There are two 2–twist-spun trefoils

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Abstract We give a short proof inspired by Carter et al [1] that the 2–twist-spun trefoil is not isotopic to its orientation reverse. The proof uses a computer calculation of the third homology group of the three colour rack. We also give a new proof using the same calculation of the well-known fact that the left and right trefoil knots are not isotopic.

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1 Introduction

In a recent paper [1] Carter, Jelsovsky, Kamada, Langford and Saito construct "state-sum" invariants of knots using a notion of quandle cohomology. As an application they prove that the twice twist-spun trefoil is not isotopic to its orientation reverse. Quandle cohomology groups are closely related to the cohomology groups of the rack space, constructed earlier by Roger Fenn and the authors [3, 4] and the state-sum invariant has a natural interpretation in terms of the canonical class in the homotopy of the rack space determined by the knot diagram.

In this paper we recover the theorem of Carter et al by a short argument based directly on the canonical class. We need to calculate the third homology group of the three-colour rack and for this we rely on a Maple worksheet [7]. The same calculation is used to prove the well-known fact that the left and right trefoil knots in $\mathbb{R}^3$ are different.

Here is an outline of the paper. In section 2 we give some background material on racks and the rack space and in section 3 we prove the main theorem, that the 2–twist-spun trefoil is not isotopic to its orientation reverse preserving orientations. In section 4 we prove that the left and right trefoils are different and we conclude in section 5 with some remarks about the proofs.

2 Background

Framings and diagrams for 2–knots in 4–space

For classical knots, the oriented and framed theories differ, for example the writhe of a knot is an invariant of framed knots but not of oriented knots. For 2–knots (ie knots of S^2 in $\mathbb{R}^4$) the distinction disappears:

Let $K \subset \mathbb{R}^4$ be an oriented, possibly knotted, 2–sphere in $\mathbb{R}^4$. The normal bundle of K has a section since K bounds a 3–manifold (a generalised "Seifert surface") and furthermore the orientations now give a canonical second section, in other words a framing of K. Thus we can confuse sections of the normal bundle and framings. Now two sections of K differ by a map $K \to S^1$ (since the normal bundle is trivial) and since $\pi_2(S^1) = 0$ any two sections are isotopic. Thus the orientations determine a canonical framing of K in $\mathbb{R}^4$.

Now consider the canonical projection $\mathbb{R}^4 \to \mathbb{R}^3$ and think of $\mathbb{R}^3$ as horizontal and the remaining $\mathbb{R}^1$ as vertical. Suppose that we are given a 2–knot K such that the projection of K on $\mathbb{R}^3$ has multiple set comprising transverse double lines and triple points. The image of the projection together with the information of the vertical order of sheets at the multiple set is called a *knot diagram* for K. The diagram determines K up to isotopy and also determines a natural framing of K corresponding to the section which is vertically up. Conversely, by the Compression Theorem [6] a 2–knot K with a section of its normal bundle can be isotoped so that the section is vertically up and hence K projects to an immersion in $\mathbb{R}^3$. By making this immersion self-transverse we now have a knot diagram for K. Furthermore, by the 1–parameter version of the Compression Theorem, isotopic framed embeddings determine isotopic knot diagrams. Combining the two facts just outlined, namely that orientation determines framing and that framing determines diagram, we have:

Lemma 2.1 *There is a natural bijection between isotopy classes of oriented embeddings of a 2–sphere in $\mathbb{R}^4$ and isotopy classes of knot diagrams in $\mathbb{R}^3$.*

The canonical class of a diagram

Full details of all the material in this section can be found in [4]. For background material on racks see [2]. Here we shall summarise the results from [4] which are needed for the main theorem.

Let R be a rack. The *rack space* BR of R is a cubical set with the set of n–cubes in bijection with R^n. We shall only need to consider cubes of dimension ≤ 3. The interpretation of such cubes is given by the pictures in figure 1.

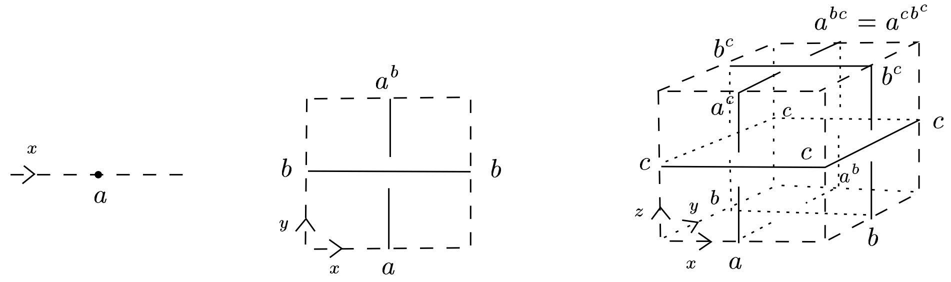

Figure 1

The figures make the boundary maps clear. Thus the back face of the 3–cube drawn (a, b, c) is glued to the 2–cube (a^b, c) and the top face is glued to (a^c, b^c).

There are similar descriptions of boundary maps from n–cubes to $(n-1)$–cubes and the general formula is:

$$\begin{aligned}
&\partial_i^0(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\
&\partial_i^1(x_1, \ldots, x_n) = (x_1^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \cdots, x_n) \quad \text{for} \quad 1 \leq i \leq n.
\end{aligned}$$

The rack space determines a topological space (also denoted BR) by gluing real cubes together using these boundary maps.

Now suppose that we are given a knot diagram in $\mathbb{R}^3$. A *labelling* of the diagram in a rack R means a labelling of sheets by elements of R so that at double lines the rule indicated in figure 2 holds (the figure shows a section transverse to the double line and the usual diagram convention has been used, namely the broken arc means the lower arc in the vertical sense; the arrow indicates the second framing vector).

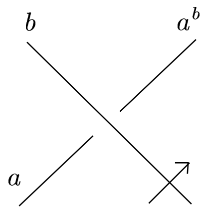

Figure 2

A labelling by R is precisely the same as a homomorphism of the fundamental rack of K to R, see [2; page 384].

Now the framing of K comprises one vector parallel to the vertical and second vector normal to the image in $\mathbb{R}^3$. Using this second vector we can provide the diagram with a bicollar with collar lines identified with I (so that the submanifold corresponds to $\frac{1}{2} \in I$ and the vector points in the positive direction along I). We can assume that the collar lines determine little cubes at triple points and normal square bundles along double lines. The squares and cubes are canonically identified with I^2 and I^3 using the vertical ordering of sheets at double lines and triple points. Now suppose the diagram is labelled in R then using the labelling these copies of I, I^2 and I^3 can be identified with cubes in BR. The diagram now determines a map of $\mathbb{R}^3$ to BR by mapping the collar lines to the 1–cubes of BR with which they are identified, the little squares along double lines to the appropriate 2–cubes and the cubes at triple points to the appropriate 3–cubes. Outside the collar all is mapped to the basepoint (the unique 0–cube) and hence we get a based map $S^3 \to BR$ by mapping infinity (the basepoint of S^3) to the basepoint of BR.

This map is almost canonical, depending only on the choice of collar, and we call it the *canonical map* determined by the diagram and the corresponding class in $\pi_3(BR)$ the *canonical class*. Given an isotopy of diagrams (or even a cobordism) labelled in R, then the canonical map varies by a homotopy by applying a similar construction to the cobordism. Thus using lemma 2.1 we have:

Lemma 2.2 *The canonical class in $\pi_3(BR)$ of a labelled 2–knot diagram is determined by*

(a) *the isotopy class of the oriented knot*

(b) *the labelling, or equivalently, the homomorphism of the fundamental rack to R.*

This is as much of the general theory from [4] that we shall need. However it is worth remarking that there is a converse construction. Given a map $S^3 \to BR$ we can use transversality to make it transverse to the 3–skeleton of BR and this means that it pulls back a diagram in $\mathbb{R}^3$ (for some surface in $\mathbb{R}^4$) labelled in R. Similarly a homotopy pulls back a cobordism of diagrams, ie, a diagram for a cobordism emdedded in $\mathbb{R}^3 \times I$.

These considerations are summarised in the following classification theorem:

Theorem 2.3 *There is a bijection between $\pi_3(BR)$ and the set of cobordism classes of diagrams in $\mathbb{R}^3$, labelled in R, of surfaces embedded in $\mathbb{R}^4$.*

See [4] for the detailed proof of this result, for the interpretation in terms of James bundles and for more general results along the same lines.

3 The 2–twist spun trefoil

Main theorem *The 2–twist spun trefoil is not (oriented) isotopic to its orientation reverse.*

Proof The proof occupies the remainder of this section. Let K be the 2–twist spun trefoil. We shall exhibit an explicit diagram for K which is labelled by the "three-colour rack" $T := \{0, 1, 2\}$ with $a^b := 2b - a \bmod 3$, ie, $a^b = c$ iff a, b, c are all the same or all different.

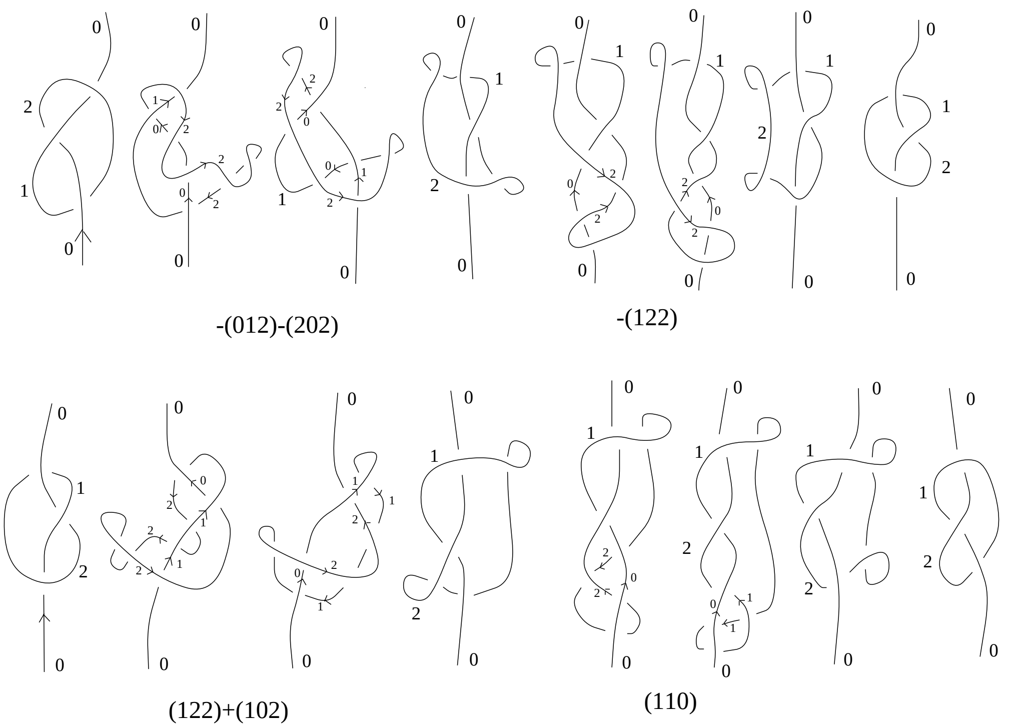

Figure 3

In figure 3 we give a series of slices of the 2–twist spun trefoil. The pictures are to be understood as follows. We are thinking of $\mathbb{R}^4$ as an "open-book decomposition" $\mathbb{R}^3_+ \times S^1$ with $x \times S^1$ identified with x for each $x \in \mathbb{R}^2 = \partial\mathbb{R}^3_+$, ie, $\mathbb{R}^2$ is the "spine" of the decomposition. Similarly we are thinking of $\mathbb{R}^3$ as an open-book decomposition with spine $\mathbb{R}^1$, ie, $\mathbb{R}^3$ is $\mathbb{R}^2_+ \times S^1$ with a similar identification for each $x \in \mathbb{R}^1 = \partial\mathbb{R}^2_+$. The projection $\mathbb{R}^4 \to \mathbb{R}^3$ is then given by the standard projection $\mathbb{R}^3_+ \to \mathbb{R}^2_+$ crossed with the identity on S^1. The figures are drawn as a sequence of diagrams in copies of $\mathbb{R}^2_+$ corresponding to embeddings in the corresponding copies of $\mathbb{R}^3_+$ and describe a diagram in $\mathbb{R}^3$ for an embedding in $\mathbb{R}^4$. The basepoint in $\mathbb{R}^4$ is chosen to lie on the spine $\mathbb{R}^2$ of the decomposition above the spine $\mathbb{R}^1$ on the decomposition of $\mathbb{R}^3$. Thus the basepoint lies above each of the diagrams drawn. The framing (not shown) is given by the left-hand rule in other words the framing vector is obtained from the orientation vector on arcs by turning to the left. Figure 3 shows one twist of the spun trefoil. To get the 2–twist spun trefoil, we repeat the sequence. The figure also shows the labelling in T. Observe that the finishing labels coincide with the starting ones with 1 and 2 interchanged, so to get the labels for the second twist, repeat the labels with 1 and 2 interchanged.

The moves from one diagram to the next should all be obvious except perhaps for the right-most pair in both rows. Here a ripple spins round in the surface to eliminate the two oppposite twists (creating no triple points in the projection).

Now let $\psi \in \pi_3(BT)$ be the canonical class of this labelled diagram for K. We shall compute $h(\psi)$ the Hurewicz image in $H_3(BT)$. Inspecting the definition of the canonical class given in section 2 above, we see that $h(\psi)$ is represented by a cycle C given as a sum of 3–cubes of BT one for each triple point of the diagram. In figure 3 we have written the 3–cubes (with sign) determined by the triple points (which appear as R3 moves in the sequence). Figure 4 gives more detail of the identification of these cubes. Figure 4 shows the cubes $-(012), -(122), (121)$ and (110) the other two ($-(202)$ and (102)) are the same as $-(012)$ and (121) respectively, with label changes.

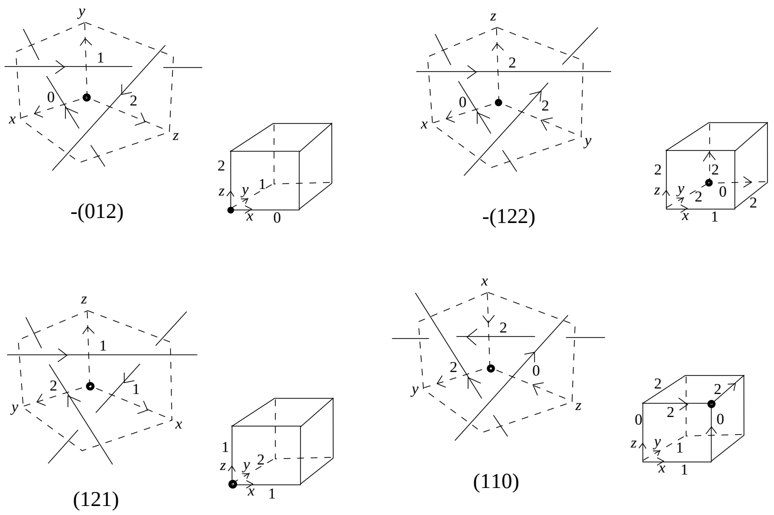

Figure 4

We can now read off the cycle which represents $h(\psi)$.

$$\begin{aligned} C = &-(012)-(202)-(122)+(121)+(102)+(110)\\ &-(021)-(101)-(211)+(212)+(201)+(220) \end{aligned}$$

We need two calculations for which details are to be found in the Maple worksheet [7].

Calculation 1 $H_3(BT) \cong \mathbb{Z} \times \mathbb{Z}_3$

Calculation 2 C represents a generator of the $\mathbb{Z}_3$–summand of $H_3(BT)$.

Lemma 3.1 *The $\mathbb{Z}_3$–summand of $H_3(BT)$ is fixed by all permutations of $T = \{0,1,2\}$.*

Proof Let Q denote the $\mathbb{Z}_3$–summand of $H_3(BT)$ and let S_3 denote the group of permutations of $\{0,1,2\}$. C represents a generator of Q and by construction is invariant under the interchange $(1,2) \in S_3$. Thus Q is fixed by $(1,2)$. Now notice that there are no symmetries of $H_3(BT)$ of order 3. It follows that the 3–cycle $(0,1,2) \in S_3$ also fixes Q. Since S_3 is generated by $(1,2)$ and $(0,1,2)$, Q is fixed by the whole of S_3. □

Corollary 3.2 *$h(\psi)$ is independent of the choice of (non-constant) labelling in T.*

Proof The labelling is clearly determined by the initial labelling of the first diagram and it follows that two different (non-constant) labels are related by a permutation of T. □

Now let K' denote K with opposite orientation. Notice that since we are labelling in an *involutory* rack T (ie one in which $a^{b^2} = a$ for all a, b) the original labels give a labelling after the orientation change. Let ψ' be the canonical class of K' using this labelling. To calculate $h(\psi')$ we observe that changing orientation reverses all arrows and from figure 5 we see that this corresponds to replacing each cube (a,b,c) by the cube (a^{bc}, b^c, c) with opposite orientation.

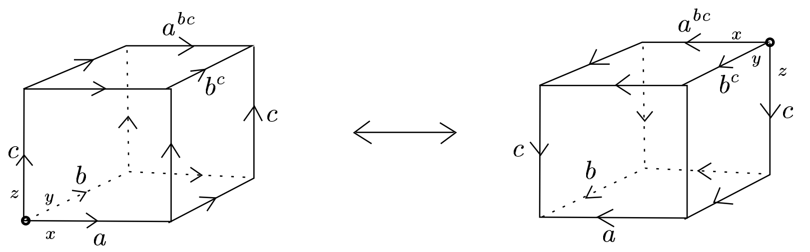

Figure 5

Performing this substitution for the cycle C given above we read off the cycle C' representing $h(\psi')$:

$$\begin{aligned} C' = &(202) + (012) + (122) - (201) - (212) - (220) \\ &(101) + (021) + (211) - (102) - (121) - (110) \end{aligned}$$

Serendipitously we notice:

Observation $C + C' = 0$

It follows that $h(\psi') = -h(\psi)$. Now suppose that K is isotopic to K' preserving orientation. Then since the fundamental racks of K, K' and the isotopy (thought of an an embedding of $S^2 \times I$ in $S^4 \times I$) are all isomorphic, the labelling of K in T induces a labelling through the isotopy which is non-trivial on K'. By corollary 3.2 we can use these labels to calculate $h(\psi)$ and $h(\psi')$. But by lemma 2.2 $\psi' = \psi$ and we have a contradiction, completing the proof of the main theorem. □

See the remarks in section 5 for explanations of some (but not all) of the concidences which occured in the above proof.

4 Trefoils left and right

We shall need a generalisation of the rack space. Let R be a rack. The *extended rack space* $B_R R$ has set of n–cubes in bijection with R^{n+1} (ie the $n+1$–cubes of BR) and the same formulae for face maps as in BR with an index shift (ie, ∂_i^ε in $B_R R$ is $\partial_{i+1}^\varepsilon$ in BR) cf [3, 1.3.2 and 3.1.2]. It follows that $H_n(B_R R) \cong H_{n+1}(BR)$ (cf [4, 5.14 and above]).

The interpretation for diagrams is labelling of both arcs and regions. More precisely, suppose that D is a diagram in $\mathbb{R}^2$ for a knot K in $\mathbb{R}^3$. An *extended labelling* of D by a rack R is a labelling of arcs and regions of D by elements of R with the rules illustrated in figure 6 for labels of adjacent regions and at crossings:

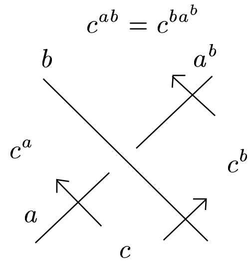

Figure 6

Given an extended labelled diagram, there is a map $\mathbb{R}^2 \to B_R R$ constructed similarly to section 2: Choose bicollars, map little squares at crossings to the appropriate 2–cube of $B_R R$ (for example a square at the crossing in figure 6 would be mapped to the 2–cube (c, a, b)) collar lines across arcs to the appropriate 1–cube (eg, a collar line across the lower left arc in figure 6 would be mapped to the 1–cube (c, a)) and regions to the 0–cube given by the label. Thus the labelling determines a canonical class $\phi \in \pi_2(B_R R)$ (based at the vertex corresponding to the label of the infinite region). An isotopy of K in $\mathbb{R}^3$ corresponds to a diagram in $\mathbb{R}^2 \times I$ to which the labelling can be canonically extended. This in turn gives a map $\mathbb{R}^2 \times I \to B_R R$, ie a homotopy of the canonical class. Thus as in section 2, the canonical class is an invariant of the isotopy class of the labelled diagram.

We now exhibit extended labellings in the three colour rack T for diagrams of the right-hand trefoil K and the left-hand trefoil K', see figure 7.

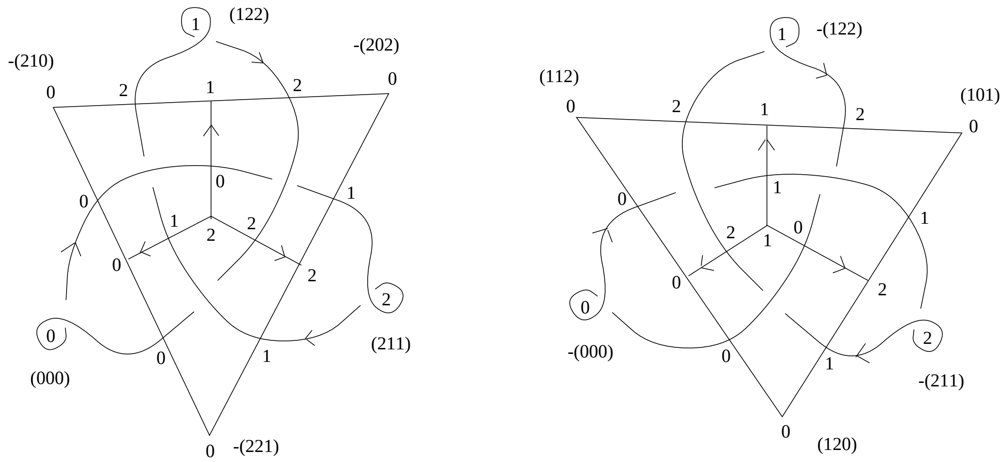

Figure 7

Let ϕ and ϕ' be the canonical classes determined by these diagrams and let $h(\phi)$ and $h(\phi')$ be their Hurewicz images in $H_2(B_T T) \cong H_3(BT) \cong \mathbb{Z} \times \mathbb{Z}_3$. We can read off representating cycles B and B' from the diagrams:

$$B = -(210) - (202) - (221) + (211) + (122) + (000)$$
$$B' = (221) + (202) + (210) - (122) - (000) - (211)$$

Observation $B = -B'$ and hence $h(\phi) = -h(\phi')$.

We need one final Maple calculation:

Calculation 3 B represents a generator of the $\mathbb{Z}_3$–summand of $H_2(B_T T)$.

Lemma 4.1 *The classes $h(\phi)$ and $h(\phi')$ are independent of the labels in T of arcs and regions in the diagrams provided that the knots themselves have non-constant labels.*

Proof We notice that both diagrams have rotational symmetry of order 3. But any two labellings which are non-constant on the knots are related by this symmetry followed by a permtuation of T. The result follows from lemma 3.1. □

We can now prove that right and left-hand trefoils are different.

Theorem 4.2 *There is no isotopy of $\mathbb{R}^3$ which carries the (unoriented) right-hand trefoil to the left-hand trefoil.*

Proof It is easy to contruct an isotopy carrying an oriented trefoil to its orientation reverse. Thus it suffices to prove that the oriented left and right trefoils are not isotopic. The diagrams drawn above both have zero writhe and if there is an isotopy between the oriented left and right trefoils then there would be a framed isotopy between the diagrams. Now the labelling on the right trefoil determines a labelling of the isotopy (thought of as a diagram of $S^1 \times I$ in $\mathbb{R}^3 \times I$) and hence a labelling of the left trefoil which is non-constant on the knot. By lemma 4.1 we can use these labels to calculate $h(\phi)$ and $h(\phi')$ and it follows that $h(\phi) = h(\phi')$ contradicting the observation made above. □

5 Some remarks

Fundamental quandles

The *involutive* fundamental quandle (ie the fundamental quandle with the operator relations $a^2 \equiv 1$ for each a) of the trefoil is the 3–colour rack/quandle T. This explains why there is a natural labelling in this rack. Furthermore using a Van Kampen argument it can be seen that the fundamental quandle of an n–twist-spun knot K is the fundamental quandle of K with the operator relations $a^n \equiv 1$ for each a. It follows that the fundamental quandle of the 2–twist-spun trefoil is T. So the labelling of the 2–twist-spun trefoil we used is the only non-trivial labelling (in a quandle).

The canonical class

The fundamental rack Γ of a classical link (ie a link of circles in S^3) together with its canonical class in $\pi_2(B\Gamma)$, is a complete invariant up to isotopy. Recall that the fundamental rack alone is a complete homeomorphism invariant but just for non-split links [2; theorem 5.2]. Thus the canonical class gives the extra information that allows the rack to classify split links and up to isotopy rather than just homeomorphism. This new result will be proved in detail in [5]; it follows from the methods of the proof of [4; theorem 5.8]. Thus the canonical class distinguishes the left and right trefoils which have the isomorphic fundamental racks. The canonical class lies in $\pi_2(B\Gamma)$ which is H_2 of the universal cover of $B\Gamma$ and therefore determines classes in every connected cover of $B\Gamma$. To distinguish the left and right trefoils we used $B_T T$ which is a 3–fold cover BT and the canonical class we used was the image of the class in the 3–fold cover of $B\Gamma$ determined in this way. This is typical of the way in which information can be extracted from the main classificatoin theorem in [5].

The $\mathbb{Z}$ factor

Notice that $H_3(BT) \cong H_2(B_T T)$ has a $\mathbb{Z}$ factor which played no role in the proofs. In fact every homology group of the rack space BR where R has a base element which acts trivially on itself (any element in a quandle will do) splits a $\mathbb{Z}$ factor which comes from the inclusion and natural projection $B* \subset BR \to B*$ where $B*$ denotes the rack space of the one element rack. Now $B*$ is a model for $\Omega(S^2)$ and has one cell in each dimension with all boundary maps trivial and hence has homology $\mathbb{Z}$ in each dimension [4; 3.3].

Now for a 2–knot the canonical class maps to $\pi_3(\Omega S^2) = \pi_4(S^3) = \mathbb{Z}_2$ which in turn maps to zero in $H_3(\Omega S^2) = \mathbb{Z}$. This explains why the $\mathbb{Z}$ factor was immaterial for the 2–twist spun trefoil. For a 1–knot the image in $H_2(\Omega S^2) = \mathbb{Z}$ is readily seen to be the writhe of the diagram. But we chose our diagrams for the left and right trefoils to have zero writhe, which explains why the $\mathbb{Z}$ factor was immaterial for this case as well.

Invariance under permutations of T

Rack spaces are *simple* in other words π_1 acts trivially on π_n for each n [4; proposition 5.2]. The proof is by diagram manipulation. Introduce a small sphere labelled by $x \in R$. Pull the sphere over the diagram (realising the change of labels corresponding to the action of x) and then pull the sphere back under the diagram and eliminate it. The same proof shows that any labelled diagram is cobordant to the diagram with labels acted on by any element of the labelling rack. Thus the operator group of the rack acts trivially on the canonical class. Now the three colour rack T has operator group equal to its automorphism group, which is the symmetric group S_3. This explains why the canonical class determined by a labelling in T is invariant under choice of labels: any rack with operator group equal to its automorphism group would have the same property. Thus the labelling used for the 2–twist-spun trefoil was immaterial. For extended labels the same proof shows invariance under change of labelling with fixed label at infinity. This together with the obvious symmetry of the choice of label at infinity in T explains why the labelling was also immaterial for the left and right trefoils.

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