The Singular Braid Monoid Embeds in a Group

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Abstract We prove that the singular braid monoid of [Ba] and [Bi2] embeds in a group. This group has a geometric interpretation as singular braids with two types of singularities which cancel.

In this paper we prove that the singular braid monoid of Baez and Birman [Ba, Bi2] embeds in a group, which we call the singular braid group. We will study this group in greater depth in a later paper. In that paper we will show that the group is torsion free and we will describe the centralisers of its natural generators. In a yet further paper, the methods of this paper will be applied to a considerably more general monoid of singular braids. We shall show that this more general monoid also embeds in a group and we shall prove analogues of the Alexander and Markov theorems for these general singular braids.

The embedding of the singular braid monoid in the singular braid group mirrors to some extent the situation of classical braids where the monoid of positive braids embeds in the full braid group (see [Gr]). However the methods which are used to prove the embedding of positive braids do not apply to the situation considered here. Neither do the various embeding theorems of monoid in groups [Ad, Ka, Re] help. As far as we are aware there does not exist a previously known algebraic method which is applicable here. Our proof relies heavily on the geometry of singular braids in space.

Recent results of Bar-Natan and Lin [Bar, L] demonstrate the importance of the singular braid monoid to braids, see also Zhu [Z]. The Birman conjecture [Bi2, FRZ] also concerns an embedding of the singular braid monoid but in the group ring of the braid group. There does not appear to be an immediate connection between the results in this paper and the conjecture.

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1. The singular braid monoid

We recall the definition of the singular braid monoid SB_n where $n \geq 1$. There are three types of generators $\sigma_i, \overline{\sigma}_i, \tau_i$ where $1 \leq i < n$ and the following relations :

(1) *commuting relations*

$$g_i h_j = h_j g_i, \quad \text{for} \quad |i-j| > 1$$

where g_i, h_i are any two of $\sigma_i, \overline{\sigma}_i, \tau_i$,

(2) *three point relations*

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \text{and} \quad \tau_i\sigma_j\sigma_i = \sigma_j\sigma_i\tau_j, \quad \text{for} \quad |i-j| = 1,$$

(3) *two point relations*

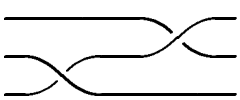
$$\overline{\sigma}_i\sigma_i = \sigma_i\overline{\sigma}_i = 1,$$

(4) *twist relations*

$$\sigma_i\tau_i = \tau_i\sigma_i.$$

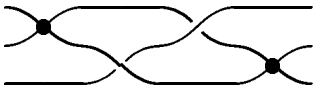
The monoid has a simple geometric interpretation. Notice that the generators and relations include the standard presentation for the classical braid group on n strings (where we have written $\overline{\sigma}_i$ instead of the more usual σ_i^{-1}). The geometric interpretation of this braid group is familiar. A geometric braid is a set of n disjoint strings in $D^2 \times I$, where a string is a smooth embedding of I. The composition $I \to D^2 \times I \to I$ is required to be the identity (where the second map is projection) and the set of strings are required to start and finish at n standard points in the interior of D^2. We shall choose D^2 to be a vertical disc (the unit disc in the (y,z)–plane), I to be horizontal (parallel to the x–axis) and the standard points to lie on the y–axis. We shall picture braids by projecting them onto the (x,y)–plane. We call a parallel copy $D^2 \times \{t\}$ of D^2 a **level**. Thus a geometric braid meets each level in n distinct points.

Two geometric braids are equivalent if they differ by an isotopy through geometric braids. The equivalence classes are in natural bijection with the elements of the braid group, where σ_i is interpreted as the geometric braid (the **positive i–twist braid**) which crosses the i-th string under the $(i+1)$-th, and $\overline{\sigma}_i$ is interpreted as the **negative i–twist braid**) which crosses the i-th string over the $(i+1)$-th. The interpretation is chosen so that on passing from left to right σ_i executes a right-handed twist and $\overline{\sigma}_i$ a left-handed twist. Composition is given by concatenation. In the following picture we fix conventions in the readers mind by illustrating the braid $\sigma_1\overline{\sigma}_2$:



Notice that we can identify a monomial in the generators $\sigma_i, \overline{\sigma}_i$ with the geometric braid obtained by concatenating the appropriate twist braids. Conversely, by general position, any geometric braid can be identified with such a monomial. Isotopy of geometric braids then corresponds precisely to the use of relations. For more details see [Bi1].

For the singular braid monoid, this geometric interpretation is extended to allow the strings to be non-disjoint. To be precise, a geometric singular braid is the same as a geometric braid except that simple intersections of one string with another are allowed called **singular crossings**. At the singular crossings the strings are required to have different tangents. The equivalence is again isotopy through geometric singular braids. It follows that the number of singular crossings is invariant under equivalence. The identification of SB_n with geometric singular braids is obtained by extending the identification of the braid group with geometric braids. The new generator τ_i corresponds to the singular braid with a single singular crossing of the i-string through the $(i+1)$-th. We picture this single crossing as a small solid blob. As for non-singular braids, we can effectively identify a geometric singular braid with a monomial in the generating set.†



Singular braid corresponding to the word $\tau_2\sigma_1\overline{\sigma}_2\tau_1$

The various relations can be interpreted as follows. The commuting relations allow parts of the braid far away from each other to pass each other.

Commuting generators

The three point relations allow a string to pass over or under a crossing (either singular or not).

Passing an arc under a double point, corresponding to the relation $\tau_2\sigma_1\sigma_2 = \sigma_1\sigma_2\tau_1$

The two point relation allows two opposite ordinary crossings to cancel. The twist relation allows a singular crossing to be turned over.

Turning a double point over, corresponding to the relation $\sigma_i\tau_i = \tau_i\sigma_i$

For more detail here see [Bi2] and [Ba].

† *The same pictures arise as representatives of the braid permutation group, [FRR]. However the relations are different.*

2. The singular braid group

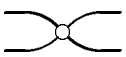
The **singular braid group** SG_n has the same generators as SB_n with one additional type of generator $\overline{\tau}_i$ where $1 \leq i < n$. There are two sets of relations :

(1) *monoid relations* the same relations as SB_n above with the additional relations obtained by substituting $\overline{\tau}_i$ for τ_i in each relevant relation,

(2) *additional two point relations*

$$\overline{\tau}_i\tau_i = \tau_i\overline{\tau}_i = 1.$$

The additional two point relation makes SG_n into a group ($\overline{\tau}_i$ being τ_i^{-1}). There is a geometric interpretation for this group obtained from the interpretation of the singular braid monoid given above. If we disallow the additional two point relation than the relations define esentially the same object as the singular braid monoid except that there are now two types of singular crossing corresponding to τ_i and $\overline{\tau}_i$. We picture the new type of crossing as an open blob :



The new crossing $\overline{\tau}_i$

We call these singular braids with two types of crossing, **extended singular braids**.

When the additional two point relation is taken into account, the equivalence of extended singular braids is enlarged to allow for the birth or death of pairs of singular crossings of different types :

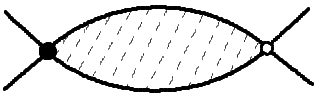
Birth and death of double points corresponding to the relation $\tau_i\overline{\tau_i} = \overline{\tau_i}\tau_i = 1$

We shall use the following notation. Let α, β be representatives of SG_n (ie α, β can be regarded either as monomials in the generating set or as extended geometric singular braids). If α, β represent the same extended singular braid (ie if they are equivalent under the relations of type (1)) then we write $\alpha \sim_M \beta$ and say that α and β are **monoid equivalent**. If α, β represent the same element of the singular braid group (ie if they are equivalent under the relations of types (1) and (2)) then we write $\alpha \sim_G \beta$ and say that thay are **group equivalent**. If α is a monomial and is obtained from another β by monoid equivalence and the birth of one pair of opposite singular crossings then we write $\beta \nearrow \alpha$. In this situation, β is obtained from α by monoid equivalence and the death of a pair of opposite singular crossings and we write $\alpha \searrow \beta$.

Notice that $\alpha \sim_G \beta$ if and only if there is a sequence $\alpha = \alpha_0, \alpha_1 \ldots, \alpha_t = \beta$ of monomials (or geometric braids) such that either $\alpha_{i-1} \searrow \alpha_i$ or $\alpha_{i-1} \nearrow \alpha_i$ for $1 \leq i \leq t$.

We need to examine geometrically the situation when $\alpha \searrow \beta$. The passage back from β to α can be described as an isotopy through extended singular braids followed by

the birth of a pair of opposite singular crossings followed by a further isotopy through extended singular braids. Immediately after the birth there is a small disc D in $D^2 \times I$ which meets each level between the two singular crossings in an arc, and which has for boundary the portions of the strings lying between the two singular crossings.

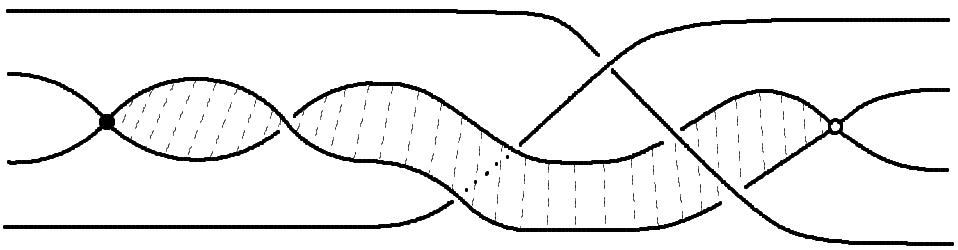


We now define a disc D, which is not assumed to be small, but which otherwise has the same properties as D. A **life disc** for a geometric braid α is a disc D embedded in $D^2 \times I$ such that :

(1) The interior of D is disjoint from α.

(2) The boundary of D lies on α and contains precisely two singular crossings p, p' of opposite types.

(3) D is contained in the levels between the levels of p and p' (inclusive). It meets the levels containing p and p' precisely in p, p' respectively and it meets each level strictly between p and p' in an arc.



The final isotopy of α may distort D somewhat, but D remains a life disc throughout the isotopy. Moreover D can be used as a guide to recover β up to isotopy. We isotope the string on one side of D across to the other side and then slide one of the singular crossings at the end of D up to kill the one at the other end. We summarise this discussion as follows :

2.1 Lemma *Suppose that α, β are geometric extended singular braids and that $\alpha \searrow \beta$. Then α contains a life disc D such that β can be recovered up to isotopy by using D as a guide for isotopy and death.*

3. The main theorem

3.1 Main theorem *The natural map $SB_n \to SG_n$ is an embedding.*

In the next section we shall prove that following result :

3.2 Proposition Diamond property *Suppose that α, β, γ are geometric extended singular braids and that $\alpha \nearrow\!\!\!\!\nearrow \beta \searrow\!\!\!\!\searrow \gamma$. Then either $\alpha \sim_M \gamma$ or there is another geometric extended singular braid β' such that $\alpha \searrow\!\!\!\!\searrow \beta' \nearrow\!\!\!\!\nearrow \gamma$.*†

We shall deduce the main theorem from the diamond property. Let α be an extended singular braid. We say that α is **irreducible** if there is no β such that $\alpha \searrow\!\!\!\!\searrow \beta$. Clearly any element of SG_n has an irreducible representative.

3.3 Corollary *Each element of SG_n has a unique irreducible representative. More precisely, suppose that α, β are irreducible and that $\alpha \sim_G \beta$, then $\alpha \sim_M \beta$.*

Proof For convenience we shall write $u \searrow v$ if $u = u_0 \searrow\!\!\!\!\searrow u_1 \searrow\!\!\!\!\searrow \ldots \searrow\!\!\!\!\searrow u_t = v$ and similarly $u \nearrow v$ if $u = u_0 \nearrow\!\!\!\!\nearrow u_1 \nearrow\!\!\!\!\nearrow \ldots \nearrow\!\!\!\!\nearrow u_t = v$. The hypothesis that $\alpha \sim_G \beta$ implies that there is a sequence $\alpha = u_o \nearrow u_1 \searrow u_2 \nearrow \ldots \searrow u_k = \beta$. (Note that the first step is an expansion and the last a collapse since α, β are irreducible.) Consider $u_o \nearrow u_1 \searrow u_2$ where $u_o \nearrow u_1$ is given by $u_o = x_o \nearrow\!\!\!\!\nearrow x_1 \nearrow\!\!\!\!\nearrow \ldots \nearrow\!\!\!\!\nearrow x_{k_o} = u_1$ and $u_1 \searrow u_2$ is given by $u_1 = y_o \searrow\!\!\!\!\searrow y_1 \searrow\!\!\!\!\searrow \ldots \searrow\!\!\!\!\searrow y_{k_1}$. So we have an intermediate stage $x_{k_o-1} \nearrow\!\!\!\!\nearrow u_1 \searrow\!\!\!\!\searrow y_1$. By the diamond property we can either replace x_{k_o-1} by y_1 or we can replace $x_{k_o-1} \nearrow\!\!\!\!\nearrow u_1 \searrow\!\!\!\!\searrow y_1$ by $x_{k_o-1} \searrow\!\!\!\!\searrow u_1' \nearrow\!\!\!\!\nearrow y_1$. In both cases the length of the first expansion reduces by one. Applying this procedure finitely many times we either obtain a new sequence which starts $\alpha = v_o \nearrow\!\!\!\!\nearrow v_1 \searrow\!\!\!\!\searrow v_2$ or we obtain a sequence $\alpha \nearrow \beta$. But the latter is impossible by irreducibilty of β. Applying the diamond property again to $\alpha = v_o \nearrow\!\!\!\!\nearrow v_1 \searrow\!\!\!\!\searrow v_2$ either $\alpha \sim_M v_2$, and we can reduce the total length of the equivalence by two, or $\alpha \searrow\!\!\!\!\searrow \gamma \nearrow\!\!\!\!\nearrow \beta$ which contradicts irreducibility again. After a finite number of such steps the total length of the equivalence has been reduced to two or one and is either $\alpha \nearrow\!\!\!\!\nearrow \gamma \searrow\!\!\!\!\searrow \beta$ or $\alpha \nearrow\!\!\!\!\nearrow \beta$. The latter contradicts irreducibility of β and the former implies $\alpha \sim_M \beta$, as required. □

Since elements in the image of SB_n in SG_n are irreducible, the corollary implies the main theorem.

† *This is similar in appearence to the Church-Rosser theorem of the λ–calculus, see [Rv].*

4. Proof of the diamond property

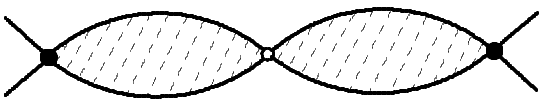
We are given geometric extended singular braids α, β, γ such that $\alpha \nearrow \beta \searrow \gamma$. By lemma 2.1 there are two life dics D and D' for β such that α (resp γ) can be recovered by using D (resp D') as a guide. Let the singular crossings on the boundary of D be p and q in order and on the boundary of D' be p', q', again in order. There are three possibilities which we consider in increasing order of difficulty :

(1) Either $q = p'$ or $p = q'$.

(2) p and q are distinct from p' and q'.

(3) $p = p'$ and $q = q'$.

Case (1) Without loss we assume that $q = p'$. Then D and D' are confined between disjoint levels apart from the level containing q. Thus the discs meet only at q. We can now isotope β so that the two discs correspond to adjacent births.



It can now be seen that $\alpha \sim_M \gamma$.

Case (2) In this case the boundaries of D and D' are disjoint and we shall isotope D by a level preserving isotopy, which fixes β, to be completely disjoint from D'. Then both deaths can be performed simultaneously and this defines the required β' such that $\alpha \searrow \beta' \nearrow \gamma$.

We can visualise the situation by looking at the intersections of the level planes with the discs D and D'. Starting at p move the level plane to q. During its travel, the intersection with D will grow from p to an arc with end-points on ∂D called **a D–arc** and finally will shrink to q. Similarly we will have a family of D'–arcs. During their existence the two families of arcs may meet. If we can move to a situation where the arcs are always disjoint then the discs will be disjoint.

Since a D–arc will never cross the end-points of a D'–arc intersections will be formed by the birth and death of pairs of intersection points in the interior of D–arcs. Call a level containing a birth or death a **critical** level and other levels **regular** levels.

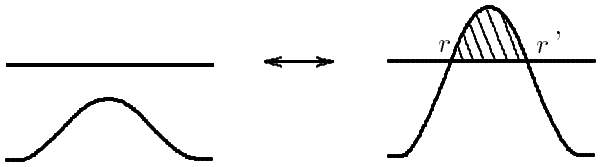
By general position we can assume that any critical level contains just one such birth or death. Define the complexity of the situation as follows. Choose one regular level beween each pair of adjacent critical levels and count the number of point of intersection of D and D' on that level. Add over all the chosen levels to get the total complexity.† We show how to reduce complexity. Complexity 0 corresponds to disjoint discs.

In order to visualise what is happening we will consider the changes taking place to the D and D'–arcs in the level sections. In the figures the horizontal arrows will represent

† *This definition is similar to the width of a knot diagram due to Gabai [Ga].*

the passage of a level section through a critical point. A vertical double arrow will represent an isotopy of the situation to something simpler. We will draw one arc, say the D'–arc, as straight and horizontal and imagine all the movement concentrated in the other, D–arc.

At the moment of birth of two intersection points, r and r', a disc is created (shown shaded below), which survives (at least) until the next critical value. Something similar happens for the death of two such points. We shall use discs such as these to push apart D and D' in order to simplify the situation.



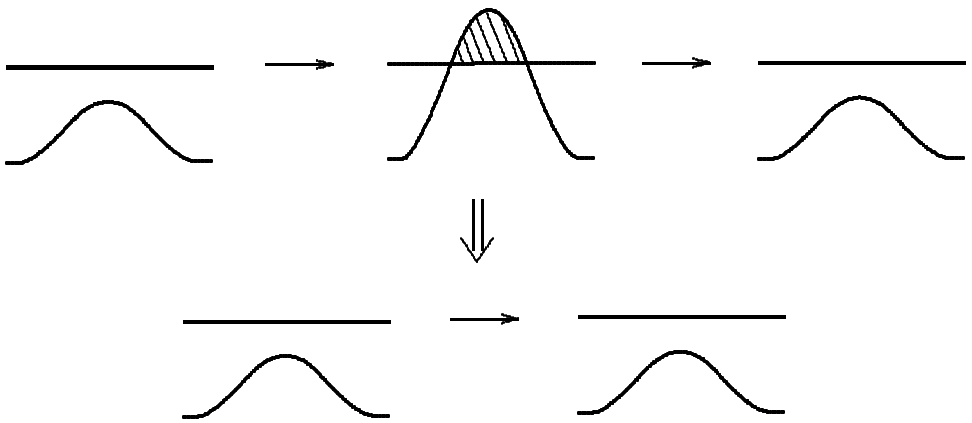
Since the first critical level is a birth and the last is a death, there must exist two adjacent critical levels such that the left level l is a birth and the right level L is a death.

It is curious that at this point the proof of the diamond property echoes its application.

There are now three possibilities according to whether the number of points involved locally in the birth and death are two, three or four in number.

(a) Two points

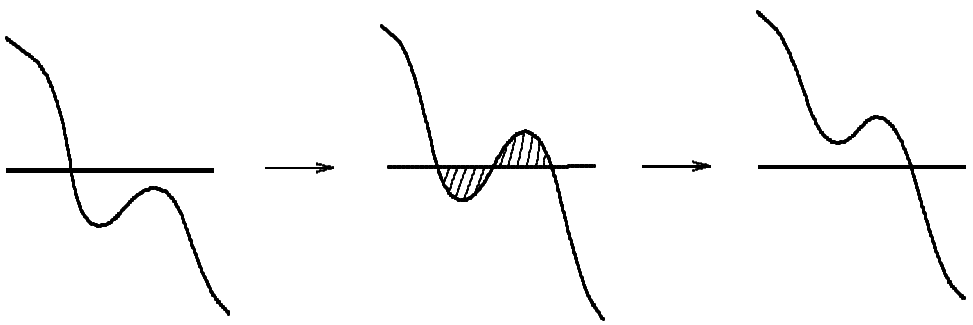
In this case there is a birth of a pair of intersections which move apart and then return together and die.



The union of the shaded discs is a 3–ball B. On the boundary of B is a simple closed curve γ which forms part of $D \cap D'$ and is the boundary on ∂B of a disc in D and a disc in D'. This ball can be used to define a level-preserving isotopy of D which has the effect of removing the curve of intersection (as indicated in the figure) and reducing complexity.

(b) Three points

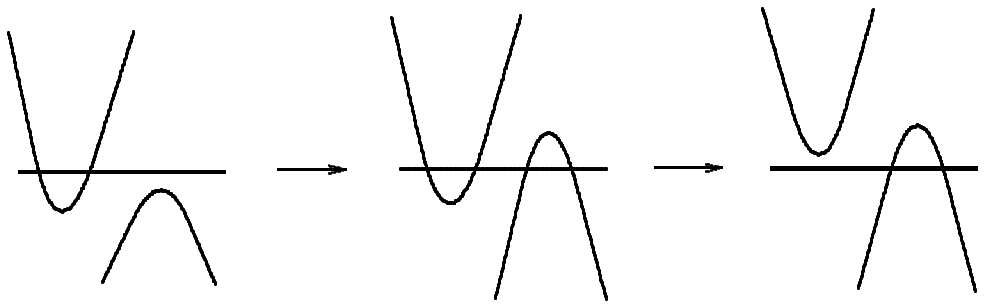
In this case the picture of what happens in the level planes is indicated below.



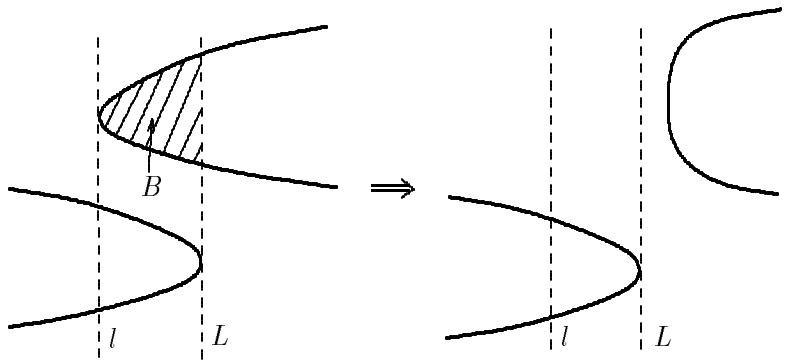
By an isotopy we can gradually reduce the maximum size of the hills and valleys (shown shaded) until there is no birth and death between l and L.

(c) Four points

In this case the births and deaths look like the following.



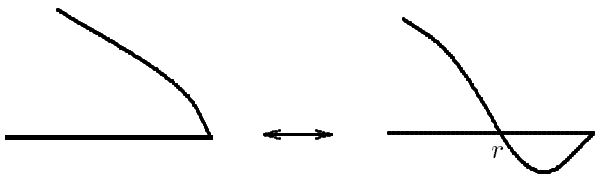
We now reduce complexity by reordering the births and deaths. This is best illustrated by looking at the projection in the (x, y)–plane to see what happens between levels l and L. The birth is pushed back beyond L over a ball B shown shaded in the diagram.



Case (3) We again look at the intersection of D and D' with the level planes. Here $p = p'$ and $q = q'$ and the two families of arcs always have the same end-points. We can assume that just after the moment of birth and just before the moment of death the interiors of the arcs are disjoint.

Intersections will appear as in case (2) but in addition there is the possibility of an intersection being born or dying at a branch point on the common boundary.

This is illustrated below where a single intersection point r is born (or dies) associated with a branch point.



Levels containing branch points will be treated as critical levels and we may assume that each critical level contains just one birth or death (interior or at a branch point).

We now define complexity exactly as in the last case and our aim is to reduce complexity to zero. When this is achieved we may assume that the discs are either disjoint except for their common boundary or, what is just as useful, that they coincide.

Looking again at an adjacent birth b to the left of a death d, we have the same possibilities as above, with the following new situations.

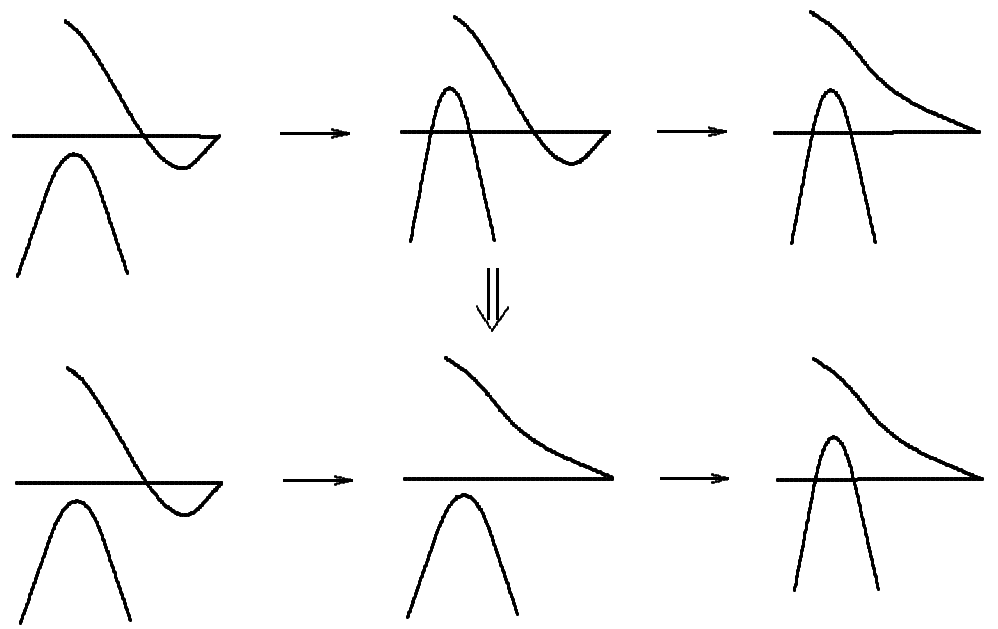
(i) The birth is at a branch point.

(ii) The death is at a branch point.

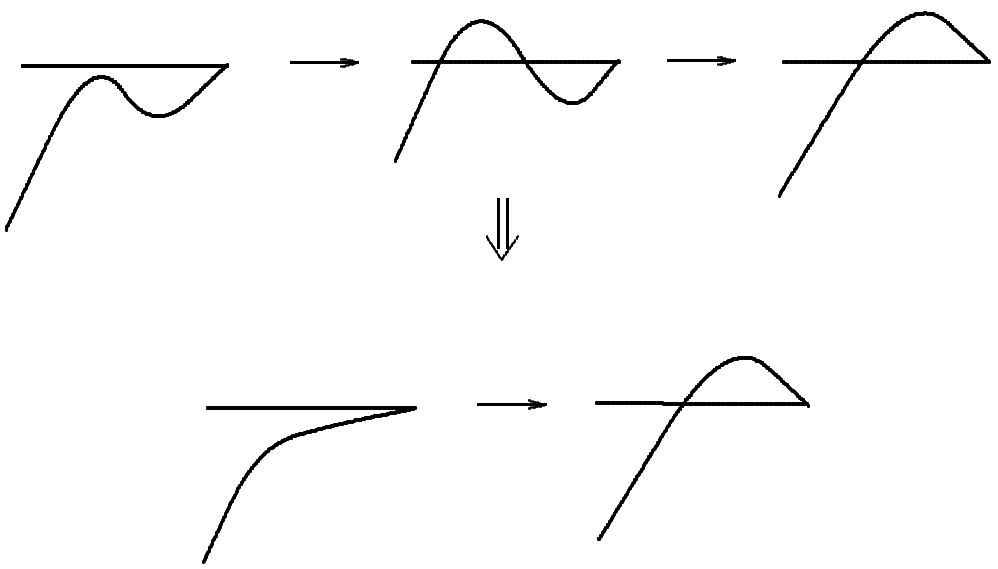
(iii) Both are at branch points.

Case (i) and (ii) are the same (with reversal of roles) so we just have to consider cases (i) and (iii).

In case (i) there are two possibilities, similar to cases (b) and (c) above with simplifications suggested by the following figure.

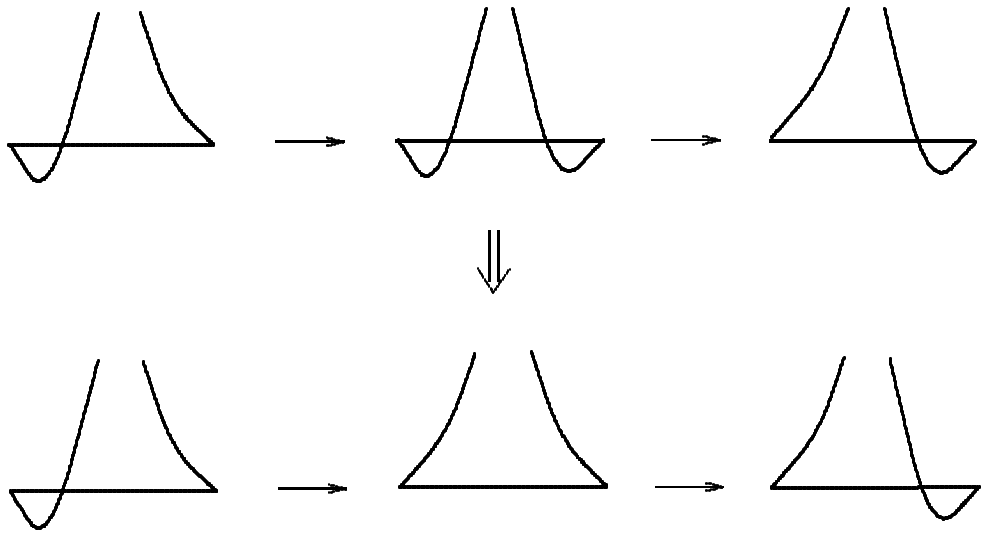


or



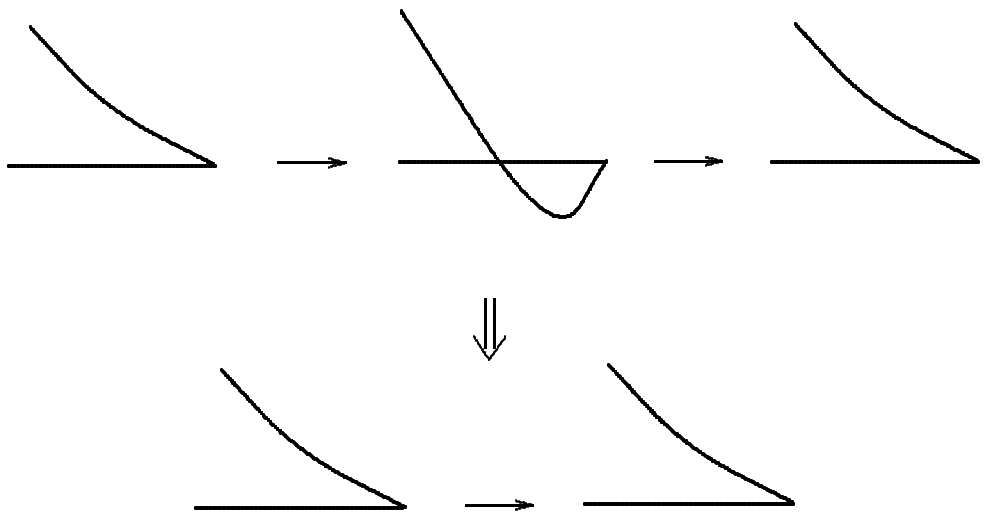
In case (iii) there are three further subcases :

(I) b and d lie on different intersection curves locally :



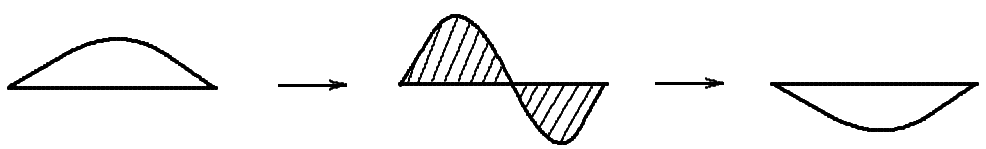
This case is similar to case (c) above and there is a simplification as indicated in the figure.

(II) b and d lie on the same curve locally and are on the same edge of D :



This case is similar to case (a) above and there is a simplification as indicated in the figure.

(III) b and d lie on the same curve locally and are on opposite edges of D :



In this case we do not reduce complexity. Instead we use the discs shaded in the figure to move the curve formed by the moving intersection point into a single level forming

an arc of intersection of D and D' at one level. The definition of complexity is now extended to take this new situation into account by ignoring this level arc. The level arcs so formed are left alone until all other modifications have been completed. At this point the discs D and D' meet only in their common boundary and a finite number of level arcs. Then D and D' can be pushed together avoiding the rest of the braid so that they coincide.

It follows that $\alpha \sim_M \beta$. □

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