trunk–map $p_Y \colon \mathcal{T}_Y(X) \to \mathcal{T}(X)$ induces a $\square$–map $N\mathcal{T}_Y X \to N\mathcal{T}X$ and (on realisation) a continuous map $B_Y X \to BX$.

More generally, an X–map $q \colon Y \to Z$ induces a $\square$–map $N\mathcal{T}_Y X \to N\mathcal{T}_Z X$. There are continuous actions of X on all the rack spaces (from the action on the trunk described in section 1) and $q_\sharp$ is equivariant.

3. **The Augmented Rack Space**

Let $\partial : X \to G$ be an augmented rack and let $\mathcal{T}_G(X)$ be the corner trunk associated to X and described in 1.4 example 3. We shall write $B_G X$ for the realisation of the nerve $\| N\mathcal{T}_G(X) \|$. This is the **augmented** rack space. The two actions of G on the trunk (the edge and vertex actions described in section 1) each induce G–actions on the augmented rack space. We shall refer to these as the edge or vertex action as appropriate. We shall see in theorem 3.7 below that $p_G \colon B_G(X) \to BX$ is the projection of a principal G–bundle.

There are two alternative augmented rack spaces given by using the trunks of 1.4 example 4 instead of example 3.

The Nerve of a Corner Trunk

The nerve of a corner trunk has a particularly simple description, and consequently the various rack spaces have nice descriptions. We need the following lemma. Edges in a trunk which share the same starting vertex will be called **co-original**. Until further notice, I^n will denote the n–cube trunk. The n edges at the origin of I^n, namely $\emptyset \to \{i\}$, where $i = 1, \ldots, n$, will be denoted e_i.

3.2 Lemma *Let S be a corner trunk and let $(x_1, x_2, \ldots, x_n)$ be a given n-tuple of co-original edges in S. Then there is a unique* Trunk*–map $\mu : I^n \to S$ such that $\mu(e_i) = x_i$ for $i = 1, \ldots, n$.*

Proof We shall use subset notation for the vertices of I^n. We construct μ on edges of I^n by induction on the size of the domain. Let $a \colon A \to A \cup \{i\}$ be an edge with $|A| \neq \emptyset$.

Write $A = B \cup \{j\}$ and consider the preferred square:

$$\begin{tikzcd}
A \arrow[r, "a"] & A \cup \{i\} \\
B \arrow[u] \arrow[r] & B \cup \{i\} \arrow[u]
\end{tikzcd} \tag{$*$}$$

The orientation of $(*)$ depends on the order of i and j. By induction μ is defined on the two edges of $(*)$ originating at B and, by the corner axiom V1, the images of these two edges determine a unique preferred square of S, and hence $\mu(a)$ has a unique value such that $(*)$ maps to a preferred square of S.

To see that $\mu(a)$ is well-defined suppose that $A = B_1 \cup \{j\} = B_2 \cup \{k\}$ and let $B = B_1 \cap B_2$ so that $A = B \cup \{j, k\}$ and suppose, without loss, that $j < k$. Consider the following diagram:

$$\begin{tikzcd}
B \cup \{i,k\} \arrow[rrr] & & & A \cup \{i\} \\
& B \cup \{k\} \arrow[ul] \arrow[r] & A \arrow[ur, "a"] & \\
& B \arrow[u] \arrow[r] \arrow[dl] \arrow[ur, phantom, "\circlearrowleft"] & B \cup \{j\} \arrow[u] \arrow[dr] & \\
B \cup \{i\} \arrow[uuu] \arrow[rrr] & & & B \cup \{i,j\} \arrow[uuu]
\end{tikzcd}$$

The missing orientation symbols can be put in depending on the order on $\{i, j, k\}$.

By induction μ is defined on the three edges originating at B and, by the corner axiom V2, the whole diagram has a unique image in S such that each square maps to a preferred square. Thus $\mu(a)$ is well defined.

Finally note that $(*)$ is a typical preferred square of I^n and hence μ is indeed a Trunk–map. $\square$

3.3 Theorem

Let S be a corner trunk. Then the nerve NS has the following description:

There is one n–cube for each co-original n–tuple $(x_1, x_2, \ldots, x_n)$ of edges in S.

Face maps are given by:

$$\partial_i^0(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$
$$\partial_i^1(x_1, \ldots, x_n) = ((x_1)^{x_i}, \ldots, (x_{i-1})^{x_i}, (x_{i+1})_{x_i}, \cdots, (x_n)_{x_i}) \quad \textit{for} \quad 1 \le i \le n.$$

Proof Lemma 3.2 provides the required bijection between n–cubes of $N(S)$ and co-original n–tuples of edges of S. We merely have to check that the formulae for the face maps are correct.

This is clear for ∂_i^0 whilst for ∂_i^1 we observe that the $n-1$ co-original edges which determine $\partial_i^1(x_1, \ldots, x_n)$ are the edges of $(x_1, \ldots, x_n)$ parallel to those which determine $\partial_i^0(x_1, \ldots, x_n)$ translated along the edge x_i. But these can be read from the following preferred squares:

$$
\begin{tikzcd}
X_i \arrow[r, "(x_j)^{x_i}"] & \bullet & & \bullet \arrow[r] & \bullet \\
X \arrow[u, "x_i"] \arrow[r, "x_j"] \arrow[ur, phantom, "\circlearrowleft"] & \bullet \arrow[u] & & X \arrow[u, "x_j"] \arrow[r, "x_i"] \arrow[ur, phantom, "\circlearrowleft"] & X_i \arrow[u, "(x_j)_{x_i}"']
\end{tikzcd}
$$

The left square gives the case $j < i$ and the right square the case $j > i$. We have labelled the *domain* and *range* of x_i as X and X_i respectively but have left irrelevant vertices and edges unlabelled.

Notice that the vertex where the edges in $\partial_i^0(x_1, \ldots, x_n)$ originate is X whilst that for $\partial_i^1(x_1, \ldots, x_n)$ is X_i. □

3.4 Examples

1. The rack space

Here the appropriate trunk, namely $\mathcal{T}(X)$ where X is a rack, has only one vertex and an n–tuple of co-original edges is precisely an n–tuple of elements of X. So we have the following description.

$N\mathcal{T}X_n = X^n$ (the n–fold cartesian product of X with itself).

Face maps are given by:

$$
\begin{aligned}
\partial_i^0(x_1, \ldots, x_n) &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\
\partial_i^1(x_1, \ldots, x_n) &= ((x_1)^{x_i}, \ldots, (x_{i-1})^{x_i}, x_{i+1}, \cdots, x_n) \quad \text{for} \quad 1 \le i \le n.
\end{aligned}
$$

The action of X on $\mathcal{T}(X)$ described in 1.4 example 3 now has a very simple formula:

$$(x_1, \ldots, x_n)^a = (x_1^a, \ldots, x_n^a)$$

That operation by a commutes with ∂_i^0 is obvious, while for ∂_i^1 note that $x_l^{x_i a} = x_l^{a x_i^a}$ which gives the l–th coordinate for $l < i$ (for $l > i$ it is again obvious).

2. The Action Rack Space

Here the trunk $\mathcal{T}_Y(X)$ has a vertex for each element of $y \in Y$ and we have:

$B_Y X_n = Y \times X_n$ and

$$
\begin{aligned}
\partial_i^0(y, x_1, \ldots, x_n) &= (y, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\
\partial_i^1(y, x_1, \ldots, x_n) &= (y^{x_i}, (x_1)^{x_i}, \ldots, (x_{i-1})^{x_i}, x_{i+1}, \cdots, x_n) \quad \text{for} \quad 1 \le i \le n.
\end{aligned}
$$

Notice that $y \xrightarrow{x_i} y^{x_i}$ is the edge $X \xrightarrow{x_i} X_i$ in the proof of theorem 3.3.

The projection map $p_Y\colon N\mathcal{T}_Y X \to N\mathcal{T}X$ is given by deleting the first coordinate, y, and, more generally, given an X–map $q\colon Y \to Z$, the induced map $q_\sharp\colon N\mathcal{T}_Y X \to N\mathcal{T}_Z X$

is given by applying q to the first coordinate. The X–action on $N\mathcal{T}_Y X$ is again easily described:

$$(y, x_1, \ldots, x_n)^a = (y^a, x_1^a, \ldots, x_n^a)$$

Commutativity with face maps is seen as in the last example. The fact that $q_\sharp$ is equivariant can also be readily seen in terms of these formulae.

3. **The Augmented Rack Space**

Here the vertices correspond to elements of G and in a similar way we have:

$N\mathcal{T}_G X_n = G \times X^n$ and

$$\partial_i^0(g, x_1, \ldots, x_n) = (g, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n),$$

$$\partial_i^1(g, x_1, \ldots, x_n) = (g^{x_i}, (x_1)^{x_i}, \ldots, (x_{i-1})^{x_i}, x_{i+1}, \cdots, x_n) \quad \text{for} \quad 1 \le i \le n.$$

The two actions of G on $N\mathcal{T}_G X$ again have simple expressions. The edge action has the formula

$$h_*(g, x_1, \ldots, x_n) = (h^{-1}g, x_1 \cdot h, \ldots, x_n \cdot h),$$

whilst the vertex action has the simpler formula

$$h_\sharp(g, x_1, \ldots, x_n) = (gh, x_1, \ldots, x_n).$$

Remarks on degeneracies

We shall not normally need to consider the nerve of a trunk as a cubical complex with degeneracies, indeed all the geometric interpretations which we shall investigate in later papers use the cubical structure *without* degeneracies. However there is a topologically equivalent version with degeneracies which can be used if degeneracies are needed (or if the reader feels happier to have degeneracies defined). We "install" degeneracies by pointing the rack or trunk.

Suppose that S is a pointed trunk then N^+S is the cubical set with degeneracies defined by:

$N^+S^n = \mathrm{Hom}^+(I^{n+}, S)$ (where I^{n+} is the pointed version of the cube trunk). General face maps are defined in the obvious way.

The observation that pointing and degeneracies are essentially equivalent can now be made. We observe that if S is an unpointed trunk then there is a natural bijection between cubes of NS and non-degenerate cubes of N^+S^+. This induces an isomorphism of CW–complexes between $\| NS \|$ and $|N^+S^+|$.

As an example consider the rack space BX which we defined to be $\| N\mathcal{T}(X) \|$. Using lemma 1.5 and the above remarks we could equally well have defined it, with degeneracies throughout, as $|N^+\mathcal{T}(X^+)|$. There is a similar remark for the action and augmented rack spaces, but not for the extended rack space, where the versions with degeneracies have an extra vertex, see the remark below lemma 1.5.

The construction in theorem 3.3 works with degeneracies. The analogue of lemma 3.2, in which S is a pointed trunk and I^{n+} and M are both pointed, is proved in exactly the same way, whilst the analogue of of the theorem in which S is a pointed corner trunk and we use the degeneracy nerve N^+, requires only the insertion of the following formula for degeneracies, which is readily checked:

$$s_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i, e, x_{i+1}, \ldots, x_n),$$

where e is the identity at the common domain of $x_1, \ldots, x_n$.

The nerve of a category

We now explore the connection of the nerve of a trunk with the nerve of a category and with the classifying spaces for groupoids and groups. At this point degeneracies are necessary because, in order to obtain the correct homotopy types for the classifying spaces of categories, groupoids and groups, using cubical sets, it is essential to use degeneracies (and the $|\,|$ realisation). We now define the **cubical nerve of a category**. There are two version with and without degeneracies which differ only in the extra degeneracy maps. The definitions are again analogous to the cubical singular complex of a space (with and without degeneracies).

Let C be a category. We define the **$\square$–nerve without degeneracies** denoted NC and the **cubical nerve with degeneracies** denoted N^+C to be the $\square$–set (respectively cubical set or $\square^+$–set) as follows.

The sets of n–cubes of both nerves are given by $NC^n = N^+C^n = \mathrm{Fun}(I^n, C)$, i.e. the set of functors from the cube category to C.

Face maps are given by $\lambda^*(f) = f \circ \lambda$ where $\lambda\colon I^p \to I^n$ is a face map (functor) in $\square$ (respectively $\square^+$) and $f\colon I^n \to C$ is an n–cube (functor).

We also use the notation BC for $|N^+C|$. This is the **classifying space** of the category C.

Notice that by adjointness $\mathrm{Fun}(I^n, C) \cong \mathrm{Hom}^+(I^{n+}, \mathrm{Trunk}(C))$ so we could equally well have defined BC to be $|N^+\,\mathrm{Trunk}(C)|$. The 2–cubes of BC come from the preferred squares of $\mathrm{Trunk}(C)$, i.e. the commuting squares in C, and in general we can regard an n–cube of BC as an n–cubical commuting diagram, i.e. a diagram of maps labelling the 1–skeleton of I^n.

We shall outline a proof of the fact that in the case of a groupoid the cubical classifying space has same homotopy type as the usual simplicial space. The proof is quite short and proceeds by constructing an analogue of the path space fibration. The definitions work for any category.

The path space fibration

Let C be a category. We choose an object $*^0$ of C as a **base vertex**. The **base complex**, denoted $*$, has one cube in each dimension, namely the unique cube (functor), denoted $*_n$, which maps every edge of I^n to the identity at the vertex $*$. We now define the **path complex** EC by $EC_n = \{x \in NC_{n+1} | \partial^1_{n+1}(x) = *_n\}$. Face maps in EC are the same as in NC but with ∂^ϵ_{n+1} omitted.

Let $p: EC \to NC$ be the projection given by $x \mapsto \partial^0_{n+1}(x)$. The **fibre** of p (i.e. $p^{-1}(*)$) is the **loop complex** ΩNC, namely cubes x of NC with $\partial^0_{n+1}(x) = \partial^1_{n+1}(x) = *_n$. The realisation $|\Omega NC|$ can be seen to be the set of loops at $*^0$ (morphisms with domain and range at $*^0$), with the discrete topology.

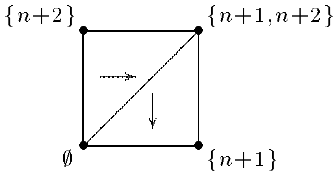
To complete the analogy with the usual path space fibration, there is a cubical contraction of EC given by using a "folded projection map" of I^{n+2} onto I^{n+1}, cf. the 'connections' of [BH1].

Define the functor $\rho: I^{n+2} \to I^{n+1}$ on vertices (in subset notation) by

$$\begin{aligned} A &\longmapsto A \\ A \cup \{B\} &\longmapsto A \cup \{n+1\}, \end{aligned}$$

where $A \subset \{1, 2, \ldots, n\}$ and $\emptyset \neq B \subset \{n+1, n+2\}$. It can readily be checked that these formulae are consistent with subsets, and hence, by lemma 2.1 define a functor.

The effect of ρ on I^{n+2} is pictured below. The last two axes only are drawn, the other n axes being all mapped by the identity. In the diagram the top left triangle is projected right onto the diagonal, and the bottom right triangle is then projected downwards onto the bottom face, which is identified with I^{n+1}:



Consider a cube $x = y \times I$ say in $EC \times I$. If y is an n–cube of EC then x can be regarded as an $(n+1)$–cube of EC and an $(n+2)$–cube of NC. Map to EC by ρ followed by the characteristic map for y. If we regard the $(n+2)$–nd direction $\emptyset \to \{n+2\}$ as the homotopy direction, these maps fit together to provide the claimed cubical contraction of EC. There is a corresponding topological contraction of $|EC|$.

If C is in fact a groupoid then both NC and EC are Kan complexes and if NC is connected then $p: EC \to NC$ is a Kan fibration and the fibre is the vertex group at $*$. Summarising we have:

3.5 Proposition *Suppose that C is a connected groupoid then BC has the homotopy type of the usual classifying space of the vertex group of C.* □

The connection between the nerves of trunks and categories

Let S be a trunk, then there is a natural □–map

$$NS \xrightarrow{\nu} N\,\mathrm{Cat}(S),$$

from the nerve of the trunk S to the nerve of the category generated by S. (Since, by adjointness, we can identify $N\,\mathrm{Cat}(S)$ with $N\,\mathrm{Trunk}\;\mathrm{Cat}(S)$, then ν is induced by the natural inclusion of trunks $S \to \mathrm{Trunk}\;\mathrm{Cat}(S)$.) It follows that there are maps of realisations

$$\| NS \| \xrightarrow{\|\nu\|} \| N\,\mathrm{Cat}(S) \| \xrightarrow{\pi} |N^+\,\mathrm{Cat}(S)| = B\,\mathrm{Cat}(S).$$

Here π is the natural map obtained by collapsing degeneracies.

Now there is a functor G from categories to groupoids which is obtained by inverting the morphisms of a given category C to obtain the groupoid category GC. If C has only one object then GC will be a group. We write $x = yz$ in the group if $x = z \circ y$ in GC. We now have natural maps:

$$\| NS \| \longrightarrow B\,\mathrm{Cat}(S) \longrightarrow BG\,\mathrm{Cat}(S).$$

Example 1 *The rack space and the associated group*

Let X be a rack and $S = \mathcal{T}(X)$. Then $\mathrm{Cat}(S)$ has one vertex and hence $G\,\mathrm{Cat}(S)$ is a group. This group is generated by the elements of X subject to the relations which come from the preferred squares, i.e. $a^b = b^{-1}ab$ for all $a, b \in X$. But this is precisely the definition of the associated group of the rack $As(X)$, see [FR; section 2]. So we have a natural map of classifying spaces

$$BX \longrightarrow BAs(X).$$

Example 2 *The augmented rack space and its group*

The last example can be generalised. Let $\partial\colon X \to G$ be an augmented rack. If we map $x \in X$ to $\partial x \in G$ then the preferred squares again map to commuting squares in G (regarded as a category with one object) by the augmentation identity. Hence we have another natural map $BX \to BG$ which we could also have defined as the composition

$$\mu\colon BX \longrightarrow BAs(X) \longrightarrow BG \tag{3.6}$$

using the homomorphism $As(X) \to G$ which comes from the universal property of the associated group [FR; proposition 2.1]. Since any rack X is an augmented rack with group either the associated or the operator group of X, this generalises example 1.

Remark We could have used degeneracies throughout the above discussion. If we replace S by the pointed trunk S^+ then we have a similar cubical map $N^+S^+ \to N^+C = NC$ which induces the identical map between realisations. In particular pointing the rack X and using degeneracies throughout gives precisely the same map of classifying spaces.

Coverings and principal bundles

We now give the promised analysis of the natural projections $N_Y X \to NX$ and $N_G X \to NX$. They are in fact projections of **coverings** in the category of $\square$–sets.

Let X be a $\square$–set and F any (discrete) set. We say that a map of $\square$–sets $p\colon E \to B$ is **locally trivial** with fibre F if for each n there is a given bijection

$$\alpha_n\colon F \times B_n \longrightarrow E_n$$

over B, i.e. so that

$$\begin{tikzcd} F \times B_n \arrow[rr, "\alpha_n"] \arrow[dr, "\pi"'] & & E_n \arrow[dl, "p_n"] \\ & B_n & \end{tikzcd}$$

commutes, where π denotes projection, and if for each $x \in B_n$, $\epsilon \in \{0,1\}$ and $1 \le i \le n$ there is another bijection $\beta_i^\epsilon(x)$: $F \to F$ so that the following square commutes:

$$\begin{tikzcd} E_n \arrow[r, "\partial_i^\epsilon"] & E_{n-1} \\ F \times B_n \arrow[u, "\alpha_n"] \arrow[r, "\overline{\beta}_i^\epsilon"] & F \times B_{n-1} \arrow[u, "\alpha_{n-1}"'] \end{tikzcd}$$

where $\overline{\beta}_i^\epsilon(f,x) = (\beta_i^\epsilon(x)f, \partial_i^\epsilon(x))$.

We also say that p is a **$\square$–covering with fibre** F.

Note that the bijection $\beta_i^\epsilon(x)$ can be thought of as the "glue" for the bundle corresponding to the face ∂_i^ϵ of the cube $x \in B_n$.

If F is in fact a group G and the β_i^ϵ are given by (left) multiplication by an element of G and if G also acts on the right of E so that α_n commutes with the action (where G acts on $G \times B_n$ by (right) multiplication on G), then we say that $p: E \to B$ is a **principal $\square$–bundle with fibre** G.

Remarks

1. If we realise a $\square$–covering or principal $\square$–bundle, then we get a genuine covering space or principal bundle (with discrete group).

2. There are other notions of $\square$–bundle (with non-discrete fibre) which we shall not need in this paper.

3.7 Theorem *Suppose that X is a rack and that Y is an X–set, then the natural projection*

$$p_Y: N_Y X \to NX$$

is a $\square$–covering with fibre Y.

Suppose that ∂: $X \to G$ is an augmented rack then the natural projection

$$p_G: N_G X \to NX$$

is a principal $\square$–bundle with fibre G.

Proof Dealing first with the action rack space, after the construction in 3.4 example 2 we can take α_n to be the identity and the β_i^ϵ can be read from the face map formulae in that example:

$$\beta_i^0(x) = \mathrm{id}. \quad \text{and} \quad \beta_i^1(x)y = y^{x_i}.$$

In the case of the augmented rack, α_n is again provided by the construction in 3.4 example 3, and there are similar formulae for the β_i^ϵ. In particular

$$\beta_i^1(x)g = g^{x_i} = (\partial x_i)^{-1}g,$$

so the β_i^ϵ are defined by left multiplication in G. Finally note that α_n commutes with right multiplication on G in $G \times NX_n$ and the vertex action of G on $N_G X$, which is again given by right multiplication on the first entry in the bar construction (see formula for $h_\sharp$ in 3.4 example 3). $\square$

It is an interesting fact that the covering $N_X X \to NX$ (of the extended rack space over the rack space) can be recovered from the principal G–bundle $N_G X \to NX$ (using any augmented rack $\partial\colon X \to G$, for example $G = As(X)$ or $G = Op(X)$ where $Op(X)$ is the operator group of X, see [FR]):

3.8 Proposition *For any augmented rack X with group G there is a $\square$–bundle isomorphism*

$$\phi\colon N_X X \to X \times_G N_G X,$$

given by $\phi(x, x_1, \ldots, x_n) = [x, (e, x_1, \ldots, x_n)]$, where we use the vertex action of G on $N_G X$ and the action on X from the definition of augmented rack.

Proof This is straightforward to check from the definitions. $\square$

The classifying map for the principal G–bundle $p_G\colon N_G X \to NX$

Let $\partial\colon X \to G$ be an augmented rack, then as we saw earlier, 3.6, there is a natural $\square$–map $\mu : NX \to NG$. We now prove that this is in fact covered by a G–bundle map. In other words:

3.9 Theorem *The $\square$–map μ is the classifying map for the principal G–bundle p_G. So there is a map $\widehat{\mu}\colon N_G X \to EG$ which covers $\mu : NX \to NG$.*

Proof We need to re-examine the path space EG. Recall that an n–cube of EG is an $(n+1)$–cube of NG with $\partial^1_{n+1} = *_n$. Thus a 1–cube (edge) is a commuting square of the following type:

$$\begin{array}{ccc} \bullet & \xrightarrow{*} & \bullet \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h^{-1}g} \\ \bullet & \xrightarrow{h} & \bullet \end{array}$$

which can be conveniently regarded as the "edge":

$$g \xrightarrow{h} h^{-1}g.$$

A 2–cube in EG is a square diagram of such "edges" (note that the labels on the vertices imply that such a square automatically commutes), and in general an n–cube is a n–cubical diagram of "edges". Thus we have a description of EG as the nerve of the trunk with these "edges" as edges and with all squares of edges preferred.

Note that the action of G on EG is given in this description by right multiplication on vertices and this makes the projection map $p: EG \to NG$ a principal $\square$–bundle which, since EG is contactible, is the classifying bundle.

We can now define the required G–bundle map $N_G X \to EG$ as induced by the Trunk–map given on edges by:

$$\begin{array}{ccc} g^x = (\partial x)^{-1} g & & (\partial x)^{-1} g \\ {\scriptstyle x}\uparrow & \overset{\widehat{\mu}}{\longmapsto} & \uparrow{\scriptstyle \partial x} \\ g & & g \end{array}$$

This covers μ and proves the theorem. $\square$

Summary

We can summarise most of the results of this section in the following two diagrams:

$$\begin{array}{ccccc} Y & \qquad & G & = & G \\ \downarrow & & \downarrow & & \downarrow \\ B_Y X & & B_G X & \xrightarrow{\widehat{\mu}} & EG \\ {\scriptstyle p_Y}\downarrow & & {\scriptstyle p_G}\downarrow & & \downarrow \\ BX & & BX & \xrightarrow{\mu} & BG \end{array}$$

In the left diagram Y is an X–set and p_Y is a covering map (fibre Y). In the right diagram X is an augmented rack with group G and p_G is the projection of a principal G–bundle of which μ is the classifying map. The rack X acts on both diagrams.

Further the action of X on the right diagram is compatible with an action of G (the edge action) which, in the fibre, is given by $h \cdot g = g^{-1}h$. This action commutes with the (vertex) action which defines the principal bundles.

Both diagrams are natural under maps of X–sets or augmented racks respectively, and such maps induce equivariant maps of diagrams under all the actions.

The right hand diagram can be applied to any rack by taking G to be either the associated group $As(X)$ or the operator group $Op(X)$.

4. Miscellaneous further properties

We first prove that the nerve of a trunk has some of the analogous properties to the classifying space of a group.

Eilenberg–Maclane properties

4.1 Proposition *For any trunk S, NS is Kan, with unique fillers, in dimensions >2, and every closed n–box has a unique filler if $n>1$.* $\square$

Proof This is immediate from the fact that a trunk map $I^n \to S$ is given by its values on the 1–skeleton of I^n. $\square$

4.2 Corollary *Let S be a trunk with nerve NS and A any $\square$–set. Then any map from the 2–skeleton of A to $N\mathcal{T}$ extends uniquely to the whole of the $\square$–set.* $\square$

Remarks (i) Proposition 4.1 is an analogue for cubical sets without degeneracies of the vanishing of higher homotopy groups.

(ii) If S has identities then NS admits degeneracies and further if $S = \mathrm{Trunk}(C)$ where C is a groupoid then NS is Kan in all dimensions and hence each component of $|NS|$ is an Eilenberg–MacLane space. (Indeed as we saw in 3.5 above, in this case each component of $|NS|$ is the classifying space for the appropriate vertex group of C, and hence an Eilenberg–MacLane space.)

$\square$–sets to trunks

We next show that the nerve functor N from trunks to $\square$–sets has an adjoint.

A $\square$–set A determines a trunk $\mathcal{T}(A)$ by taking as vertices the set A^0, as edges the set A^1, and for typical preferred squares the sequence $\partial_2^0 a, \partial_1^0 a, \partial_2^1 a, \partial_1^1 a$ where $a \in A^2$.

4.3 Proposition *The functors $\mathcal{T}$ from $\square$–sets to trunks and N from trunks to $\square$–sets are adjoint. More precisely given a $\square$–set A and a trunk S there is a natural bijection between the set of $\square$–maps $\mathrm{Hom}(A, NS)$ and the set of Trunk–maps $\mathrm{Hom}(\mathcal{T}(A), S)$.*

Proof There is an obvious forgetful map $\phi\colon \mathrm{Hom}(A, NS) \to \mathrm{Hom}(\mathcal{T}(A), S)$, but this is a bijection by 4.2. $\square$

The adjoint to the identity on $\mathcal{T}(A)$ is a $\square$–map $\iota_A\colon A \to N\mathcal{T}(A)$ which can be defined explicitly as follows. Suppose $a \in A_n$ then $\iota_A(a)\colon I^n \to \mathcal{T}(A)$ is given by $\iota_A(a)\lambda = \lambda^* a$ where λ is either a vertex or an edge.

4.4 Proposition *For any $\square$–set A, $\iota_A\colon A \to N\mathcal{T}(A)$ is an isomorphism on the 1–skeleton and induces an isomorphism of fundamental groups.* $\square$

The adjoint to the identity on NS is a Trunk–map $\kappa_S\colon \mathcal{T}(NS) \to S$ which can be seen to be a natural isomorphism of trunks.

Remark The trunk and nerve functors here are analogous to the truncation and coskeleton functors found in [D].

The fundamental groupoid of a $\square$–set

Let X be a $\square$–set then the **fundamental groupoid** of X, $\pi(X)$, has the following combinatorial description. The object set is X_0. Given objects $x, y \in X_0$ the morphisms from x to y are equivalence classes of sequences of directed edges (elements of X_1) starting at x and ending at y under the equivalence relation generated by pushing across squares (elements of X_2).

Now $\pi(X)$ can be identified with the fundamental groupoid of $\| X \|$ after restricting the object set to the set of vertices, also denoted X_0, in $\| X \|$. The vertex groups $\pi_1(X, x)$ of $\pi(X)$, where $x \in X_0$, are canonically isomorphic to the groups $\pi_1(\| X \|, x)$.

Consider a rack X and X–set Y. Then the action of the rack X on Y induces an action of the associated group $As(X)$ on Y.

4.5 Proposition *$\pi_1(B_Y X, y)$ can be identified with the stabiliser of y in $As(X)$.*

Proof A closed sequence of edges in $B_Y X$ gives a word in $As(X)$ by composition of labels. By looking at the effect on the labels on the vertices, it is seen to lie in the stabiliser of y. Finally the squares correspond to defining relations in $As(X)$ (see the example below 3.5). $\square$

Topological invariants of the rack space

In the case that Y is a one–point set, proposition 4.5 says that the fundamental group of the rack space is the associated group of the rack. In general the determination of the higher homotopy groups and other algebraic topological invariants of rack spaces is non-trivial. Here is a summary of the information that we have to date. Details will appear in other papers.

In the case that X is a **classical rack** (the fundamental rack of a link in S^3) then π_2 is a free abelian group of rank the number of non-trivial maximal irreducible sublinks of the link, see [FRS].

The rack spaces are all **simple** spaces (π_1 acts trivially on π_n for $n > 1$), see [FRS].

The rack space of the **trivial** rack ($a^b = a$ for all a, b) of n elements has the homotopy type of $\Omega(\bigvee_n S^2)$.

The homology of the the extended rack space $B_X X$ is the same as the reduced homology of the rack space BX but with a shift of one dimension.

Rack Spaces and Crossed Complexes

We finish by investigating the connection between the rack space and the classifying space for crossed modules constructed by Brown and Higgins, see [BH2] and for the equivalent cubical version see [BH3].

Recall that an augmented rack X with group G is a **crossed module** if X is a group, $\partial: X \to G$ is a homomorphism and $a.\partial b = b^{-1}ab$ for all $a, b \in X$. Notice that a crossed module is naturally pointed as an augmented rack.

Crossed complexes

The **nerve** N^+C of a crossed complex C is described in [BH2, BH3]. We recall the definition. To a filtered space

$$\mathbb{Y} : Y_0 \subset Y_1 \subset \ldots \subset Y_n \subset \ldots \subset Y$$

there is an associated homotopy crossed complex $\pi\mathbb{Y}$, which consists for $n = 1$ of the fundamental groupoid $\pi_1\mathbb{Y} = \pi_1(Y_1, Y_0)$, and for $n \geq 2$ of the family $\pi_n\mathbb{Y}$ of relative homotopy groups $\pi_n(Y_n, Y_{n-1}, v)$, $v \in Y_0$, with the usual boundaries $\partial: \pi_n\mathbb{Y} \to \pi_{n-1}\mathbb{Y}$ and action of $\pi_1\mathbb{Y}$ on $\pi_n\mathbb{Y}$. The formal properties satisfied by $\pi\mathbb{Y}$ define the notion of **crossed complex**, and there is a category of crossed complexes.

The n–cube as a crossed complex

Let $\mathbb{I}^n$ be the n–cube with its skeleton filtration. Generators of $\pi_m\mathbb{I}^n$ may be identified with Trunk–maps $\mu: I^m \to I^n$. Define **axes** $\alpha_i: I^1 \to I^m$ by $\alpha_i = \delta^0_m \ldots \delta^0_{i+1}\delta^0_{i-1} \ldots \delta^0_1$. Then $\mu\alpha_i$ is an element of the groupoid $\pi_1\mathbb{Y}$ and ∂_m is given by the following formulae.

$$\partial_m\mu = \begin{cases} \sum_{i=1}^m (-1)^i(\mu\delta^0_i - (\mu\delta^1_i)^{-\mu\alpha_i}) & \text{for } m > 3 \\ (\mu\delta^1_3)^{-\mu\alpha_3} + \mu\delta^0_2 + (\mu\delta^1_3)^{-\mu\alpha_1} - \mu\delta^0_3 - (\mu\delta^1_2)^{-\mu\alpha_2} - \mu\delta^0_1 & \text{for } m = 3 \\ \mu\delta^0_1 + \mu\delta^1_2 - \mu\delta^1_1 - \mu\delta^0_2 & \text{for } m = 2 \end{cases}$$

Here exponentiation denotes the action of the fundamental groupoid.

The order of terms is important in case $m < 4$ and all three formulae may be regarded as special cases of the following $2m$–term expression.

$$\ldots \mu\delta^0_4 + (\mu\delta^1_3)^{-\mu\alpha_3} + \mu\delta^0_2 + (\mu\delta^1_1)^{-\mu\alpha_1} \ldots - (\mu\delta^1_4)^{-\mu\alpha_4} - \mu\delta^0_3 - (\mu\delta^1_2)^{-\mu\alpha_2} - \mu\delta^0_1$$

The nerve of a crossed complex

The **nerve** N^+C of a crossed complex C is the cubical set given by defining N^+C_n to be the set of maps of crossed complexes from $\pi\mathbb{I}^n$ to C, with face and degeneracy maps given (as usual) by composition.

4.6 Remark It is convenient to think of an n–cube of NC as a labelling by elements of C of the faces of a standard n–cube so that the labels on the boundary of a face 'add up', according to the above formulae, to the boundary of the label on that face.

Let C_1 be a group. The complex N^+C is Kan and so one can define its **loop complex** ΩN^+C (as before) by

$$\Omega N^+C_n = \{x \in N^+C^{n+1} | \partial^\epsilon_{n+1}x = e, \epsilon = 0, 1\}$$

where $e: \pi\mathbb{I}^n \to C$ is the trivial map.

Crossed modules and crossed complexes

A crossed module given as $\partial: X \to G$ determines a crossed complex C by setting:

$$C_n = \begin{cases} X & \text{if } n = 2 \\ G & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $\partial_2: C_2 \to C_1$ is $\partial: X \to G$.

The Brown–Higgins classifying space of the crossed module is then the realisation of the nerve of C.

Let $\mathrm{Cat}_G X$ be the category with object set G and morphisms of the form $g \xrightarrow{x} g^x$ where $g \in G$, $x \in X$ and $g^x = (\partial x)^{-1} g$.

4.7 Theorem *Let C be the crossed complex associated with the crossed module $\partial: X \to G$ then there is an isomorphism of cubical sets*

$$\psi : N^+(\mathrm{Cat}_G X) \cong \Omega N^+ C.$$

The proof of theorem 4.7 will be given presently.

Notice that $N^+(\mathrm{Cat}_G X)$ can be identified with $N^+ \,\mathrm{Trunk}(\mathrm{Cat}_G X)$ and that the definition of $\mathrm{Trunk}(\mathrm{Cat}_G X)$ is very similar to the definition of the augmented rack trunk $\mathcal{T}_G(X)$ given in 1.4 example 3. Indeed the sets of vertices and edges are exactly the same whilst the set of preferred squares is larger. It follows that there is a natural map

$$N^+\mathcal{T}_G(X) \longrightarrow \Omega N^+ C.$$

Further if $\partial: X \to G$ is any augmented rack then there is a natural map of G–racks to the associated crossed module (see [FR, p.356]) and again we have $N^+\mathcal{T}_G(X) \longrightarrow \Omega N^+ C$.

Proof of theorem 4.7

An n–cube of $N^+\mathrm{Trunk}(\mathrm{Cat}_G X)$ can be thought of as a standard n–cube with vertices labelled in G and edges labelled in X. But as we saw in remark 4.6, an n–cube of N^+C is also a labelled n–cube and an n–cube of the loop complex is an $(n+1)$–cube of N^+C with trivial labels at top and bottom—thinking of the last direction as 'up'. Intuitively the map ψ is given by multiplying an n–cube of $N^+(\mathrm{Cat}_G X)$ by the interval I and labelling the resulting edges and 'squares' accordingly. Details now follow.

Let $f \in N^+(\mathrm{Cat}_G X)_n$, so $f: I^n \to \mathrm{Cat}_G X$ is a functor.

We need to define $\psi(f): \pi\mathbb{I}^{n+1} \to C$ so that

$$\psi(f) \circ \pi\delta^{\epsilon}_{n+1}: \pi\mathbb{I}^n \to \pi\mathbb{I}^{n+1} \to C$$

is trivial for $\epsilon \in \{0,1\}$. Let $\alpha = \psi(f)_2$, $\beta = \psi(f)_1$ denote the components of $\psi(f)$ in dimensions 2 and 1. We have to define α and β so that the following diagram commutes and determines a map of crossed complexes. Since $\pi_2\mathbb{I}^{n+1}$ is a free crossed module over the groupoid $\pi_1\mathbb{I}^{n+1}$ generated by the elements determined by the 2-cells of I^{n+1}, it will be sufficient to define α and β so that $\alpha\partial_3 = 0$, and $\partial_2\alpha = \beta\partial_2$ on the 2-cell generators.

$$\begin{array}{ccccc} \pi_3\mathbb{I}^{n+1} & \xrightarrow{\partial_3} & \pi_2\mathbb{I}^{n+1} & \xrightarrow{\partial_2} & \pi_1 I^{n+1} \\ \downarrow & & \downarrow \alpha & & \downarrow \beta \\ 0 & \longrightarrow & X & \xrightarrow{\partial} & G \end{array}$$

Definition of α and β

Given $\lambda: I^m \to I^n$ define the suspension $s\lambda: I^{m+1} \to I^{n+1}$ by $s\lambda(A, \epsilon) = (\lambda A, \epsilon)$ for $\epsilon \in \{0,1\}$, and vertex $A \in I^m$. Notice that $s\delta_j^\epsilon = \delta_j^\epsilon$.

For $\mu: I^2 \to I^{n+1}$ a generator define

$$\alpha(\mu) = \begin{cases} (f\lambda)e_1 & \text{if } \mu = s\lambda \\ \text{identity} & \text{if } \mu \text{ is not a suspension} \end{cases}$$

where e_1 is the edge in I^1.

For $\mu: I^1 \to I^{n+1}$ a generator define

$$\beta(\mu) = \begin{cases} (f\lambda)e_0 & \text{if } \mu = s\lambda \\ \text{identity} & \text{if } \mu \text{ is not a suspension} \end{cases}$$

where e_0 is the object in I^0.

The composition $\alpha\partial_3 = 0$

Let μ be a generator of $\pi_3\mathbb{I}^{n+1}$. Then

$$\begin{aligned} \alpha\partial_3\mu &= \alpha[(\mu\delta_3^1)^{-\mu\alpha_3} + \mu\delta_2^0 + (\mu\delta_1^1)^{-\mu\alpha_1} - \mu\delta_3^0 - (\mu\delta_2^1)^{-\mu\alpha_2} - \mu\delta_1^0] \\ &= \alpha(\mu\delta_3^1)^{-\beta(\mu\alpha_3)} + \alpha(\mu\delta_2^0) + \alpha(\mu\delta_1^1)^{-\beta(\mu\alpha_1)} - \alpha(\mu\delta_3^0) - \alpha(\mu\delta_2^1)^{-\beta(\mu\alpha_2)} - \alpha(\mu\delta_1^0) \end{aligned}$$

and if μ is not a suspension $\alpha\partial_3(\mu) = 0$ so suppose $\mu = s\lambda$ then

$$\begin{aligned} \alpha\partial_3\mu &= \alpha[\mu\delta_2^0 + \mu\delta_1^1 - \mu\delta_2^1 - \mu\delta_1^0] \\ &= f\lambda\delta_2^0 e_1 + f\lambda\delta_1^1 e_1 - f\lambda\delta_2^1 e_1 - f\lambda\delta_1^0 e_1. \end{aligned}$$

Now suppose that $f\lambda e_1$ is the square

$$\begin{array}{ccc} g^u & \xrightarrow{v} & g^{xy} \\ \uparrow u & \circlearrowleft & \uparrow y \\ g & \xrightarrow{x} & g^x \end{array}$$

where $uv = xy$ then

$$\alpha\partial_3\mu = xyv^{-1}u^{-1} = 0.$$

The equality $\partial\alpha = \beta\partial_2$

It is enough to consider a generator $\mu = s\lambda: I^2 \to I^{n+1}$. Then $\partial\alpha\mu = \partial f\lambda e_1$ and

$$\begin{aligned} \beta\partial_2\mu &= \beta[\mu\delta_1^0 + \mu\delta_2^1 - \mu\delta_1^1 - \mu\delta_2^0] \\ &= \beta[\mu\delta_1^0 - \mu\delta_1^1] \\ &= f\lambda\delta_0^1 e_0 - f\lambda\delta_1^1 e_0. \end{aligned}$$

Suppose $f\lambda e_0$ is the edge $g \xrightarrow{x} g^x$ then

$$\beta\partial_2\mu = g(g^x)^{-1} = \partial x = \partial f\lambda e_1.$$

Now it follows from the description in terms of labels that ψ is a bijection on sets of cubes. Therefore to complete the proof it remains to show that ψ is cubical map.

ψ is a cubical map

The map $\delta_i^\epsilon\colon I^n \to I^{n+1}$ induces $\delta_i^\epsilon\colon \pi\mathbb{I}^n \to \pi\mathbb{I}^{n+1}$. From definitions $(\partial_i^\epsilon \alpha(f))\mu = \alpha(f)(\delta_i^\epsilon \mu)$. If μ is not a suspension then $\delta_i^\epsilon \mu$ is not a suspension and if $\mu = s\lambda$ then $\delta_i^\epsilon \mu = s\delta_i^\epsilon \lambda$. So if $\mu = s\lambda$ then

$$\begin{aligned} \alpha(f)(\delta_i^\epsilon \mu) &= (\delta_i^\epsilon \lambda)^* f e_1 \\ &= \lambda^*(\partial_i^\epsilon f) e_1 \\ &= \alpha(f)(\delta_i^\epsilon f)\mu. \end{aligned}$$

If μ is not a suspension $\alpha(f)(\delta_i^\epsilon \mu) = \text{id.} = \alpha(\partial_i^\epsilon f)\mu$. Similar considerations apply to β.

Finally it is not hard to see that the association of α, β to f induces the required bijection. $\square$

References

[B] **E.Brieskorn**, *Automorphic sets and singularities*, Contemporary maths., 78 (1988) 45–115

[BH1] **R.Brown and P.Higgins**, *On the Algebra of Cubes*, Jour. Pure and App. Alg., 21 (1981) 233–260

[BH2] **R.Brown and P.Higgins**, *The classifying space of a crossed complex*, Math. Proc. Cambridge Philos. Soc., 110 (1991) 95–120

[BH3] **R.Brown and P.Higgins**, *The classifying space of a crossed complex*, U. Coll. of N. Wales, Math. preprint 89.06 (1989)

[BRS] **S.Buoncristiano, C.Rourke and B.Sanderson**, *A geometric approach to homology theory*, London Math. Soc. Lecture Note Series, no. 18 C.U.P. (1976)

[CW] **J.C.Conway and G.C.Wraith**, correspondence, 1959

[D] **J.Duskin**, *Simplicial methods and the interpretation of "triple" cohomology*, Memoirs of the AMS, volume 3, issue 2, 163(1973).

[FR] **R.Fenn and C.Rourke**, *Racks and links in codimension two*, J. of Knot Theory and its Ramifications, 1 (1992), 343–406

[FRS] **R.Fenn, C.Rourke and B.Sanderson**, *An introduction to species and the rack space*, Proceedings of the Topology conference, Erzurum, Turkey (1992)

[FY] **P.Freyd and D.Yetter**, *Braided compact closed categories with applications to low dimensional topology*, Advances in math. 77 (1989) 156–182

[H] **S.Hintze**, *Polysets, □-sets and semi-cubical sets*, MPhil thesis, Warwick (1973)

[J] **D.Joyce**, *A classifying invariant of knots; the knot quandle*, J. Pure Appl. Alg., 23 (1982) 37–65

[Ja] **I.James**, *On the suspension triad*, Ann. Math. 63 (1956) 191–247

[K] **L.Kauffman**, *Knot-crystals - classical knot theory in modern guise*, Knots and Physics, World Scientific (1991) 186–204

[Ka] **D.M.Kan**, *Abstract homotopy I*, Proc Nat. Acad. Sci. 41 (1955) 1092–1096

[Mv] **S. Matveev**, *Distributive groupoids in knot theory*, Math. USSR Sbornik, 47 (1984) 73–83

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