The Princeton notes on the Hauptvermutung

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Preface

The homotopy Hauptvermutung is the conjecture that a (topological) homeomorphism between two PL (= piecewise linear) manifolds may be continuously deformed to a PL homeomorphism.

These notes contain a proof, due to Casson and Sullivan, of the homotopy Hauptvermutung for simply connected manifolds under the hypothesis of 'no 2-torsion in H^4'. They were written in 1968 at the Institute for Advanced Study, Princeton and reissued in the Warwick Lecture Note Series in 1972. Nearly 25 years later there is still no other complete account available, hence their appearance in a more permanent form in this volume.

The connection with the subsequent solution of the isotopy Hauptvermutung by Kirby and Siebenmann [2, 3] is outlined in a coda. The two theories combine to give a fibration

$$K(\mathbb{Z}_2,3) \simeq TOP/PL \longrightarrow G/PL \longrightarrow G/TOP \simeq \Omega^{4n}(G/PL)$$

and the following theorem.

Theorem. *Suppose that $h : Q \longrightarrow M$ is a (topological) homeomorphism between PL manifolds of dimension at least five, whose restriction to ∂M is PL. Then there is an obstruction $\theta \in H^3(M, \partial M; \mathbb{Z}_2)$ which vanishes if and only if h is isotopic to a PL homeomorphism keeping ∂M fixed. If in addition M is 1–connected then h is homotopic to a PL homeomorphism if and only if $\delta\theta \in H^4(M, \partial M; \mathbb{Z})$ is zero.*

When M is not 1–connected the solution to the homotopy Hauptvermutung is bound to be more complicated (see the final remark in the coda).

More detail on the relationship of the results proved here with later results is to be found in the paper of Ranicki at the start of this volume.

The Princeton notes consist of three papers written by Armstrong, Rourke and Cooke, presented as three chapters, and a coda. The first chapter, written by Armstrong, gives an account of the Lashof-Rothenberg proof for 4–connected manifolds, and includes the 'canonical' Novikov splitting theorem used in the main argument. The second, by Rourke, contains the geometry of the main proof, and deals with simply connected manifolds which satisfy $H^3(M; \mathbb{Z}_2) = 0$. The treatment follows closely work of Casson on the global formulation of Sullivan theory.

This approach to the Hauptvermutung was the kernel of Casson's fellowship dissertation [1] and a sketch of this approach was communicated to Rourke by Sullivan in the Autumn of 1967. The remainder of the chapter contains an outline of an extension of the proof to the weakest hypothesis (M simply connected and $H^4(M;\mathbb{Z})$ has no elements of order 2), and some side material on block bundles and relative Sullivan theory. The final chapter, written by Cooke, gives the details of the extension mentioned above. It contains part of Sullivan's analysis of the homotopy type of G/PL and its application in this context. (The other part of this analysis is the verification that G/PL and BO have the same homotopy type 'at odd primes', see [5]). Sullivan's original arguments (outlined in [4, 5]) were based on his 'Characteristic Variety Theorem', and the present proof represents a considerable simplification on that approach.

Sadly George Cooke is no longer with us. We recall his friendship and this collaboration with much pleasure, and dedicate these notes to his memory.

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CHAPTER I

The Hauptvermutung according to Lashof and Rothenberg

By M. A. Armstrong

§1. Introduction

The aim of Chapter I is to prove the following result.

(1.1) Theorem. *Let $h : Q \longrightarrow M$ be a topological homeomorphism between two closed PL manifolds of dimension at least five. If M is 4–connected, then h is homotopic to a PL homeomorphism.*

The approach is due to Lashof and Rothenberg [11]. Our treatment differs only in that we write with the specialist less in mind, and prefer to emphasise the geometry throughout rather than enter a semi-simplicial setting. The theorem can be refined, but we shall examine only the version given above. Stronger results, due to Casson and Sullivan, are presented in Chapters II and III written by Rourke and Cooke.

In this introduction we shall present a bird's eye view of the proof of (1.1), referring the reader to later sections for more detail. Suppose then that M and Q are PL manifolds, and that we are presented with a (topological) homeomorphism h from Q to M. We shall assume throughout that our manifolds are closed (compact without boundary) and 4–connected. The first step is to use h to construct a PL $\mathbb{R}^k$-bundle over Q, and a topological trivialization of this bundle. Second, by reference to Browder-Novikov theory, we show that if the given trivialization is properly homotopic to a PL trivialization, then h is homotopic to a PL homeomorphism. The problem of homotoping the topological trivialization to a PL trivialization will then occupy the remainder of the argument. Use of Browder-Novikov surgery necessitates the simple connectivity of our manifolds; the solution of the trivialization problem will require that the manifolds be 4–connected.

Identify M with $M \times \{0\} \subseteq M \times \mathbb{R}^n$ for some large integer n, and think of h as a (topological) embedding of Q in $M \times \mathbb{R}^n$. If we are in the stable range, in other words if n is at least $m+2$, a result of Gluck [6] provides an ambient isotopy $\{H_t\}$ of $M \times \mathbb{R}^n$ which moves h to a PL embedding

$$e \ = \ H_1 h \ : \ Q \longrightarrow M \times \mathbb{R}^n \ .$$

Further, in this range, work of Haefliger and Wall [7] shows that the new embedding has a PL normal disc bundle. Taking the pullback gives a PL n-disc bundle over

Q and an extension of e to a PL embedding $e : E \longrightarrow M \times \mathbb{R}^n$ of its total space onto a regular neighbourhood V of $e(Q)$ in $M \times \mathbb{R}^n$. Let $E \xrightarrow{\pi} Q$ denote the associated $\mathbb{R}^n$-bundle. Choose a closed n-dimensional disc $D \subseteq \mathbb{R}^n$ centred on the origin, and of sufficiently large radius so that V is contained in the interior of the 'tube' $M \times D$. Then $M \times D \backslash \mathrm{int}(V)$ is an h-cobordism between $M \times \partial D$ and ∂V and, by the h-cobordism theorem, this region is PL homeomorphic to the product $\partial V \times [0,1]$. Therefore we may assume that $e : E \longrightarrow M \times \mathbb{R}^n$ is onto.

So far we have produced the following diagram

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "\pi"'] & M \times \mathbb{R}^n \arrow[d, "p_1"] \\
Q \arrow[r, "h"] & M
\end{tikzcd}$$

which commutes up to homotopy, with $e : E \longrightarrow M \times \mathbb{R}^n$ a PL homeomorphism, and $\pi : E \longrightarrow Q$ a PL $\mathbb{R}^n$-bundle. We claim that e is stably isotopic to a topological bundle equivalence. Certainly the composition

$$H_1^{-1} e \ : \ E \longrightarrow M \times \mathbb{R}^n$$

provides a topological normal bundle for the embedding $h : Q \longrightarrow M \times \mathbb{R}^n$. On the other hand this embedding has a natural normal bundle given by

$$h \times 1 \ : \ Q \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^n \ .$$

These two are stably isotopic (see for example Hirsch [8] combined with Kister [9]). More precisely, if r is at least $(m+1)^2 - 1$ then the associated normal bundles of

$$h \ : \ Q \longrightarrow M \times \mathbb{R}^n \times \mathbb{R}^r$$

are isotopic. Therefore

$$e \times 1 : E \times \mathbb{R}^r \longrightarrow M \times \mathbb{R}^n \times \mathbb{R}^r$$

is isotopic to a topological bundle equivalence. Let g denote this equivalence, write

$$E' \ = \ E \times \mathbb{R}^r \ \ (k = n + r) \ ,$$

and consider

$$\begin{tikzcd}
E' \arrow[r, "e \times 1"] \arrow[r, bend left, "g"] \arrow[d, "\pi p_1"'] & M \times \mathbb{R}^k \arrow[d, "p_1"] & Q \times \mathbb{R}^k \arrow[l, "h \times 1"'] \arrow[d, "p_1"] \\
Q \arrow[r, "h"] & M & Q \arrow[l, "h"']
\end{tikzcd}$$

The composite

$$t \;=\; (h^{-1}\times 1)g \;:\; E' \longrightarrow Q\times\mathbb{R}^k$$

is topological trivialization of the PL $\mathbb{R}^k$-bundle $\pi p_1 : E' \longrightarrow Q$.

Assume for the moment that t is properly homotopic to a PL trivialization. (We remind the reader that a proper map is one for which the inverse image of each compact set is always compact.) Then the inverse of this new trivialization followed by $e\times 1$ gives a PL homeomorphism

$$f \;:\; Q\times\mathbb{R}^k \longrightarrow M\times\mathbb{R}^k$$

which is properly homotopic to

$$h\times 1 \;:\; Q\times\mathbb{R}^k \longrightarrow M\times\mathbb{R}^k\ .$$

Now let $\lambda : M\longrightarrow Q$ be a PL map which is homotopic to h^{-1}, so that the composition

$$Q\times\mathbb{R}^k \xrightarrow{\ f\ } M\times\mathbb{R}^k \xrightarrow{\ \lambda\times 1\ } Q\times\mathbb{R}^k$$

is homotopic to the identity via a proper homotopy

$$F \;:\; Q\times\mathbb{R}^k\times I \longrightarrow \mathbb{R}^k\ .$$

Notice that both $F_0 = (\lambda\times 1)f$ and $F_1 = \mathrm{id} : Q\times\mathbb{R}^k\longrightarrow\mathbb{R}^k$ are PL and transverse regular to the submanifold $Q\times\{0\}$ of $Q\times\mathbb{R}^k$. Also $F_0^{-1}(Q\times\{0\})$ is PL homeomorphic to M, and $F_1^{-1}(Q\times\{0\})$ is the submanifold $Q\times\{0\}\times\{1\}$ of $Q\times\mathbb{R}^k\times\{1\}$. Using the relative simplicial approximation theorem of Zeeman [22] and the transverse regularity theorem of Williamson [20], we may assume without loss of generality that F is itself PL and transverse regular to $Q\times\{0\}\subseteq Q\times\mathbb{R}^k$. Let W denote the compact manifold $F^{-1}(Q\times\{0\})$. Then W is a proper submanifold of $Q\times\mathbb{R}^k\times I$ which has a trivial normal bundle (the pullback of the natural normal bundle of $Q\times\{0\}\subseteq Q\times\mathbb{R}^k$ under $F|_W$). Embed $Q\times\mathbb{R}^k$ in a sphere S^N of high dimension, and extend this embedding in the obvious way to an embedding of $Q\times\mathbb{R}^k\times I$ in S^{N+1}. If ν_Q denotes the normal bundle of $Q\times\{0\}$ in S^N, and ν_W that of W in S^{N+1}, then our map $F|_W : W\longrightarrow Q\times\{0\}$ extend to a bundle map $\nu_W\longrightarrow\nu_Q$.

Summarizing, we have produced a PL manifold W, whose boundary consists of the disjoint union of M and Q, and a PL map $F : W\longrightarrow Q$ such that:

(i) $F|_Q$ is the identity;
(ii) $F|_M$ is a homotopy equivalence;
(iii) F pulls back the stable PL normal bundle of Q to that of W.

In this situation we may apply the surgery results of Browder and Novikov [2, 3, 4, 14] to alter W and F, though not ∂W or $F|_{\partial W}$, until F becomes a homotopy equivalence. The net result is an h-cobordism W' between M and Q, together with a deformation retraction $F' : W'\longrightarrow Q$. The h-cobordism theorem provides a PL

homeomorphism $G : M \times I \longrightarrow W'$, and $F'G$ is then a PL homotopy between the maps

$$F_0G_0 \ , \ F_1G_1 \ : \ M \longrightarrow Q \ .$$

Now $F_0 : M \longrightarrow Q$ consists of a PL automorphism of M, followed by a PL map from M to Q that is homotopic to h^{-1}. Also, $F_1 : Q \longrightarrow Q$ is the identity map, and both of G_0, G_1 are PL homeomorphisms. Therefore h^{-1} is homotopic to a PL homeomorphism. Consequently h is also homotopic to a PL homeomorphism. This completes our outline of the proof of Theorem 1.1.

Remarks. 1. If the dimension of Q is even there is no obstruction to performing surgery. However, when the dimension is odd, there is an obstruction which must be killed and which, in the corresponding smooth situation, would only allow us to produce an h-cobordism between M and the connected sum of Q with an exotic sphere. Lack of exotic PL spheres means that, in the PL case, killing the surgery obstruction does not alter the boundary components of W.
2. In the terminology of Sullivan [18, 19] the PL bundle $\pi : E \longrightarrow Q$ together with the fibre homotopy equivalence

$$\begin{tikzcd} E \arrow[r, "(h^{-1} \times 1)e"] \arrow[d, "\pi"'] & Q \times \mathbb{R}^n \arrow[d, "p_1"] \\ Q \arrow[r, equal] & Q \end{tikzcd}$$

is a characteristic $(F/PL)_n$-bundle for h^{-1}, and is classified by a homotopy class of maps from Q to F/PL. Our work in stably moving e to a topological bundle equivalence can be reinterpreted as factoring this class through TOP/PL. The final step (deforming the topological trivialization through fibre homotopy equivalences to a PL trivialization) amounts to proving that the associated composite map

$$Q \longrightarrow TOP/PL \longrightarrow F/PL$$

is homotopically trivial. This will be the setting in the Chapters II, III by Rourke and Cooke.

Conversations with Colin Rourke were invaluable during the preparation of these notes, and I would like to thank him for his help.

§2. Splitting theorems

At the end of §1 we were left with a PL $\mathbb{R}^k$-bundle $E' \longrightarrow Q$, a topological trivialization $t : E' \longrightarrow Q \times \mathbb{R}^k$ and the problem of exhibiting a proper homotopy between t and a PL trivialization. Triangulate Q in some way. We can now examine the restriction of the bundle to each simplex and try to push through an inductive argument. More precisely, let Δ be a simplex of the triangulation and $E'(\Delta)$ the part of the bundle over Δ. Our problem reduces to that of constructing, inductively, a proper homotopy between $t|_{E'(\partial\Delta)}$ and a PL bundle equivalence $E'(\partial\Delta)\longrightarrow\partial\Delta\times\mathbb{R}^k$, in such a way that it extends to one that moves $t|_{E'(\Delta)}$ to a PL bundle equivalence $E'(\Delta)\longrightarrow\Delta\times\mathbb{R}^k$. This is the motivation for the 'splitting theorems' below.

Maps between bounded manifolds will, without further mention, be assumed to be maps of pairs (that is to say, they should carry boundary to boundary). Let M be a compact topological manifold of dimension m, W a PL manifold of dimension $m+k$, and $h : W\longrightarrow M\times\mathbb{R}^k$ a proper homotopy equivalence.

Definition. A **splitting** for $h : W\longrightarrow M\times\mathbb{R}^k$ consists of a compact PL manifold N, a PL homeomorphism $s : N\times\mathbb{R}^k\longrightarrow W$ and a proper homotopy ϕ from $hs : N\times\mathbb{R}^k\longrightarrow M\times\mathbb{R}^k$ to $\lambda\times 1 : N\times\mathbb{R}^k\longrightarrow M\times\mathbb{R}^k$, where λ is the homotopy equivalence given by the composition

$$N \xrightarrow{\times 0} N\times\mathbb{R}^k \xrightarrow{hs} M\times\mathbb{R}^k \xrightarrow{\text{proj.}} M\ .$$

The splitting will be denoted by the ordered triple (N, s, ϕ). Remember, under our convention, h, s, and the proper homotopy ϕ all preserve boundaries. When h has a splitting we shall simply say that h **splits**. A splitting (N', s', ϕ') of $h| : \partial W\longrightarrow\partial M\times\mathbb{R}^k$ **extends** to one for h if there is a splitting (N, s, ϕ) for $h : W\longrightarrow M\times\mathbb{R}^k$ such that $\partial N = N'$, $s|_{\partial N\times\mathbb{R}^k} = s'$ and $\phi|_{\partial N\times\mathbb{R}^k\times I} = \phi'$.

(2.1) Splitting theorem. *Let W be a PL manifold, M a compact topological manifold and $h : W\longrightarrow M\times\mathbb{R}^k$ a homeomorphism. Then h splits if M is simply connected, and is either a closed manifold of dimension at least five, or has a simply connected boundary and dimension at least six.*

The proof of this theorem will occupy §4. We shall construct the splitting of h using a very concrete construction due to Novikov, and we shall call a splitting a **Novikov splitting** if it arises in this way. There is a relative version of the theorem for Novikov splittings.

(2.2) Relative splitting theorem. *Let W be a PL manifold, M a compact simply connected topological manifold of dimension at least five, and $h : W\longrightarrow M\times\mathbb{R}^k$ a homeomorphism. Then any Novikov splitting for $h|_{\partial W}$ extends to a Novikov splitting for h.*

A proof of this relative version is given in §5.

Remarks. (1) In our applications to the trivialization problem, the relative split-

ting theorem (2.2) will be applied in situations where M is either a cell of dimension at least six, or M is a cell of dimension five which is already supplied with a rather special splitting over the boundary. The Poincaré Conjecture will then tell us that the associated manifold N is a PL cell. Our hypothesis of 4–connectivity will enable us to avoid any reference to the splitting theorem over cells of dimension less than five.
(2) For $k = 1$ both theorems come directly from work of Siebenmann [16]. His arguments will not be repeated here, though his results are summarized in the next section. The manifold N will occur in a very natural way as the boundary of a collar neighbourhood of an end of $M \times \mathbb{R}$. For higher values of k, ideas of Novikov allow us to produce a situation which is ripe for induction. Siebenmann's results are applied a second time in the inductive step.

§3. Siebenmann's collaring theorems

In later sections we shall rely heavily on results from Siebenmann's thesis [16]. For completeness we sketch the necessary definitions and theorems. We remark that Siebenmann works entirely in the smooth category, however (as he notes) there are analogous PL techniques, and we shall interpret all the results in PL fashion.

An **end** $\mathcal{E}$ of a Hausdorff space X is a collection of subsets which is maximal under the properties:

(i) Each member of $\mathcal{E}$ is a non-empty open connected set with compact frontier and non-compact closure;
(ii) If $A_1, A_2 \in \mathcal{E}$ then there exists $A_3 \in \mathcal{E}$ such that $A_3 \subseteq A_1 \cap A_2$.
(iii) The intersection of the closures of all the sets in $\mathcal{E}$ is empty.
A subset U of X is a **neighbourhood** of $\mathcal{E}$ if it contains some member of $\mathcal{E}$.

Our spaces are at worst locally finite simplicial complexes. For these one can show:

(i) The number of ends of X is the least upper bound of the number of components of $X \backslash K$, where K ranges over all finite subcomplexes of X.
(ii) The number of ends of X is an invariant of the proper homotopy type of X.

A compact space has no ends; $\mathbb{R}$ has two ends and $\mathbb{R}^n$ has one end when $n \geq 2$; if X is compact then $X \times \mathbb{R}$ has two ends; the universal covering space of the wedge of two circles has uncountably many ends. Think of a compact manifold with non-empty boundary. Removing a boundary component M creates one end, and this end has neighbourhoods which are homeomorphic to $M \times [0, 1)$. Indeed the end has 'arbitrary small' neighbourhoods of this type, in the sense that every neighbourhood contains one of these so called collar neighbourhoods.

Let W be a non-compact PL manifold. A **collar** for an end $\mathcal{E}$ of W is a connected PL submanifold V of W which is a neighbourhood of $\mathcal{E}$, has compact boundary, and is PL homeomorphic to $\partial V \times [0,1)$. In what follows we look for conditions on an end which guarantee the existence of a collar.

Given an end $\mathcal{E}$ of W, let $\{X_n\}$ be a sequence of path connected neighbourhoods of $\mathcal{E}$ whose closures have empty intersection. By selecting a base point x_n from each X_n, and a path which joins x_n to x_{n+1} in X_n, we obtain an inverse system

$$\mathcal{S} \ : \ \pi_1(X_1, x_1) \xleftarrow{f_1} \pi_1(X_2, x_2) \xleftarrow{f_2} \cdots \ .$$

Following Siebenmann, we say that π_1 is **stable** at $\mathcal{E}$ if there is a sequence of neighbourhoods of this type for which the associated inverse system induces isomorphisms

$$\mathrm{im}(f_1) \xleftarrow{\cong} \mathrm{im}(f_2) \xleftarrow{\cong} \cdots \ .$$

When π_1 is stable at $\mathcal{E}$, define $\pi_1(\mathcal{E})$ to be the inverse limit of an inverse system $\mathcal{S}$ constructed as above. One must of course check that this definition is independent of all the choices involved.

Recall that a topological space X is **dominated** by a finite complex K if there are maps

$$K \underset{g}{\overset{f}{\rightleftarrows}} X$$

together with a homotopy

$$fg \ \simeq \ 1 \ : \ X \longrightarrow X \ .$$

Let $\mathcal{D}$ be the collection of all those spaces which are of the homotopy type of a CW complex and dominated by a finite complex.

Definition. An end $\mathcal{E}$ of W is **tame** if π_1 is stable at $\mathcal{E}$ and, in addition, there exist arbitrarily small neighbourhoods of $\mathcal{E}$ that lie in $\mathcal{D}$.

The **reduced projective class group** $\widetilde{K}_0(\mathbb{Z}[G])$ is the abelian group of stable isomorphism classes of finitely generated projective $\mathbb{Z}[G]$-modules.

(3.1) The collaring theorem. *Let $\mathcal{E}$ be a tame end of a PL manifold which has compact boundary and dimension at least six. There is an obstruction in $\widetilde{K}_0(\mathbb{Z}[\pi_1(\mathcal{E})])$ which vanishes if and only if $\mathcal{E}$ has a collar.*

The corresponding relative version involves the ends of a PL manifold W whose boundary is PL homeomorphic to the interior of a compact PL manifold. (So in particular the ends of its boundary all have collars.) A **collar** for an end $\mathcal{E}$

of W is now connected PL submanifold neighbourhood V of $\mathcal{E}$ such that:

(i) The frontier bV of V in W is a compact PL submanifold of W (this frontier may itself have a boundary); and
(ii) V is PL homeomorphic to $bV \times [0, 1)$.

(3.2) Relative collaring theorem. *Let $\mathcal{E}$ be a tame end of a PL manifold which has dimension at least six, and whose boundary is PL homeomorphic to the interior of a compact PL manifold. Then $\mathcal{E}$ has a collar provided an obstruction in $\widetilde{K}_0(\mathbb{Z}[\pi_1(\mathcal{E})])$ vanishes. Further, the collar of $\mathcal{E}$ can be chosen to agree with any preassigned collars of those ends of ∂W which are 'contained' in $\mathcal{E}$.*

Remarks on the proof of (3.1). A tame end is always isolated (in the sense that it has a neighbourhood which is not a neighbourhood of any other end), and its fundamental group is finitely presented. Given a tame end $\mathcal{E}$ of W, it is easy to produce a neighbourhood V of $\mathcal{E}$ which is a connected PL manifold having compact boundary and only one end. The idea is then to modify V so that the inclusion of ∂V in V becomes a homotopy equivalence, when V must be a collar by Stallings [17]. Preliminary modifications ensure that:

(i) ∂V is connected,
(ii) the homomorphisms $\pi_1(\mathcal{E}) \longrightarrow \pi_1(V)$, and $\pi_1(\partial V) \longrightarrow \pi_1(V)$ induced by inclusion are isomorphisms and
(iii) the homology groups $H_i(\widetilde{V}, \widetilde{\partial V})$ are zero for $i \neq n-2$, where $n = \dim(W)$.

Here $\widetilde{V}$ denotes the universal covering space of V and, by (ii), the part of $\widetilde{V}$ which sits over ∂V is precisely the universal cover $\widetilde{\partial V}$ of ∂V. At this stage $H_{n-2}(\widetilde{V}, \widetilde{\partial V})$ turns out to be a finitely generated projective $\mathbb{Z}[\pi_1(\mathcal{E})]$-module. The class of this module in $\widetilde{K}_0(\mathbb{Z}[\pi_1(\mathcal{E})])$ is the obstruction mentioned in the statement of (3.1). When this module is stably free we can modify V further so that $H_*(\widetilde{V}, \widetilde{\partial V})$ is zero, and the inclusion of ∂V in V is then a homotopy equivalence.

The following result will be needed later. Let M be a compact topological manifold of dimension m, W a PL manifold of dimension $m+1$, and $h : W \longrightarrow M \times \mathbb{R}$ a proper homotopy equivalence of pairs.

(3.3) Theorem. *The ends of W are tame.*

Proof. Since h is a proper homotopy equivalence W has exactly two ends. Let g be a proper homotopy inverse for h, and $\mathcal{E}$ the end whose neighbourhoods contain sets of the form $g(M \times [t, \infty))$.

(a) Given a path connected neighbourhood X of $\mathcal{E}$, choose t so that $g(M \times [t, \infty))$ is contained in X. Write $\alpha(X)$ for the homomorphism from $\pi_1(M)$ to $\pi_1(X)$ induced by the composite map

$$M \xrightarrow{\times t} M \times [t, \infty) \xrightarrow{g} X ,$$

and note that $\alpha(X)$ is a monomorphism because hg is homotopic to the identity map of $M \times \mathbb{R}$.

Begin with a path connected neighbourhood X_1 of $\mathcal{E}$. If F is a proper homotopy from gh to the identity map of W, choose a path connected neighbourhood X_2 of E which lies in the interior of X_1 and satisfies $F(X_2 \times I) \subseteq X_1$. If $f_1 : \pi_1(X_2) \longrightarrow \pi_1(X_1)$ is induced by inclusion we have a commutative diagram

$$\begin{tikzcd} \pi_1(X_2) \arrow[rr, "f_1"] & & \pi_1(X_1) \\ & \pi_1(M) \arrow[ul, "\alpha(X_2)"] \arrow[ur, "\alpha(X_1)"'] & \end{tikzcd}$$

We claim that $\mathrm{im}(\alpha(X_1)) = \mathrm{im}(f_1)$, so that $\alpha(X_1)$ is an isomorphism from $\pi_1(M)$ to $\mathrm{im}(f_1) \subseteq \mathrm{im}(\alpha(X_1))$. We now select X_3 in the interior of X_2 with the property $F(X_3 \times I) \subseteq X_2$, and so on. The inverse system

$$\mathcal{S} \ : \ X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \longleftarrow \ \cdots$$

then shows that π_1 is stable at $\mathcal{E}$, and that $\pi_1(\mathcal{E})$ is isomorphic to $\pi_1(M)$.

(b) We quote the following lemma from Siebenmann [16].

Lemma. *Let Z be connected CW complex which is the union of two connected sub-complexes Z_1, Z_2. If $Z_1 \cap Z_2, Z \in \mathcal{D}$, and if both $\pi_1(Z_1), \pi_1(Z_2)$ are retracts of $\pi_1(Z)$, then $Z_1, Z_2 \in \mathcal{D}$.*

From part (a) we know that π_1 is stable at $\mathcal{E}$, and that $\pi_1(\mathcal{E})$ is finitely presented. To see the latter, remember that $\pi_1(\mathcal{E}) \cong \pi_1(M)$ and that M is a compact topological manifold, and therefore dominated by a finite complex. Hence $\pi_1(M)$ is a retract of a finitely presented group and is itself finitely presented. Assume for simplicity that M is closed. Given a neighbourhood X of $\mathcal{E}$, Siebenmann's methods allow us to construct a connected PL submanifold neighbourhood V inside X such that the homomorphisms $\pi_1(\mathcal{E}) \longrightarrow \pi_1(V)$, $\pi_1(\partial V) \longrightarrow \pi_1(V)$ induced by inclusion are both isomorphisms. Then $\pi_1(V)$ and $\pi_1(W \backslash \mathrm{int}(V))$ are both isomorphic to $\pi_1(W)$. To complete the proof of (3.3) we simply apply the lemma, taking $Z = W$, $Z_1 = V$ and $Z_2 = W \backslash \mathrm{int}(W)$.

One can define π_r to be stable at $\mathcal{E}$ in exactly the same way as for π_1. Having done this the first part of the above proof is easily modified to give:

(3.4) Addendum. *If $\mathcal{E}$ is an end of W then, for each r, π_r is stable at $\mathcal{E}$ and*

$$\pi_r(\mathcal{E}) \cong \pi_r(M) \cong \pi_r(W) .$$

§4. Proof of the splitting theorem

We consider the splitting theorem (2.1) in its simplest form. As before let W be a PL manifold, M a closed simply connected topological manifold of dimension at least five and $h : W \longrightarrow M \times \mathbb{R}^k$ a homeomorphism. We must show that h splits.

Let T^k denote the k-dimensional torus (the cartesian product of k copies of the circle), and let

$$D = \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k \,\|\, |x_j| \leq 1, 1 \leq j \leq k\} .$$

Starting from an embedding of $S^1 \times \mathbb{R}$ in $\mathbb{R}^2$ we can inductively define embeddings

$$T^{k-1} \times \mathbb{R} \subseteq \mathbb{R}^k$$

for which the universal covering projection

$$e = \exp \times 1 : \mathbb{R}^{k-1} \times \mathbb{R} \longrightarrow T^{k-1} \times \mathbb{R}$$

is the identity on a neighbourhood of D. We leave the details to the reader.

If P denotes $h^{-1}(M \times T^{k-1} \times \mathbb{R})$, then P is an open subset of W and therefore inherits a PL structure from W. Write h_1 for the restriction of h to P, and consider the pullback from :

$$\begin{tikzcd}
\widetilde{P} \arrow[r, dashed, "\widetilde{h}_1"] \arrow[d, dashed, "p"'] & M \times \mathbb{R}^k \arrow[d, "1 \times e"] \\
P \arrow[r, "h_1"] & M \times T^{k-1} \times \mathbb{R}
\end{tikzcd}$$

Since M is simply connected, $\widetilde{P}$ is just the universal cover of P, and p the associated covering projection. Let i denote the inclusion map of P in W.

(4.1) Theorem. *There is a PL homeomorphism $d : \widetilde{P} \longrightarrow W$ such that:*
(i) *$d = ip$ on a neighbourhood of $\widetilde{h}_1^{-1}(M \times D)$, and*
(ii) *hd is isotopic to $\widetilde{h}_1$ keeping a neighbourhood of $\widetilde{h}_1^{-1}(M \times D)$ fixed.*

Remark. In view of (4.1) we shall be able to restrict ourselves to the problem of splitting $\widetilde{h}_1 : \widetilde{P} \longrightarrow M \times \mathbb{R}^k$.

Proof of 4.1. Diagramatically we have

$$\begin{tikzcd}
W \arrow[drr, "h"] & & \\
 & \widetilde{P} \arrow[ul, dashed, "d"'] \arrow[r, dashed, "\widetilde{h}_1"] \arrow[d, dashed, "p"'] & M \times \mathbb{R}^k \arrow[d, "1 \times e"] \\
 & P \arrow[uul, "i"] \arrow[r, "h_1"] & M \times T^{k-1} \times \mathbb{R}
\end{tikzcd}$$

The map $1 \times e : M \times \mathbb{R}^k \longrightarrow M \times T^{k-1} \times \mathbb{R}$ is the identity on $M \times D_\epsilon$, for some $\epsilon > 1$, where

$$D_\epsilon \;=\; \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k \,\|\, |x_j| \leq \epsilon, 1 \leq j \leq k\} \;.$$

Therefore ip PL embeds $\widetilde{h}_1^{-1}(M \times D_\epsilon)$ in W. Now $\widetilde{h}_1^{-1}(M \times D_\epsilon)$ has only one end; it is clearly tame and its fundamental group is trivial because M is simply connected. The collaring theorem provides a compact PL submanifold B of $\widetilde{P}$ such that

$$\widetilde{h}_1^{-1}(M \times \mathrm{int}(D_\epsilon)) \;=\; B \cup \partial B \times [0,1) \;\;,\;\; \widetilde{h}_1^{-1}(M \times D) \subseteq \mathrm{int}(B) \;.$$

Consider the PL manifold $\widetilde{P}\backslash\mathrm{int}(B)$. Again we have one simply connected end and, if V is a collar of this end, the region $\widetilde{P}\backslash(\mathrm{int}(B) \cup \mathrm{int}(V))$ is an h-cobordism. Hence by Stallings [17] there is a PL homeomorphism $\gamma : \widetilde{P} \longrightarrow B \cup \partial B \times [1,0)$ which is the identity on B. At this stage $ip\gamma : \widetilde{P} \longrightarrow W$ is a PL embedding that agrees with ip on B. By the same trick, applied this time in W, we can 'expand' $ip\gamma$ to provide a PL homeomorphism $d : \widetilde{P} \longrightarrow W$ which satisfies (i).

To deal with property (ii) it is sufficient to show that

$$\psi \;=\; hd\widetilde{h}_1^{-1} \;:\; M \times \mathbb{R}^k \longrightarrow M \times \mathbb{R}^k$$

is isotopic to the identity keeping a neighbourhood of $M \times D$ fixed. Write

$$\psi(m,x) \;=\; (\psi^1(m,x), \psi^2(m,x))$$

and use the 'Alexander isotopy' defined by

$$\begin{aligned}
\psi_0 \;&=\; \text{identity} \;, \\
\psi_t(m,x) \;&=\; \left(\psi^1(m,tx), \frac{1}{t}\psi^2(m,tx)\right) \;\; (0 < t \leq 1) \;.
\end{aligned}$$

This completes the proof of (4.1).

We make a couple of assertions concerning proper maps, leaving the reader to fill in the details.

(4.2) Assertion. *Let A and B be compact spaces. A map $f : A \times \mathbb{R}^n \longrightarrow B \times \mathbb{R}^n$*

is proper if and only if given an arbitrarily large positive real number ϵ there is a positive δ such that $|p_2 f(a,x)| > \epsilon$, for all $a \in A$ and $x \in \mathbb{R}^n$ with $|x| > \delta$.

(4.3) Assertion. *A bundle map between two bundles which have locally compact base spaces and a locally compact fibre is proper if and only if the corresponding map of base spaces is proper.*

(4.4) Theorem. *Let P be a PL manifold of dimension $m+r+1$ and $h : P \longrightarrow M \times T^r \times \mathbb{R}$ a proper homotopy equivalence. Then h splits.*

Proof. Since h is a proper homotopy equivalence, P has exactly two ends. Both are tame by (3.3), and their fundamental groups are free abelian of rank r. There is no obstruction to collaring because $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^r])$ is the trivial group (see [1]). Let U and V be disjoint collars of the two ends. Addendum (3.4) can be used to see that $P \backslash (\mathrm{int}(U) \cup \mathrm{int}(V))$ is an h-cobordism, and therefore $P \backslash (\mathrm{int}(U) \cup V)$ is PL homeomorphic to $\partial U \times [0,1)$. But $P \backslash (\mathrm{int}(U) \cup V)$ is also PL homeomorphic to $P \backslash \mathrm{int}(U)$. Collecting together this information we find there is a PL homeomorphism $s : \partial U \times \mathbb{R} \longrightarrow P$. Let

$$g \;=\; hs \;:\; \partial U \times \mathbb{R} \longrightarrow M \times T^r \times \mathbb{R}\,,$$

and write

$$g(u,x) \;=\; \big(g_1(u,x), g_2(u,x)\big)$$

where $g_2(u,x) \in \mathbb{R}$. As g is a proper map, the limit of $g_2(u,x)$ as x tends to $+\infty$ is either $+\infty$ or $-\infty$ simultaneously for all $u \in U$. We assume s chosen so as to give the positive limit. The map

$$\begin{aligned} \phi \;:\; \partial U \times \mathbb{R} \times [0,1] &\longrightarrow M \times T^r \times \mathbb{R}\,; \\ (u,x,t) &\longrightarrow \big(g_1(u,tx), (1-t)x + t g_2(u,x)\big) \end{aligned}$$

is a homotopy between g and $\lambda \times \mathrm{id}_{\mathbb{R}}$, where $\lambda(u) = g_1(u,0)$. Using (4.2) one easily checks that ϕ is a proper map. Therefore $(\partial U, s, \phi)$ splits h.

Returning to the terminology of 4.1 we obtain the next step in the proof of the splitting theorem (2.1):

(4.5) Theorem. *The homeomorphism $\widetilde{h}_1 : \widetilde{P} \longrightarrow M \times \mathbb{R}^k$ splits.*

Proof. Apply (4.4) repeatedly to construct a **tower** for $h_1 : P \longrightarrow M \times T^{k-1} \times \mathbb{R}$ as illustrated below.

$$
\begin{tikzcd}
 & & \overline{P}_k \times \mathbb{R} \arrow[r, "h_k"] & M \times \mathbb{R} \arrow[d] & \\
 & & & M \times S^1 & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
 & \overline{P}_2 \times \mathbb{R} \arrow[r, "s_2"] & P_2 \arrow[r, "h_2"] & M \times T^{k-2} \times \mathbb{R} \arrow[r] \arrow[d] & M \times T^{k-2} \\
\overline{P}_1 \times \mathbb{R} \arrow[r, "s_1"] & P \arrow[r, "h_1"] & M \times T^{k-1} \times \mathbb{R} \arrow[r] & M \times T^{k-1} &
\end{tikzcd}
$$

We say a few words in case the reader starts operating at the wrong end of the diagram. Start with the homeomorphism $h_1 : P \longrightarrow M \times T^{k-1} \times \mathbb{R}$. Split this using (4.4) to obtain a compact PL manifold $\overline{P}_1$, a PL homeomorphism $s_1 : \overline{P}_1 \times \mathbb{R} \longrightarrow P$ and a proper homotopy ϕ^1 from $h_1 s_1$ to $\lambda_1 \times 1$, where λ_1 is the homotopy equivalence given by the composite

$$\overline{P}_1 \xrightarrow{\times 0} \overline{P}_1 \times \mathbb{R} \xrightarrow{h_1 s_1} M \times T^{k-1} \times \mathbb{R} \xrightarrow{\text{proj.}} M \times T^{k-1} .$$

Now induce $h_2 : P_2 \longrightarrow M \times T^{k-2} \times \mathbb{R}$ as the pullback

$$
\begin{tikzcd}
P_2 \arrow[r, dashed, "h_2"] \arrow[d, dashed] & M \times T^{k-2} \times \mathbb{R} \arrow[d, "\mathrm{id}_{M \times T^{k-2}} \times \exp"] \\
\overline{P}_1 \arrow[r, "\lambda_1"] & M \times T^{k-1}
\end{tikzcd}
$$

Then h_2 is a proper homotopy equivalence by Assertion (4.3). Split again using (4.4) to produce $(\overline{P}_2, s_2, \phi^2)$, and so on. The process terminates after k steps.

For each r let $\widetilde{P}_r$ denote the universal covering space of P_r. There are induced bundle maps

$$\begin{tikzcd}
\widetilde{P}_{r+1} \arrow[r, dashed, "\widetilde{s}_r"] \arrow[d] & \widetilde{P}_r \arrow[r, dashed, "\widetilde{h}_r"] \arrow[d, dashed] & M \times \mathbb{R}^{k-r+1} \arrow[d, "1 \times \exp \times 1"] \\
\overline{P}_r \times \mathbb{R} \arrow[r, "s_r"] & P_r \arrow[r, "h_r"] & M \times T^{k-r} \times \mathbb{R}
\end{tikzcd}$$

Note that

$$\widetilde{P}_{k+1} \;=\; \overline{P}_k \quad , \quad \widetilde{P}_k \;=\; P_k \quad , \quad \widetilde{s}_k \;=\; s_k \quad , \quad \widetilde{h}_k \;=\; h_k \; .$$

Let

$$N \;=\; \overline{P}_k \;=\; \widetilde{P}_{k+1} \; ,$$

and let $\widetilde{s}$ denote the composition

$$N \times \mathbb{R}^k \xrightarrow{\widetilde{s}_k \times 1} \widetilde{P}_k \times \mathbb{R}^{k-1} \xrightarrow{\widetilde{s}_{k-1} \times 1} \cdots \xrightarrow{\widetilde{s}_2 \times 1} \widetilde{P}_2 \times \mathbb{R} \xrightarrow{\widetilde{s}_1} \widetilde{P}$$

where $\widetilde{s}_r \times 1$ stands for $\widetilde{s}_r \times \mathrm{id}_{\mathbb{R}^{r-1}}$. Then N is a compact PL manifold and

$$\widetilde{s} \;:\; N \times \mathbb{R}^k \longrightarrow \widetilde{P}$$

is a PL homeomorphism. We are left to construct a proper homotopy $\widetilde{\phi}$ between $\widetilde{h}_1\widetilde{s}$ and the usual product $\widetilde{\lambda} \times \mathrm{id}_{\mathbb{R}^k}$. For each r the tower construction provides a proper homotopy ϕ^r from $h_r s_r$ to $\lambda_r \times \mathrm{id}_{\mathbb{R}}$. These lift to proper homotopies (use (4.3) again) from $\widetilde{h}_r\widetilde{s}_r$ to $\widetilde{h}_{r+1} \times \mathrm{id}_{\mathbb{R}}$, which in turn induce proper homotopies

$$\begin{aligned}
\widetilde{h}_1\widetilde{s} \;=\; \widetilde{h}_1\widetilde{s}_1(\widetilde{s}_2 \times 1)\cdots(\widetilde{s}_k \times 1) \;&\simeq\; (\widetilde{h}_2\widetilde{s}_2 \times 1)\cdots(\widetilde{s}_k \times 1) \\
&\simeq\; \cdots \\
&\simeq\; \widetilde{h}_k\widetilde{s}_k \times 1 \\
&\simeq\; \widetilde{\lambda} \times 1 \; .
\end{aligned}$$

If $\widetilde{\phi}$ denotes the composite proper homotopy from $\widetilde{h}_1\widetilde{s}$ to $\widetilde{\lambda} \times 1$, then $(N, \widetilde{s}, \widetilde{\phi})$ splits $\widetilde{h}_1$. This completes the proof of (4.4).

Proof of the splitting theorem (2.1). By (4.1) and (4.4) we have the following situation

$$N \times \mathbb{R}^k \xrightarrow{\widetilde{s}} \widetilde{P} \xrightarrow{d} W \xrightarrow{h} M \times \mathbb{R}^k$$

where hd is isotopic to $\widetilde{h}_1 : \widetilde{P} \longrightarrow M \times \mathbb{R}^k$. Let

$$s \;=\; d\widetilde{s} \;:\; N \times \mathbb{R}^k \longrightarrow W \;,$$

and construct a proper homotopy ϕ from hs to $\lambda \times \mathrm{id}_{\mathbb{R}^k}$ as the composition

$$hs \;=\; hd\widetilde{s} \;\simeq\; \widetilde{h}_1\widetilde{s} \;\simeq\; \widetilde{\lambda} \times \mathrm{id}_{\mathbb{R}^k} \;\simeq\; \lambda \times \mathrm{id}_{\mathbb{R}^k} \;.$$

The triple (N, s, ϕ) is a splitting for $h : W \longrightarrow M \times \mathbb{R}^k$, as required.

We have proved the splitting theorem when M is a closed simply connected manifold of dimension at least five. Exactly the same process goes through for compact, simply connected, manifolds of dimension at least six which have a simply connected boundary. All maps and homotopies must now preserve boundaries, and the relative collaring theorem is needed for the bounded analogues of (4.1) and (4.4).

§5. Proof of the relative splitting theorem

Let W be a PL manifold, M a compact topological manifold (which may have boundary), and $h : W \longrightarrow M \times \mathbb{R}^k$ a homeomorphism.

Definition. A splitting of h is a **Novikov splitting** if it can be obtained by the construction presented in §4.

More precisely, a splitting (N, s, ϕ) of h is a Novikov splitting if (keeping the previous notation) we can find a PL homeomorphism $d : \widetilde{P} \longrightarrow W$ satisfying the hypotheses of (4.1), plus a tower for $h_1 : P \longrightarrow M \times T^{k-1} \times \mathbb{R}$, such that $N = \overline{P}_k$, $s = d\widetilde{s}$ and ϕ can be constructed from the tower homotopies and the isotopy of (4.1) in the manner described earlier.

Note that it makes sense to speak of a Novikov splitting for $h : W \longrightarrow M \times \mathbb{R}^k$ even when M is not simply connected. Of course, in this case, the covering spaces involved are no longer universal coverings. For example, $\widetilde{P}$ becomes the cover of P which corresponds to the subgroup $\pi_1(M) \lhd \pi_1(M \times T^{k-1} \times \mathbb{R})$.

In the special case where M is a PL manifold, and h is a PL homeomorphism, then $(M, h^{-1}, h^{-1} \times 1)$ is a splitting for h and will be called the **natural splitting**.

It is a Novikov splitting. Just take $d = h^{-1}\widetilde{h}_1 : \widetilde{P} \longrightarrow W$ and use

$$\begin{aligned} \overline{P}_1 &= M \times T^{k-1} \ , \quad s_1 = h_1^{-1} \ , \\ P_r &= M \times T^{k-r} \times \mathbb{R} \ , \quad \overline{P}_r = M \times T^{k-r} \ , \\ h_r &= s_r = \text{identity} \quad (r > 1) \ , \end{aligned}$$

as a tower for $h_1 : P \longrightarrow M \times T^{k-1} \times \mathbb{R}$.

For the remainder of this section we shall assume that M is simply connected and has dimension at least five. Given a Novikov splitting for $h|_{\partial W}$, we must show that it extends to a Novikov splitting for h. There are two essential ingredients in the construction of a Novikov splitting, namely a suitable PL homeomorphism $d : \widetilde{P} \longrightarrow W$ and a tower for $h_1 : P \longrightarrow M \times T^{k-1} \times \mathbb{R}$. We therefore need relative versions of (4.1) and (4.4).

(5.1) Theorem. *Suppose that $d' : \partial\widetilde{P} \longrightarrow \partial W$ is a PL homeomorphism which satisfies :*

(i) *$d' = ip|_{\partial\widetilde{P}}$ on a neighbourhood of $\widetilde{h}_1^{-1}(\partial M \times D)$, and*

(ii) *$(h|_{\partial W})d'$ is isotopic to $\widetilde{h}_1|_{\partial\widetilde{P}}$ keeping a neighbourhood of $\widetilde{h}_1^{-1}(\partial M \times D)$ fixed.*

Then there is a PL homeomorphism $d : \widetilde{P} \longrightarrow W$ which satisfies (i) *and* (ii) *of (4.1), such that $d|_{\partial\widetilde{P}} = d'$ and the isotopy of hd extends that of $(h|_{\partial W})d'$.*

Proof. Proceeding essentially as in (4.1) we use the relative collaring theorem (3.2) to construct a PL homeomorphism $\overline{d} : \widetilde{P} \longrightarrow W$ such that $d = ip$ on a neighbourhood of $h_1^{-1}(M \times D)$, and hd is isotopic to $\widetilde{h}_1$ keeping a neighbourhood of $\widetilde{h}_1^{-1}(M \times D)$ fixed. Along the way we write

$$\widetilde{P} = B \cup bB \times [0,1) \ , \quad \partial\widetilde{P} = B' \cup \partial B' \times [0,1)$$

where B is a compact PL submanifold of $\widetilde{P}$ which meets $\partial\widetilde{P}$ transversally, $B' = B \cap \partial\widetilde{P}$, bB is the frontier of B in $\widetilde{P}$, and:

$$\begin{aligned} &\widetilde{h}_1^{-1}(M \times D) \subseteq \operatorname{int}(B) \ , \\ &\overline{d} = ip \ \text{ on a neighbourhood of } B \ , \\ &\overline{d}|_{\partial\widetilde{P}} = d' \ \text{ on a neighbourhood of } B' \ . \end{aligned}$$

Since $\overline{d}|_{\partial B' \times [0,1)}$, $d'|_{\partial B' \times [0,1)}$ are collars of $d'(\partial B')$ in $\partial W \backslash d'(\operatorname{int}(B'))$, there is a PL ambient isotopy of ∂W which moves $\overline{d}|_{\partial\widetilde{P}}$ so as to agree with d' whilst keeping $d'(B')$ fixed. Extend this ambient isotopy to an ambient isotopy H of all of W which keeps $\overline{d}(B)$ fixed, and let

$$d = H_1\overline{d} : \widetilde{P} \longrightarrow W \ .$$

Then by construction we have $d|_{\partial\widetilde{P}} = d'$ and $d = ip$ on a neighbourhood of $\widetilde{h}_1^{-1}(M \times D)$.

Take the given isotopy from $(h|_{\partial W})d'$ to $\widetilde{h}_1|_{\partial\widetilde{P}}$ and extend it over $\widetilde{P}$, keeping a neighbourhood of $\widetilde{h}_1^{-1}(M \times D)$ fixed, to an isotopy from hd to a PL homeomorphism $g : \widetilde{P} \longrightarrow M \times \mathbb{R}^k$. Then

$$\psi \; = \; g\widetilde{h}_1^{-1} \; : \; M \times \mathbb{R}^k \longrightarrow M \times \mathbb{R}^k$$

is the identity on $\partial M \times \mathbb{R}^k$ and on a neighbourhood of $M \times D$. The Alexander isotopy constructed for (4.1) slides ψ to the identity, keeping $\partial M \times \mathbb{R}^k$ and a neighbourhood of $M \times D$ fixed. Therefore g is isotopic to $\widetilde{h}_1$ leaving $\partial\widetilde{P}$ and a neighbourhood of $\widetilde{h}_1^{-1}(M \times D)$ fixed. Combining this isotopy with that from hd to g gives the required result.

(5.2) Theorem. *Let P be a PL manifold of dimension $m+r+1$ and $h : P \longrightarrow M \times T^r \times \mathbb{R}$ a proper homotopy equivalence of pairs. Then any splitting of $h|_{\partial P}$ extends to a splitting of h.*

Proof. Let (N', s', ϕ') be a splitting of $h|_{\partial P}$. Clearly P has two ends, and $s'|_{N' \times [1,\infty)}$ provides a collar of those ends of ∂P contained by one of the ends of P. Using the relative collaring theorem (3.2) we can extend this collar to a collar of the whole end. If N denotes the base of the extended collar, then $\partial N = N'$. In exactly the same way we can produce a (disjoint) collar of the other end which is compatible with $s'|_{N' \times [-1,-\infty)}$. As in (4.4) we have an h-cobordism (this time between manifolds with boundary) sandwiched by the two collars. A version of Stallings [17] for manifolds with boundary provides a PL homeomorphism $s : N \times \mathbb{R} \longrightarrow P$ such that $s|_{N' \times \mathbb{R}} = s'$. If we can find a proper homotopy ϕ between hs and the usual product $\lambda \times \mathrm{id}_{\mathbb{R}}$ which extends ϕ', then (N, s, ϕ) is the required splitting of h. We can certainly extend ϕ' to a proper homotopy between hs and some map $g : N \times \mathbb{R} \longrightarrow M \times T^r \times \mathbb{R}$. Then, proceeding as in (4.4), we can construct a proper homotopy from g to $\lambda \times \mathrm{id}_{\mathbb{R}}$ which fixes $N' \times \mathbb{R}$. The composition of the two homotopies gives ϕ.

A proof of our relative splitting theorem (2.2) may now be obtained simply by reworking the material of §4, allowing M to have boundary and using (5.1) and (5.2) in place of (4.1) and (4.4).

We end this section with the observation that in the special case when M and h are both PL, we can extend the **natural splitting** of $h|_{\partial W}$ to a splitting of h.

§6. The trivialization problem

This final section will be devoted to a proof of the following result.

(6.1) Theorem. *Let X be a compact 4-connected polyhedron of dimension m, $\pi : E \longrightarrow X$ a PL $\mathbb{R}^k$-bundle with $k \geq m+2$, and $t : E \longrightarrow X \times \mathbb{R}^k$ a topological trivialization. Then t is properly homotopic to a PL trivialization.*

As a direct corollary we have a solution to the bundle trivialization problem proposed in §1. A proof of (6.1) will therefore complete our arguments.

Proof of (6.1). Triangulate X in some way, let K denote the 4–skeleton of the triangulation, and CK the cone on K. Since X is 4–connected, the inclusion map of K into X extends to a map d from all of CK to X. Using d we can pull back the diagram

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[d] & X \times \mathbb{R}^k \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd}$$

to give a bundle over CK and trivialization

$$\begin{tikzcd} E^* \arrow[r, "t^*"] \arrow[d] & CK \times \mathbb{R}^k \arrow[d] \\ CK \arrow[r, equal] & CK \end{tikzcd}$$

As CK is contractible, we can find a PL bundle equivalence

$$f \ : \ E^* \longrightarrow CK \times \mathbb{R}^k \ .$$

Now let $E(K)$ denote the part of E which lies over K, and extend

$$K \times \mathbb{R}^k \xrightarrow{t^{-1}} E(K) \xrightarrow{f} K \times \mathbb{R}^k$$

to a map $g : X \times \mathbb{R}^k \longrightarrow X \times \mathbb{R}^k$ as follows. Use the contractibility of CK again to produce a map $r : X \longrightarrow CK$ which extends the identity on K, and define

$$g(x,u) \ = \ (x, p_2 f t^{*-1}(r(x),u)) \ .$$

Then g is a bundle equivalence and is homotopic to the identity via a proper homotopy. Therefore $gt : E \longrightarrow X \times \mathbb{R}^k$ is a topological trivialization of our bundle which is properly homotopic to t and which, by construction, is PL over K.

We now apply the splitting process over each simplex Δ of X, in other words we split $t : E(\Delta) \longrightarrow \Delta \times \mathbb{R}^k$, taking care that the splittings fit together to give a splitting of $t : E \longrightarrow X \times \mathbb{R}^k$. Since t is already PL over the 4–skeleton of X, we may use the natural splitting over each simplex of K. These splittings are of course compatible, in the sense that the natural splitting over a simplex restricts to the natural splitting over any face. Having rid ourselves of low dimensional problems in this way, we work on the remaining simplexes inductively in order of increasing dimension. The relative splitting theorem allows us to construct a Novikov splitting over each simplex which, when restricted to a face is the splitting constructed earlier. Suppose (B, s, ϕ) is the splitting over Δ. We observe that B is PL homeomorphic to Δ. If $\Delta \in K$, then $B = \Delta$, and for the other simplexes we can use the Poincaré Conjecture noting that, in the special case of a 5-simplex, we know the boundary is already standard. Therefore we have a compatible system of PL homeomorphisms

$$s \ : \ \Delta \times \mathbb{R}^k \longrightarrow E(\Delta) \ \ (\Delta \in X)$$

and a compatible family of proper homotopies $\phi : \Delta \times \mathbb{R}^k \longrightarrow \Delta \times \mathbb{R}^k$ from ts to $\lambda \times \mathrm{id}_{\mathbb{R}^k}$.

A homeomorphism from a ball to itself, which is the identity on the boundary, is isotopic to the identity keeping the boundary fixed. Therefore, again taking the simplexes in some order of increasing dimension, we can inductively homotope the λ's to the identity. Combining all these homeomorphisms and homotopies gives a PL homeomorphism

$$s \ : \ X \times \mathbb{R}^k \longrightarrow E$$

together with a proper homotopy from ts to the identity. Hence t is homotopic to s^{-1} via a proper homotopy that is fixed over K. Although s^{-1} sends $\Delta \times \mathbb{R}^k$ to $E(\Delta)$, for each $\Delta \in X$, it is not at this stage a bundle map. If Γ is a PL section of $E \xrightarrow{\pi} X$ there is an ambient isotopy H of E such that $H_1 s^{-1}(\Delta \times \{0\}) = \Gamma(\Delta)$ for every simplex Δ of X. To construct H we use the Unknotting Theorem [21] inductively. For the inductive step we have a situation where $s^{-1}(\Delta \times \{0\})$ and $\Gamma(\Delta)$ are two embeddings of Δ into $E(\Delta)$ which agree on $\partial\Delta$, and which are therefore ambient isotopic keeping $\partial\Delta$ fixed. The section Γ now has two normal bundles in E, namely the bundle structure of E itself, and that given by s^{-1}. The stable range uniqueness theorem for PL normal bundles [7] provides a PL ambient isotopy G of E such that

$$G_1 H_1 s^{-1} \ : \ X \times \mathbb{R}^k \longrightarrow E$$

is fibre preserving. Therefore $G_1 H_1 s^{-1}$ is a PL trivialization of $E \xrightarrow{\pi} X$ which is properly homotopic to t, and the proof of (6.1) is complete.

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CHAPTER II

The Hauptvermutung according to Casson and Sullivan

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§1. Introduction

Suppose M and Q are PL manifolds and $h : Q \longrightarrow M$ is a (topological) homeomorphism. The Hauptvermutung asserts that in this situation there is a PL isomorphism $g : Q \longrightarrow M$. The purpose of Chapter II is to give a proof of the Hauptvermutung for a large class of manifolds:

(1.1) Main theorem. *Let M, Q, h be as above. Suppose :*
(1) *M and Q are closed of dimension ≥ 5 or bounded of dimension ≥ 6.*
(2) *$\pi_1(M) = \pi_1(\partial M) = 0$. (Assumed for each component of M and ∂M.)*
(3) *$H^3(M; \mathbb{Z}_2) = 0$.*
Then h is homotopic to a PL isomorphism.

We shall also indicate a proof that condition (3) can be weakened to the following:

(3′) *$H^4(M; \mathbb{Z})$ has no elements of order 2.*

Various refinements of the theorem are possible. One can weaken condition (2) to $\pi_1(M) = \pi_1(\partial M)$ if ∂M is connected and non-empty and h is a simple homotopy equivalence. One can keep submanifolds, on which h is already a PL isomorphism, fixed during the homotopy. Precise statements of these refinements are given in §7.

The rest of the introduction consists of a broad outline of the proof of the main theorem followed by a guide to the rest of the chapter.

Outline of the Proof. According to Sullivan [41], h is homotopic to a PL isomorphism if and only if a certain map $q_h : M_0 \longrightarrow G/PL$ is null-homotopic (where $M_0 = M$ if $\partial M \neq \emptyset$ and $M \backslash \{\text{pt.}\}$ if $\partial M = \emptyset$) and from the definition of q_h if follows that q_h factors via TOP/PL.

$$\begin{tikzcd} & TOP/PL \arrow[dr, "i"] & \\ M_0 \arrow[ur] \arrow[rr, "q_h"] & & G/PL \end{tikzcd}$$

The spaces PL, TOP and G will be defined in §2 and an account of the result from

[41] which we use is given in §3, where a more general result is proved. The main tools here are Browder-Novikov-Wall style surgery and the h-cobordism theorem.

From the above diagram we can assert that q_h is null-homotopic if we know that i is null-homotopic. In fact we prove that i factors via $K(\mathbb{Z}_2, 3)$

$$\begin{tikzcd} & K(\mathbb{Z}_2,3) \arrow[dr] & \\ TOP/PL \arrow[ur] \arrow[rr, "i"] & & G/PL \end{tikzcd}$$

From this factoring it follows that the obstruction to homotoping q_h to zero is an element of $H^3(M_0; \mathbb{Z}_2)$ and, using condition (3), the result follows.

The proof of the factorization of i has two main steps:

Step 1. Construct a periodicity map

$$\mu \ : \ G/PL \longrightarrow \Omega^{4n}(G/PL)$$

where $\Omega^n(X)$ denotes the n-th loop space on X. μ is defined to be the composite

$$G/PL \xrightarrow{\alpha} (G/PL)^{\mathbb{CP}^{2n}} \xrightarrow{\sigma} \Omega^{4n}(G/PL)$$

where X^Y is the space of maps $Y \longrightarrow X$, and α is defined by $\alpha x(y) = x$ for all $y \in \mathbb{CP}^{2n}$ (complex projective $2n$-space) and σ is a canonically defined surgery obstruction (see §4). The periodicity map μ has the property that $\mu_* : \pi_k(G/PL) \longrightarrow \pi_{k+4n}(G/PL)$ is an isomorphism for $k \neq 4$ and multiplies by 2 for $k = 4$. (Recall that $\pi_n(G/PL) = 0$ for n odd, $\mathbb{Z}$ if $n = 4k$ and $\mathbb{Z}_2$ if $n = 4k + 2$, essentially Kervaire and Milnor [19], see §4). The proof that μ_* has these properties follows from a product formula for surgery obstructions which is proved by Rourke and Sullivan [36]. It follows that the homotopy-theoretic fibre of μ is a $K(\mathbb{Z}_2, 3)$.

Step 2. Prove that the composite

$$\sigma' \ : \ (TOP/PL)^{\mathbb{CP}^{2n}} \longrightarrow (G/PL)^{\mathbb{CP}^{2n}} \xrightarrow{\sigma} \Omega^{4n}(G/PL)$$

is null-homotopic. This result can be elucidated as follows: the surgery obstruction for a map $\mathbb{CP}^{2n} \longrightarrow G/PL$ is an "index" obstruction (see §4) and can be measured in terms of the Pontrjagin numbers of a certain bundle over $\mathbb{CP}^{2n}$. Using the fact that the map comes from a map of $\mathbb{CP}^{2n} \longrightarrow TOP/PL$, it follows from Novikov [27] that the obstruction is zero. Thus to prove the result it is only necessary to prove Novikov's result in a canonical form, and this is done by using Siebenmann's thesis [37]. In fact we never mention Pontrjagin classes but prove the result directly using Siebenmann. Details are in §5.

Now consider the diagram

$$\begin{tikzcd} TOP/PL \arrow[r, "i"] \arrow[d, "\alpha'"'] & G/PL \arrow[d, "\alpha"] \\ (TOP/PL)^{\mathbb{CP}^{2n}} \arrow[r] \arrow[dr, "\sigma'"'] & (G/PL)^{\mathbb{CP}^{2n}} \arrow[d, "\sigma"] \\ & \Omega^{4n}(G/PL) \end{tikzcd}$$

$\sigma'\alpha' \simeq *$, consequently i factors via the fibre of $\sigma\alpha$ which is $K(\mathbb{Z}_2, 3)$, as required.

Guide to the rest of Chapter II

§2 collects most of the notation and basic definitions which we use. Two important definitions here are the semi-simplicial complexes $HT(M)$ and $NM(M)$. These are (roughly) the space of homotopy triangulations of M and the space of "normal maps" onto M. A normal map is a degree 1 map $f : M_1 \longrightarrow M$ covered by a bundle map from the normal bundle of M_1 to some bundle over M (the terminology "normal map" is Browder's). $HT(M)$ should not be confused with the **set** of PL equivalence classes of homotopy triangulations of M, which we denote $Ht(M)$. (This set was called $PL(M)$ by Sullivan [41] – a notation which we consider should be reserved for the space of PL isomorphisms of M.)

In §3 we prove two basic homotopy equivalences:

$$NM(M) \ \simeq \ (G/PL)^M$$

(which is true in general) and

$$HT(M) \ \simeq \ NM(M)$$

if M is bounded of dimension ≥ 6 and $\pi_1(M) = \pi_1(\partial M) = 0$. The classification of homotopy triangulations

$$Ht(M) \ \cong \ [M_0, G/PL]$$

follows at once. We conclude §3 by defining the characteristic map $q_h : M_0 \longrightarrow G/PL$ (see sketch above) and proving that it factors via TOP/PL if h is a homeomorphism.

The main result of §4 is the homotopy equivalence

$$(G/PL)^M \ \simeq \ (G/PL)^{M_0} \times \Omega^n(G/PL)$$

in case M is closed of dimension ≥ 6. The "canonical surgery obstruction" map

$$\sigma \ : \ (G/PL)^M \longrightarrow \Omega^n(G/PL)$$

is then projection on the second factor. We deduce the properties of μ mentioned earlier.

§5 completes the proof of proving Step 2 above. We start from a refined version of the Novikov–Siebenmann splitting theorem and construct a map

$$\lambda : (TOP/PL)^M \longrightarrow HT(M)$$

which commutes with maps of both spaces into $(G/PL)^M$ – the first induced by inclusion, the second defined in §3. The result then follows easily from the definition of σ.

In the remaining sections, we consider improvements to the main theorem. In §6 we sketch the proof of the weakening of condition (3).

In §7 three refinements are proved:

(a) Replace condition (2) (simple connectivity) by the condition that $\pi_1(\partial M) \longrightarrow \pi_1(M)$ is an isomorphism **and** h is a simple homotopy equivalence. A corollary (using Connell and Hollingsworth [7]) is that the Hauptvermutung holds for manifolds with 2-dimensional spines.
(b) Assume, instead of a topological homeomorphism $h : Q \longrightarrow M$, that M and Q are (topologically) h-cobordant.
(c) Assume that $h : Q \longrightarrow M$ is a cell-like map (cf. Lacher [21]) rather than a homeomorphism.

In §8 we prove a theorem on homotopy triangulations of a block bundle. Two corollaries are:

1. A relative Hauptvermutung; that is, if $h : (M_1, Q_1) \longrightarrow (M, Q)$ is a homeomorphism of pairs and Q is a submanifold either of codim 0 or codim ≥ 3, then h is homotopic to a PL isomorphism of pairs.
2. The embedding theorem first proved by Casson-Sullivan and independently by Haefliger [11] and Wall.

I am indebted to Chris Lacher for pointing out §7(b) and a crucial step in §7(c), and to Greg Brumfiel and George Cooke for patiently explaining §6 . Chapter III by Cooke supplies more detail for §6.

§2. Notation and basic definitions

We refer to Rourke and Sanderson [35] for the definition of the PL category. Objects and maps in this category will be prefixed "PL".

The following are standard objects in the category. $\mathbb{R}^n = \mathbb{R}^1 \times \cdots \times \mathbb{R}^1$, Euclidean n-space. Δ^n, vertices $\{v_0, \cdots, v_n\}$, the standard n-simplex. The face map $\partial_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ is the simplicial embedding which preserves order and fails to cover v_i and $\Lambda_{n,i} = c\ell(\partial\Delta^n \backslash \partial_i\Delta^{n-1})$. The n-cube $I^n = [-1,+1]^n$ and $I = [0,1]$, the unit interval.

Semi-simplicial objects.

We work always without degeneracies – by Rourke and Sanderson [33] they are irrelevant to our purposes, and we shall not then have to make arbitrary choices to define them.

Let Δ denote the category whose objects are Δ^n, $n = 0, 1, \dots$ and morphisms generated by ∂_i^n. A **Δ-set**, **pointed Δ-set**, **Δ-group** is a contravariant functor from Δ to the category of sets, pointed sets, groups.

Δ-maps, Δ-homomorphisms, etc. are natural transformations of functors. Ordered simplicial complexes are regarded as Δ-sets in the obvious way. A Δ-set X satisfies the **(Kan) extension condition** if every Δ-map $\Lambda_{(n,i)} \longrightarrow X$ possesses an extension to Δ^n.

We now define the various Δ-groups and Δ-monoids that we need:

PL_q : typical k-simplex is a zero and fibre preserving PL isomorphism

$$\sigma \ : \ \Delta^k \times \mathbb{R}^q \longrightarrow \Delta^k \times \mathbb{R}^q$$

(i.e $\sigma|_{\Delta^k \times \{0\}} = \text{id.}$ and σ commutes with projection on $\mathbb{R}^q$).

$\widetilde{PL}_q$: typical k-simplex is a zero and block preserving PL isomorphism

$$\sigma \ : \ \Delta^k \times \mathbb{R}^q \longrightarrow \Delta^k \times \mathbb{R}^q$$

(i.e. $\sigma|_{\Delta^k \times \{0\}} = \text{id}$ and $\sigma(K \times \mathbb{R}^q) = K \times \mathbb{R}^q$ for each subcomplex $K \subset \Delta^k$).

Face maps are defined by restriction and $PL_q, \widetilde{PL}_q$ form Δ-groups by composition.

G_q : typical k-simplex is a zero and fibre preserving homotopy equivalence of pairs

$$\sigma \ : \ (\Delta^k \times \mathbb{R}^q, \Delta^k \times \{0\}) \longrightarrow (\Delta^k \times \mathbb{R}^q, \Delta^k \times \{0\})$$

(i.e. $\sigma^{-1}(\Delta^k \times \{0\}) = \Delta^k \times \{0\}$ and $\sigma|_{\Delta^k \times (\mathbb{R}^q \backslash \{0\})}$ has degree ± 1).

$\widetilde{G}_q$: typical k-simplex is a zero and block preserving homotopy equivalence of pairs

$$\sigma \ : \ (\Delta^k \times \mathbb{R}^q, \Delta^k \times \{0\}) \longrightarrow (\Delta^k \times \mathbb{R}^q, \Delta^k \times \{0\}) \ .$$

Again face maps are defined by restriction and G_q, $\widetilde{G}_q$ form Δ-monoids by composition.

Inclusions $i : PL_q \subset PL_{q+1}$ etc. are defined by $i(\sigma) = \sigma \times$ id. (write $\mathbb{R}^{q+1} = \mathbb{R}^q \times \mathbb{R}^1$) and the direct limits are denoted $PL, \widetilde{PL}, G, \widetilde{G}$.

The notation used here differs from that used in Rourke and Sanderson [32], where these complexes were called $PL_q(\mathbb{R})$ etc. However, as we never use the other complexes, no confusion should arise.

G/PL_q and $\widetilde{G}_q/\widetilde{PL}_q$ are the complexes of right cosets (i.e. a k-simplex of G_q/PL_q is an equivalence class of k-simplexes of G_q under $\sigma_1 \sim \sigma_2$ iff $\sigma_1 = \sigma_3 \circ \sigma_2$ where $\sigma_3 \in PL_q$).

The following basic properties of the complexes defined so far will be used (cf. Rourke and Sanderson [33] for notions of homotopy equivalence, etc.).

(2.1) Proposition.
(a) *$G_q \subset \widetilde{G}_q$ is a homotopy equivalence for all q.*
(b) *The inclusions $PL_q \subset \widetilde{PL}_q$, $PL_q \subset PL$, $\widetilde{PL}_q \subset \widetilde{PL}$ and $G_q \subset G$ are all $(q-1)$-connected.*
(c) *$PL \subset \widetilde{PL}$ is a homotopy equivalence.*
(d) *The map $\widetilde{G}_q/\widetilde{PL}_q \longrightarrow \widetilde{G}/\widetilde{PL}$ induced by inclusion is a homotopy equivalence for $q > 2$.*
(e) *The complexes G_q/PL_q (resp. $\widetilde{G}_q/\widetilde{PL}_q$) are classifying for PL bundles with fibre $(\mathbb{R}^q, \{0\})$ and with a fibre homotopy trivialization (resp. open block bundles with a block homotopy trivialization – i.e. a trivialization of the associated fibre space).*

Remark. A "PL bundle with a fibre homotopy trivialization" means a pair (ξ^q, h) where ξ^q/K is a PL fibre bundle with base K and fibre $(\mathbb{R}^q, \{0\})$, and

$$h \ : \ E_0(\xi^q) \ = \ E(\xi)\backslash K \longrightarrow K \times (\mathbb{R}^q \backslash \{0\})$$

is a fibre map with degree ± 1 on each fibre (cf. Dold [9]). Such pairs form a bundle theory with the obvious definitions of induced bundle, Whitney sum, etc. (see Sullivan [41]). A PL block bundle with a block homotopy trivialization can be defined in a similar way.

Proof of 2.1. Parts (a) to (d) were all proved in Rourke and Sanderson [32], the following notes will help the reader understand the status of these results:
(a) is proved by an easy "straight line" homotopy.
(b) the first two parts depend on Haefliger and Poenaru [13] – the second part is explicit in Haefliger and Wall [14] and the first part is a translation of their main result. The third part is a straight analogue of the smooth stability theorem and the fourth part is classical homotopy theory (James [17]).
(c) follows from (b).
(d) is a translation of the stability theorem of Levine [23] using the transverse

regularity of Rourke and Sanderson [31] and Williamson [45] – special arguments are necessary in low dimension – see §7.

(e) follows from the fibrations

$$G_q/PL_q \longrightarrow BPL_q \longrightarrow BG_q \ ,$$
$$\widetilde{G}_q/\widetilde{PL}_q \longrightarrow B\widetilde{PL}_q \longrightarrow B\widetilde{G}_q$$

(see Rourke and Sanderson [34]) and can also be proved by a simple direct argument analogous to Rourke and Sanderson [30; §5].

TOP_q is the topological analogue of PL_q i.e. a k-simplex is a zero and fibre preserving homeomorphism $\Delta^k \times \mathbb{R}^q \longrightarrow \Delta^k \times \mathbb{R}^q$. TOP is the direct limit of

$$i \ : \ TOP_q \subset TOP_{q+1} \subset \dots \ .$$

The stability theorem for TOP_q is for weaker than 2.1(b), but one can define a stable K-theory of topological bundles (see Milnor [26]).

(2.2) Proposition. *The complex TOP_q/PL_q classifies PL bundles with a topological trivialization.*

The proof is the same as 2.1(e).

All the complexes defined so far satisfy the extension condition – this follows easily from the existence of a PL isomorphism $\Lambda_{k,i} \times I \longrightarrow \Delta^k$.

Function spaces.

Let X be a Δ-set with the extension condition and P a polyhedron. A **map** of P in X is an ordered triangulation K of P and a Δ-map $K \longrightarrow X$. A typical k-simplex of the Δ-set X^P is a map $P \times \Delta^k \longrightarrow X$ where the triangulation K of $P \times \Delta^k$ contains $P \times \partial_i \Delta^{k-1}$ as a subcomplex, each i. Face maps are then defined by restriction. For connections with other definitions see Rourke and Sanderson [33].

When X is pointed, denote by $* \subset X$ the subset consisting of base simplexes (or the identity simplexes in case X is a Δ-group or monoid). X^P is then pointed in the obvious way.

Relative function spaces are defined in a similar way. In particular the nth loop space of X is defined by

$$\Omega^n(X) \ = \ (X, *)^{(I^n, \partial I^n)} \ .$$

(Δ^k, n)-manifolds.

Let $\mathcal{M}$ denote the category with objects: PL manifolds; morphisms: inclusions of one PL manifold in the boundary of another. A (Δ^k, n)-**manifold** is a lattice in $\mathcal{M}$ isomorphic to (and indexed by) the lattice of faces of Δ^k, where the isomorphism is graded and decreases dimension by n. (I.e. each s-face of Δ^k indexes an $(n+s)$-manifold.) If $M^{n,k}$ is a (Δ^k, n)-manifold then the element indexed by $\sigma \in \Delta^k$ is denoted M_σ.

Examples.
(1) If M^n is an n-manifold then $M^n \times \Delta^k$ is a (Δ^k, n)-manifold in the obvious way, with $M_\sigma = M \times \sigma$.
(2) A (Δ^0, n)-manifold is an n-manifold.
(3) A (Δ^1, n)-manifold is a cobordism of n-manifolds, possibly with boundary.
(4) If $M^{n,k}$ is a (Δ^k, n)-manifold then the $(\Delta^k, n-1)$-manifold $\partial M^{n,k}$ is defined by

$$\begin{aligned}(\partial M)_{\Delta^k} &= c\ell\big(\partial(M_{\Delta^k})\backslash \bigcup_i M_{\partial_i \Delta^{k-1}}\big) \ , \\ (\partial M)_\sigma &= (\partial M)_{\Delta^k} \cap M_\sigma \ .\end{aligned}$$

Thus, in example (3), $\partial M^{n,1}$ is the corresponding cobordism between the boundaries.

Now we come to two basic definitions:

The Δ-set $HT(M)$.

A **map** of (Δ^k, n)-manifolds $f : M^{n,k} \longrightarrow Q^{n,k}$ is a map $f : M_{\Delta^k} \longrightarrow Q_{\Delta^k}$ such that $f(M_\sigma) \subset Q_\sigma$ for each $\sigma \in \Delta^k$. A **homotopy equivalence** of (Δ^k, n)-manifolds is a map h such that $h|_{M_\sigma} : M_\sigma \longrightarrow Q_\sigma$ is a homotopy equivalence for each $\sigma \in \Delta^k$.

Let M^n be a PL manifold (possibly with boundary). A k-simplex of the Δ-set $HT(M)$ ("homotopy triangulations of M") is a homotopy equivalence of pairs

$$h \ : \ (Q^{n,k}, \partial Q^{n,k}) \longrightarrow (M^n \times \Delta^k, \partial M^n \times \Delta^k)$$

where $Q^{n,k}$ is some (Δ^k, n)-manifold. (I.e. $h(\partial Q)_{\Delta^k} \subset \partial M^n \times \Delta^k$ and $h|_{\partial Q^{n,k}}$ is also a homotopy equivalence.)

Face maps are defined by restriction and it is easy to prove that $HT(M)$ satisfies the extension condition.

The Δ-set $NM(M)$.

A typical k-simplex is a **normal map** $f : Q^{n,k} \longrightarrow M \times \Delta^k$. I.e. f is a (Δ^k, n)-map, has degree 1 on each pair $(Q_\sigma, \partial Q_\sigma) \longrightarrow (M \times \sigma, \partial M \times \sigma)$, and is covered by a map of PL bundles:

$$\begin{array}{ccc} E(\nu_Q) & \xrightarrow{\widehat{f}} & E(\xi) \\ \downarrow & & \downarrow \\ Q & \xrightarrow{f} & M \times \Delta^k \end{array}$$

where ν_Q is the (stable) normal bundle of Q and ξ is some PL bundle on $M \times \Delta^k$. Face maps are defined by restriction and it is again easy to check that $NM(M)$ satisfies the extension condition.

The (stable) **normal bundle** of $Q^{n,k}$ is the normal bundle of an embedding $Q^{n,k} \subset I^{n+N} \times \Delta^k$ (N large) of (Δ^k, n)-manifolds. The normal bundle of $Q^{n,k}$ restricts to the normal bundle of $Q_\sigma \subseteq I^{n+N} \times \sigma$ for each σ. To find such a bundle, use general position to embed, and apply Haefliger and Wall [14].

Reducibility.

Let $\xi^N / Q^{n,k}$ be a bundle. $T(\xi)$, the Thom space, is said to be **reducible** if there is a map $f : I^{n+N} \times \Delta^k \longrightarrow T(\xi)$ which respects the lattice structure and such that

$$f| \ : \ (I^{n+N} \times \sigma, \partial(I^{n+N} \times \sigma)) \longrightarrow (T(\xi|M_\sigma), T(\xi|\partial M_\sigma))$$

has degree 1 for each σ. Thus f gives a simultaneous reduction of all the Thom spaces in the lattice.

For example, the Thom construction shows $T(\nu_Q)$ is reducible, in fact has a canonical choice of reduction map.

Notice that, in the definition of $NM(M), T(\xi)$ is reducible. This is because the Thom isomorphism is natural and f has degree 1. Indeed $T(\widehat{f})$ and the canonical reduction of $T(\nu_Q)$ give a reduction of $T(\xi)$.

The Δ-sets $(G/PL)_M$ and $(TOP/PL)_M$.

Finally we define two Δ-sets which, although essentially the same as the function spaces $(G/PL)^M$ and $(TOP/PL)^M$, have certain advantages for some of our constructions.

A k-simplex of the Δ-set $(G/PL)_M$ (resp. $(TOP/PL)_M$) is a stable PL bundle $\xi^N/M \times \Delta^k$ together with a fibre homotopy trivialization (resp. a topological trivialization). Face maps are defined by restriction and the extension condition is easily verified.

(2.3) Proposition. *There are homotopy equivalences*

$$\kappa \ : \ (G/PL)_M \longrightarrow (G/PL)^M \ ,$$
$$\kappa' \ : \ (TOP/PL)_M \longrightarrow (TOP/PL)^M$$

which commute with the natural maps

$$(TOP/PL)^M \xrightarrow{i_*} (G/PL)^M \ , \ (TOP/PL)_M \xrightarrow{j} (G/PL)_M \ .$$

This follows at once from 2.1(e) and 2.2, commutativity being obvious. One of the advantages of the new sets is that they have an easily described H-space structure. For example, the map

$$m \ : \ (G/PL)_M \times (G/PL)_M \longrightarrow (G/PL)_M \ ;$$
$$(\xi_1^{N_1}, t_1; \xi_2^{N_2}, t_2) \ \longmapsto \ (\xi_1^{N_1} \oplus \xi_2^{N_2}, t_1 \oplus t_2)$$

endows $(G/PL)_M$ with a homotopy commutative H-space structure with homotopy unit. [To make precise sense of $t_1 \oplus t_2$, when ξ_1 and ξ_2 are stable bundles, regard the range of each as $M \times \Delta^k \times \mathbb{R}^\infty$ and choose a homeomorphism $\mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ by alternating coordinates.]

"$\times$", in the displayed formula above, means categorical direct product (as always when we are dealing with Δ-sets). For sets with the extension condition $X \times Y$ has the same homotopy type as $X \otimes Y$ (cf. Rourke and Sanderson [33]).

§3. An account of Sullivan theory

In this section we prove the following results:

(3.1) Theorem. *If M^n is any PL manifold (with or without boundary) then there is a homotopy equivalence*

$$r \ : \ NM(M^n) \longrightarrow (G/PL)^{M^n} \ .$$

(3.2) Theorem. *If M^n, $n \geq 6$ is a PL manifold with non-empty boundary, and $\pi_1(M) = \pi_1(\partial M) = 0$, then there is a homotopy equivalence*

$$i \ : \ HT(M) \longrightarrow NM(M) \ .$$

Theorem 3.1 is proved by transverse regularity (the usual argument) while Theorem 3.2 is proved by surgery. This breaks the usual Browder-Novikov argument into its two basic components. Combining the two results we have:

(3.3) Corollary. *If M is as in Theorem 3.2 then there is a homotopy equivalence*

$$q \;:\; HT(M) \longrightarrow (G/PL)^M \;.$$

To obtain a theorem for closed manifolds, suppose M is closed and let $M_0 = M\backslash\{disc\}$, with $n \geq 6$. Define a Δ-map

$$c \;:\; HT(M_0) \longrightarrow HT(M)$$

by adding a cone over the boundary. (The Poincaré theorem is used here).

(3.4) Theorem. *c is a homotopy equivalence.*

(3.5) Corollary. *If M is closed of dimension ≥ 6 and $\pi_1(M) = 0$, then there is a homotopy equivalence $HT(M) \longrightarrow (G/PL)^{M_0}$.*

Classification of homotopy equivalences.

Now let M^n ($n \geq 5$) be a PL manifold, $\pi_1(M) = \pi_1(\partial M) = 0$ (and if $n = 5$ assume $\partial M = \emptyset$). Define the set $Ht(M)$ (homotopy triangulations of M) as follows:

A representative is a homotopy equivalence $h : (Q^n, \partial Q^n) \longrightarrow (M^n, \partial M^n)$ where Q^n is some PL manifold. $h_1 \sim h_2$ if there is a PL isomorphism $g : Q_1 \longrightarrow Q_2$ such that

$$\begin{tikzcd} Q_1 \arrow[rr, "g"] \arrow[dr, "h_1"'] & & Q_2 \arrow[dl, "h_2"] \\ & M & \end{tikzcd}$$

is homotopy commutative. If follows immediately from the h-cobordism theorem that $Ht(M) = \pi_0(HT(M))$.

(3.6) Corollary. *Let $n \geq 6$ and write $M_0^n = M^n$ if $\partial M \neq \emptyset$ and $M_0^n = M\backslash\{disc\}$ if $\partial M = \emptyset$. ($\pi_1(M) = \pi_1(\partial M) = 0$, as usual.) Then there is a bijection*

$$q_* \;:\; Ht(M) \longrightarrow [M_0, G/PL]$$

where [,] denotes homotopy classes.

This follows immediately from 3.3 and 3.4. For the case $n = 5$ of the main theorem we need

(3.7) Addendum. *If $n = 5$ and $\partial M = \emptyset$ then there is an injection*

$$q_* \;:\; Ht(M) \longrightarrow [M, G/PL] \;.$$

This will be proved using methods similar to those used for 3.2 - 3.4. Following these proofs, we shall make two remarks about q_* which clarify the properties needed for the main theorem.

Proof of Theorem 3.1. We define homotopy inverses $r_1 : NM(M) \longrightarrow (G/PL)_M$ and $-r_2 : (G/PL)_M \longrightarrow NM(M)$. Then $r = \kappa \circ r_1$ is the required equivalence.

Definition of r_1.

0-simplexes.

Let $f : Q^n \longrightarrow M^n$ be a degree 1 map covered by a bundle map:

$$\begin{tikzcd} E(\nu_Q) \arrow[r, "\widehat{f}"] \arrow[d] & E(\xi) \arrow[d] \\ Q^{n,k} \arrow[r, "f"] & M \end{tikzcd}$$

Then, as remarked earlier, $T(\xi)$ has a prescribed reduction. Let

$$u \ : \ (I^{n+N}, \partial I^{n+N}) \longrightarrow (T(\xi), T(\xi|\partial M))$$

be this reduction. Then the uniqueness theorem of Spivak [38] (see the proof given in Wall [43]) says that there is a fibre homotopy equivalence $g : \xi \longrightarrow \nu_M$ such that

(3.8)
$$\begin{tikzcd} & (I^{n+N}, \partial I^{n+N}) \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ (T(\xi), T(\xi|_{\partial M})) \arrow[rr, "T(g)"] & & (T(\nu_M), T(\nu_{\partial M})) \end{tikzcd}$$

is homotopy commutative, where u' is the canonical reduction of ν_M. This diagram determines g up to fibre homotopy, see Wall [43].

Now g determines a stable fibre homotopy trivialization of $[\nu_M] - [\xi]$ (where $[\xi]$ denotes the element of the K-theory corresponding to ξ, etc.). This defines r_1 on 0-simplexes.

In general r_1 is defined by induction and the fact that all the choices made above were only within prescribed homotopy classes implies that a choice over the $(k-1)$-skeleton extends to the k-skeleton.

Definition of $r_2 : (G/PL)_M \longrightarrow NM(M)$. We now define a map r_2. r_1 and $-r_2$ are homotopy inverses ($-r_2$ means r_2 composed with inversion in $(G/PL)_M$, which, as an H-space, possesses a homotopy inverse.) It will be easy to verify that r_1 and $-r_2$ are in fact homotopy inverses and we leave this verification to the reader.

0–simplexes.

Let (ζ, g_1) be a 0-simplex of $(G/PL)_M$, meaning that ζ/M is a PL bundle and $g_1 : \zeta/M \longrightarrow M \times \mathbb{R}^N$ is a fibre homotopy trivialization. Adding ν_M to both sides we have a fibre homotopy equivalence $g : \xi \longrightarrow \nu_M$, where $\xi = \nu_M \oplus \zeta$, and hence a homotopy equivalence $T(g) : T(\xi) \longrightarrow T(\nu_M)$. Thus the canonical reduction $t' : I^{n+N} \longrightarrow T(\nu_M)$ determines a reduction t of $T(\xi)$. By Rourke and Sanderson [31] and Williamson [45] we may assume that t is transverse regular to M and hence $t^{-1}(M)$ is a PL manifold Q and $t|_Q : Q \longrightarrow M$ is covered by a map $t| : \nu_Q \longrightarrow \xi$ of bundles. This defines a 0-simplex of $NM(M)$.

In general the same argument applies to define r_2 on k-simplexes extending a given definition on $(k-1)$-simplexes. The only change needed is that one uses the relative transverse regularity theorem.

Proof of Theorem 3.2.

Definition of i.

0-simplexes. Let $h : (Q, \partial Q) \longrightarrow (M, \partial M)$ be a homotopy equivalence and let ν_Q denote the normal bundle of Q. Let h' be a homotopy inverse of h and let ξ/M be $(h')^*\nu_Q$ then h is covered by a bundle map $\widehat{h} : E(\nu_Q) \longrightarrow E(\xi)$. This defines a 0-simplex of $NM(M)$.

Again, since all choices were within canonical classes, this definition on 0-simplexes yields an inductive definition on k-simplexes. Notice that i is an embedding. Thus to prove i is a homotopy equivalence we only have to prove that $NM(M)$ deformation retracts on $HT(M)$. We prove the following assertion:

Assertion. Suppose $f : Q^{n,k} \longrightarrow M^n \times \Delta^k$ is a degree 1 map (covered by the usual bundle map, as always) and suppose $f|_{Q_\sigma}$ is a homotopy equivalence for each proper face $\sigma < \Delta^k$. Then f is bordant rel $\bigcup_{\sigma \in \partial \Delta^k} Q_\sigma$ to a homotopy equivalence, (and the bordism is covered by the usual bundle map, extending the given map over $Q^{n,k}$).

The assertion implies that a typical relative homotopy element is zero and hence the result (for more detail see Rourke and Sanderson [33]).

Proof of the assertion. The bordism is constructed as the trace of a finite number of surgeries of $(Q_{\Delta^k}, (\partial Q)_{\Delta^k})$ (each surgery being covered by a map of bundles). The construction here is familiar (see Browder [1], Novikov [28], Wall [42]) so we shall not repeat it. Here is a lemma, proved in Wall [44, 1.1] which allows one to do the surgery.

(3.9) Lemma. *Every element $\alpha \in \ker(\pi_i(Q_{\Delta^k}) \longrightarrow \pi_i(M \times \Delta^k))$ gives rise to a well-defined regular homotopy class of immersions of $T = S^i \times D^{n+k-i}$ in Q. We can use an embedding of T in Q to perform surgery on α iff the embedding lies in this class.*

By 3.9 we can immediately perform surgery up to just below the middle dimension. To kill middle-dimensional classes use the method of surgery of relative classes in Wall [42, 44]. (For a direct proof that surgery obstructions are zero on a boundary see Rourke and Sullivan [36].) This completes 3.2.

Proof of 3.4. Observe that c is an embedding and so we have to construct a deformation retract of $HT(M)$ on $c(HT(M))$. To construct this on 0-simplexes one has to prove that any homotopy triangulation is homotopic to one which is a PL homeomorphism on the inverse image of a disc $D^n \subset M^n$. This is easy. In general we have to prove the following assertion, which follows from the splitting theorem of Browder [2].

Assertion. Suppose $f : Q^{n,k} \longrightarrow M^n \times \Delta^k$ is a (Δ^k, n)-homotopy equivalence and M^n is closed. Suppose $f|_{f^{-1}(D^n \times \partial\Delta^k)}$ is a PL homeomorphism. Then f is homotopic rel $\bigcup_{\sigma \in \partial\Delta^k} Q_\sigma$ to a map which is a PL homeomorphism on $f^{-1}(D^n \times \Delta^k)$.

Proof of 3.7. $q : HT(M) \longrightarrow (G/PL)^M$ is defined as before. Suppose $h_1 : M_1 \longrightarrow M$ and $h_2 : M_2 \longrightarrow M$ are vertices in $HT(M)$ which map into the same component of $(G/PL)^M$. Then the proof of 3.1 yields a cobordism between M_1 and M_2 (covered by the usual map of bundles). By taking bounded connected sum with a suitable Kervaire manifold (cf. Browder and Hirsch [5]) we may assume that the surgery obstruction vanishes and hence this cobordism may be replaced by an h-cobordism. So h_1 and h_2 lie in the same component of $HT(M)$.

Two remarks on q_*.

(1) $HT(M)$ and $(G/PL)^{M_0}$ are both based sets and, examining the proofs of 3.1 and 3.2, we see that both r and i can be chosen to preserve base-points. So q_* is base-point preserving and we can rephrase 3.6 and 3.7, as follows.

(3.10) Corollary. *Given a homotopy equivalence $h : M_1 \longrightarrow M$ there is defined, up to homotopy, a map $q_h : M_0 \longrightarrow G/PL$ with the property that $q_h \simeq *$ iff h is homotopic to a PL isomorphism (in case n=5, interpret M_0 as M).*

(2) Suppose $h : M_1 \longrightarrow M$ is a (topological) homeomorphism then h determines a topological isomorphism $\nu(h) : \nu_{M_1} \longrightarrow \nu_M$. This can be seen as follows. Regard M as imbedded in a large-dimensional cube I^{N+n}. Use Gluck [10] to ambient isotope h to a PL embedding; reversing this isotopy and using the (stable) uniqueness theorem for topological normal bundles (Hirsch [15] and Milnor [26]) we obtain the required isomorphism of ν_{M_1} with ν_M.

Now $\nu(h)$ determines a topological trivialization $t(h)$ of $[\nu_M] - [(h^{-1})^*(\nu_{M_1})]$ and hence a map $t_h : M \longrightarrow TOP/PL$ by 2.2.

Proposition 3.11. *The diagram*

$$\begin{tikzcd} M \arrow[rr, "t_h"] \arrow[dr, "q_h"'] & & TOP/PL \arrow[dl, "i"] \\ & G/PL & \end{tikzcd}$$

commutes up to homotopy.

Proof. From the definition of $\nu(h)$ is clear that

$$\begin{tikzcd} & (I^{n+N}, \partial I^{n+N}) \arrow[dl, "u_1"'] \arrow[dr, "u"] & \\ (T(\nu_{M_1}), T(\nu_{\partial M_1})) \arrow[rr, "T(\nu_h)"] & & (T(\nu_M), T(\nu_{\partial M})) \end{tikzcd}$$

commutes up to homotopy, where u_1 and u are the canonical reductions.

Comparing with diagram 3.8 we see that $\nu(h)$ coincides, up to fibre homotopy, with g and hence the fibre homotopy trivialization of $[\nu_M] - [(h^{-1})^*(\nu_{M_1})]$ which determines q_h (see below 3.8) coincides up to fibre homotopy with $t(h)$, as required.

§4. Surgery obstructions

In this section we define the "canonical surgery obstruction"

$$\sigma \ : \ (G/PL)^M \longrightarrow \Omega^n(G/PL)$$

and the periodicity map

$$\mu \; : \; (G/PL) \longrightarrow \Omega^{4n}(G/PL)$$

mentioned in the introduction.

Throughout the section M denotes a closed PL n-manifold, $n \geq 6$, and $M_0 = M\backslash\{\text{disc}\}$.

(4.1) Proposition. *The restriction map* $p : (G/PL)^M \longrightarrow (G/PL)^{M_0}$ *is a fibration with fibre* $\Omega^n(G/PL)$.

Proof. To prove that p has the homotopy lifting property one has to prove that a map of $M \times I^n \cup M_0 \times I^n \times I \longrightarrow G/PL$ extends to $M \times I^n \times I$ (see §2 for the notion of a map of a polyhedron in a Δ-set) and this follows at once from the generalized extension condition proved in Rourke and Sanderson [33]. Thus the result will follow if we know that p is onto, it is clear that the fibre will be $\Omega^n(G/PL)$.

Consider the diagram

$$\begin{tikzcd} HT(M) \arrow[r, "q'"] & (G/PL)^M \arrow[d, shift right, "p"'] \\ HT(M_0) \arrow[u, "\simeq"', "c"] \arrow[r, "q", "\simeq"'] & (G/PL)^{M_0} \arrow[u, dashed, shift right, "s"'] \end{tikzcd}$$

c and q were defined in §3 and q' is defined exactly as q. Then $q \simeq pq'c$ straight from the definitions. Let $s = q'cq^{-1}$ where q^{-1} is a (homotopy) inverse to q, then $ps \simeq id$ and the result follows.

(4.2) Theorem. $(G/PL)^M$ *is homotopy equivalent to the product* $\Omega^n(G/PL) \times (G/PL)^{M_0}$.

Proof. Regard $\Omega^n(G/PL)$ as a subset of $(G/PL)^M$ by inclusion as the fibre. Define

$$d \; : \; \Omega^n(G/PL) \times (G/PL)^{M_0} \longrightarrow (G/PL)^M \; ; \; (x,y) \; \longmapsto \; \kappa m(\kappa^{-1}x, \kappa^{-1}sy)$$

where κ^{-1} is a homotopy inverse of κ and m, κ are as defined in §2 (below 2.3).

We assert that the diagram

$$\begin{tikzcd}
 & \Omega^n(G/PL) \arrow[dl, "\subseteq"'] \arrow[dr, "\subseteq"] & \\
\Omega^n(G/PL) \times (G/PL)^{M_0} \arrow[rr, "d"] \arrow[dr, "\pi_2"'] & & (G/PL)^M \arrow[dl, "p"] \\
 & (G/PL)^{M_0} &
\end{tikzcd}$$

commutes up to homotopy. The result follows by comparing the two exact homotopy sequences, using the five-lemma.

The top triangle commutes since $m(\ ,*) \simeq \mathrm{id}$. To prove that the bottom triangle commutes, let $x^k \in \Omega^n(G/PL)$ be a k-simplex. Then $\kappa^{-1}x$ is a pair (ξ, t), where $\xi/M \times \Delta^k$ is a PL bundle and t is a fibre homotopy trivialization. Since x lies in $\Omega^n(G/PL)$, t is the identity on $\xi|_{M_0 \times \Delta^k}$. Now $m(\kappa^{-1}x, \kappa^{-1}sy)$ is the pair $(\xi \oplus \xi_1, t \oplus t_1)$ where $\xi_1/M \times \Delta^k$ is another bundle. Moreover

$$p_1 m(\kappa^{-1}x, \kappa^{-1}sy) \;=\; p_1 m(*^k, \kappa^{-1}sy)\ ,$$

where $p_1 : (G/PL)_M \longrightarrow (G/PL)_{M_0}$ is induced by restriction. Since it is clear that p, p_1 commute with κ, the result follows.

Definition. The composite

$$\sigma\ :\ (G/PL)^M \xrightarrow{d^{-1}} \Omega^n(G/PL) \times (G/PL)^{M_0} \xrightarrow{\pi_1} \Omega^n(G/PL)$$

is the **canonical surgery obstruction**, where d^{-1} is some homotopy inverse to d.

We now recall the more usual surgery obstructions. The connection with σ will be established in 4.4.

The surgery obstruction of a class $\alpha \in [M, G/PL]$.

According to 3.1 α can be interpreted as a bordism class of normal maps:

$$\begin{tikzcd}
E(\nu_{M_1}) \arrow[r, "\widehat{f}"] \arrow[d] & E(\xi) \arrow[d] \\
M_1^n \arrow[r, "f"] & M^n
\end{tikzcd}$$

One can then associate to α a surgery obstruction in the following groups:

$$s(\alpha) = \begin{cases} I(M) - I(M_1) \in 8\mathbb{Z} & n = 4k \\ 0 & \text{if } n \text{ is odd} \\ K(f) \in \mathbb{Z}_2 & \text{if } n = 4k+2 . \end{cases}$$

Here $I(\ )$ denotes index and $K(\ )$ the Kervaire obstruction. Direct definitions of $K(f)$ are given by Browder [4] and Rourke and Sullivan [36]. The methods of Browder [1] and Novikov [28], translated into the PL category, imply that $s(\alpha) = 0$ iff the bordism class of $(f, \widehat{f})$ contains a homotopy equivalence, $n \geq 5$.

Computation of $\pi_n(G/PL)$.

If M^n is the sphere S^n, then $\xi_0 = \xi|_{S^n \setminus \{disc\}}$ is trivial and so $\nu_{M_1}|_{M_1 \setminus \{disc\}}$ is trivial and in fact has, up to equivalence, a well-defined trivialization given by trivializing ξ_0. This recovers the theorem (see also Rourke and Sanderson [32] and Sullivan [41]):

(4.3) Theorem. *$\pi_n(G/PL) \cong P_n$, the group of almost framed cobordism classes of almost framed PL n-manifolds.*

The surgery obstruction give maps $s : \pi_n(G/PL) \longrightarrow 8\mathbb{Z}, 0$, or $\mathbb{Z}_2$ which are injective for $n \geq 5$ by the Browder-Novikov theorem quoted above (using the Poincaré theorem). Moreover in this range s is surjective, since all obstructions are realized by suitable Kervaire or Milnor manifolds (see Kervaire [18] and Milnor [25]). So we have

$$\pi_n(G/PL) = \begin{cases} 8\mathbb{Z} & \text{if } n = 4k \\ 0 & \text{if } n \text{ is odd} \\ \mathbb{Z}_2 & \text{if } n = 4k+2. \end{cases}$$

To compute $\pi_n(G/PL)$ for $n < 5$ it is necessary to use the braid of the triple $O \subset PL \subset G$ (see Levine [23], also Rourke and Sanderson [32]) and known homotopy groups. Then the above formulae hold for $n < 5$ as well. However there is a distinct singularity because the generator of P_4 has index 16 (Rohlin [29]) instead of 8 for P_{4k}, $k > 1$ (cf. Milnor [26]).

We now prove:

(4.4) Theorem. *The map $\sigma : (G/PL)^M \longrightarrow \Omega^n(G/PL)$ induces the surgery obstruction function*

$$\sigma_* = s : [M, G/PL] = \pi_0(G/PL)^M \longrightarrow \pi_0(\Omega^n(G/PL)) = \pi_n(G/PL) = P_n$$

for $n \geq 5$.

Proof. We first make two observations.

(1) The surgery obstruction is additive (under connected sums). The connected sum of normal maps $f : M_1 \longrightarrow M, g : Q_1 \longrightarrow Q$ is a normal map $f\#g : M_1\#Q_1 \longrightarrow M\#Q$ with surgery obstruction

$$s(f\#g) \;=\; s(f) + s(g) \;.$$

(2) The action of an element in $\pi_0(\Omega^n(G/PL))$ on an element in $[M, G/PL]$ given by multiplication in $(G/PL)^M$ corresponds to taking connected sum of the associated normal maps. (For the normal map corresponding to a vertex in $\Omega^n(G/PL)$ – after inclusion in $(G/PL)^M$ – is the identity outside a disc in M, and we can assume that the other situation is the identity in this disc.)

Now let $f : M \longrightarrow G/PL$ be a vertex of $(G/PL)^M$ and let $\alpha \in \pi_n(G/PL)$ be the class $-s(f)$. Let $\alpha_0 \in \Omega^n(G/PL)$ be a corresponding vertex. Then

$$\sigma_*[m_1(\alpha^0, f)] \;=\; [m_2(\sigma\alpha^0, \sigma f)] \;=\; \sigma_*[\alpha^0] + \sigma_*[f] \;,$$

with m_1 and m_2 the multiplications in $(G/PL)^M$ and $\Omega^n(G/PL)$ respectively. But

$$[m_1(\alpha^0, f)] \;=\; [*]$$

by choice of α and observations (1) and (2), so that $\sigma_*[f] = -\sigma_*[\alpha_0]$. The composite

$$\Omega^n(G/PL) \subset (G/PL)^M \xrightarrow{\sigma} \Omega^n(G/PL)$$

is the identity (by definition); hence $\sigma_*[\alpha^0] = \alpha$ and the result follows.

Remarks. (1) 4.4 fails for $n = 4$ (we cannot even define σ in this case). Indeed if one considers the composition

$$\pi_4(G/PL) \xrightarrow{i_*} [M^4, G/PL] \xrightarrow{s} \mathbb{Z}$$

(s denotes the surgery obstruction) then $si_*(\pi_4(G/PL)) = 16\mathbb{Z}$, as remarked above, while s maps **onto** $8\mathbb{Z}$ for suitable choice of M. This follows from the fact that $24\mu/\mathbb{CP}^2$ is fibre homotopy trivial, where μ is the normal bundle of $\mathbb{CP}^2$ in $\mathbb{CP}^3$. Then the surgery obstruction of the corresponding map of $\mathbb{CP}^2$ into G/PL is 24(Hirzebruch index of μ) = 24.
(2) It has been obvious for some time that $\pi_n(G/PL) \longrightarrow \Omega_n^{PL}(G/PL)$, where $\Omega_n^{PL}(\ )$ denotes PL bordism, is a monomorphism (see e.g. observation (2) above). In fact it follows at once from 4.4 (and the fact that surgery obstructions are bordism invariants) that $\pi_n(G/PL)$ splits as a direct summand of $\Omega_n^{PL}(G/PL)$ for $n \geq 5$ (see also Sullivan [41]). This remark also fails for $n = 4$ for the same reasons as remark (1).

The periodicity map.

We now define the periodicity map mentioned in the introduction.

Definition. μ is the composite

$$\mu : G/PL \xrightarrow{\alpha} (G/PL)^{\mathbb{CP}^{2n}} \xrightarrow{\sigma} \Omega^{4n}(G/PL)$$

where α is a semi-simplicial approximation to the map (of spaces) defined in the introduction (see Rourke and Sanderson [33]).

(4.5) Theorem. *The map* $\mu_* : \pi_i(G/PL) \longrightarrow \pi_i(\Omega^{4n}(G/PL)) = \pi_{i+4n}(G/PL)$ *is an isomorphism for* $i \neq 4$ *and is the inclusion* $16\mathbb{Z} \longrightarrow 8\mathbb{Z}$ *for* $i = 4$.

Proof. By Theorem 4.4 we have to consider the composition

$$[S^i, G/PL] \xrightarrow{\alpha_*} [S^i \times \mathbb{CP}^{2n}, G/PL] \xrightarrow{s} \pi_{i+4n}(G/PL)$$

As mentioned earlier, an element $\beta \in [S^i, G/PL]$ is represented by a normal map

$$\begin{tikzcd} E(\nu_M) \arrow[r, "\widehat{f}"] \arrow[d] & E(\xi) \arrow[d] \\ M^i \arrow[r, "f"] & S^i \end{tikzcd}$$

and then $\alpha_*(\beta)$ is represented by the normal map

$$\begin{tikzcd} E(\nu_M \times \nu_{\mathbb{CP}^{2n}}) \arrow[r, "\widehat{f} \times 1"] \arrow[d] & E(\xi \times \nu_{\mathbb{CP}^{2n}}) \arrow[d] \\ M^i \times \mathbb{CP}^{2n} \arrow[r, "f \times 1"] & S^i \times \mathbb{CP}^{2n} \end{tikzcd}$$

(this is easily checked from the proof of 3.1). So it is necessary to know how surgery obstructions behave under cartesian product. A complete answer is provided by Rourke and Sullivan [36]. Using the fact that $I(\mathbb{CP}^{2n}) = 1$ the required result follows.

§5. The 'canonical Novikov homotopy'

We now complete the proof of the Main Theorem (1.1) by proving:

(5.1) Theorem. *The composite*

$$(TOP/PL)^M \xrightarrow{i_*} (G/PL)^M \xrightarrow{\sigma} \Omega^n(G/PL)$$

is null-homotopic, assuming M is closed of dimension $n \geq 5$.

Proof. We construct a map $\lambda : (TOP/PL)^M \longrightarrow HT(M)$ such that

$$\begin{tikzcd} (TOP/PL)^M \arrow[rr, "i_*"] \arrow[dr, "\lambda"'] & & (G/PL)^M \\ & HT(M) \arrow[ur, "q'"'] & \end{tikzcd}$$

is homotopy commutative. But by the definition of σ we have

$$\begin{tikzcd} HT(M) \arrow[r, "q'"] \arrow[d, "(\sigma^{-1}q') \times *"'] & (G/PL)^M \arrow[d, "\sigma"] \\ (G/PL)^{M_0} \times \Omega^n(G/PL) \arrow[r, "\pi_2"] & \Omega^n(G/PL) \end{tikzcd}$$

homotopy commutative, so that $\sigma q' \simeq *$ and the theorem follows.

Construction of λ. In fact we shall construct a map $\lambda_1 : (TOP/PL)_M \longrightarrow HT(M)$ so that

$$\begin{tikzcd} (TOP/PL)_M \arrow[rr, "j"] \arrow[dr, "\lambda_1"'] & & (G/PL)_M \\ & HT(M) \arrow[ur, "q_1"'] & \end{tikzcd}$$

commutes up to homotopy, where $q_1 = r_1 \circ i$ (see §3). The result then follows by 2.3.

The main tool in the construction of λ_1 is a refined version of the Novikov-Siebenmann splitting theorem. In what follows all maps of bounded manifolds are assumed to carry boundary to boundary.

Definition. Suppose $h : W \longrightarrow M \times \mathbb{R}^k$ is a topological homeomorphism, W and M being PL manifolds. We say h **splits** if there is a PL isomorphism $g : M_1 \times \mathbb{R}^k \longrightarrow W$ such that the composite $hg : M_1 \times \mathbb{R}^k \longrightarrow M \times \mathbb{R}^k$ is properly homotopic to $f \times \mathrm{id}_{\mathbb{R}^k}$, where f is the composite

$$M_1 \subset M_1 \times \mathbb{R}^k \xrightarrow{hg} M \times \mathbb{R}^k \xrightarrow{\pi_1} M .$$

A splitting of h is a triple (M_1, g, H) where M_1, g are as above and H is a proper homotopy between hg and $f \times \mathrm{id}_{\mathbb{R}^k}$. Two splittings (M_1, g, H) and (M_1', g', H')

are **equivalent** if there is a PL isomorphism $e : M_1 \longrightarrow M_1'$ such that diagrams (a) and (b) commute up to isotopy and proper homotopy respectively:

$$\begin{tikzcd} M_1 \times \mathbb{R}^k \arrow[rr, "g"] \arrow[dr, "e \times \{\mathrm{id}\}"'] & & W \\ & M_1' \times \mathbb{R}^k \arrow[ur, "g'"'] & \end{tikzcd}$$

(a)

$$\begin{tikzcd} M_1 \times \mathbb{R}^k \times I \arrow[rr, "H"] \arrow[dr, "e \times \{\mathrm{id}\}"'] & & M \times \mathbb{R}^k \\ & M_1' \times \mathbb{R}^k \times I \arrow[ur, "H'"'] & \end{tikzcd}$$

(b)

(5.4) Theorem. *Suppose given h as above and M is closed of dimension ≥ 5 or bounded of dimension ≥ 6 with $\pi_1(M) = \pi_1(\partial M) = 0$. Then there is a well-defined equivalence class of splittings of h which we call 'Novikov splittings'. If M is bounded then the restriction of a Novikov splitting of h to ∂M is a Novikov splitting of $h|_{\partial M}$.*

Remark. The second half of 5.4 implies that any Novikov splitting of $h|_{\partial W}$ extends to a Novikov splitting of h.

Theorem 5.4 is proved by constructing a **tower** of interpolating manifolds for

$$h| \ : \ h^{-1}(M \times T^{k-1} \times \mathbb{R}) \longrightarrow M \times T^{k-1} \times \mathbb{R}$$

(cf. Novikov [27]), and applying inductively Siebenmann's 1-dimensional splitting theorem [37] (translated into the PL category using the techniques of Rourke and Sanderson [30, 31]). Full details are to be found in Lashof and Rothenberg [22] or Chapter I of these notes.

Definition of λ_1 on 0-simplexes.

Suppose $\sigma^0 \in (TOP/PL)_M$ is a 0-simplex. Then σ^0 is a pair (ξ, h) where ξ^k/M is a PL bundle and $h : E(\xi^k) \longrightarrow M \times \mathbb{R}^k$ is a topological trivialization. Let

$g : M_1 \times \mathbb{R}^k \longrightarrow E(\xi)$ be a Novikov splitting for h. Then the composite $f : M_1 \longrightarrow M$ (as in the definition above) is a homotopy equivalence and hence a 0-simplex $\lambda_1(\sigma^0) \in HT(M)$.

In general λ_1 is defined by induction on dimension. The general definition is similar to that for 0-simplexes except that one uses the relative version of 5.4.

Commutativity of 5.3.

In §3 $q_1 = r_1 \circ i$ was only defined up to homotopy. We shall prove that the definition of q_1 could have been chosen so that (5.3) commutes precisely. We prove this for 0-simplexes. The general proof is similar.

Let $\sigma^0 \in (TOP/PL)_M$ as above. Then we have:

$$\begin{tikzcd} M_1 \times \mathbb{R}^k \arrow[rr, "g", "\cong"'] \arrow[dr, "f \times \{\mathrm{id}\}"'] & & E(\xi) \arrow[dl, "h"] \\ & M \times \mathbb{R}^k & \end{tikzcd} \tag{5.5}$$

commuting up to proper homotopy. Add the (stable) bundle ν_{M_1} to all the terms in (5.5) and observe that $(g^{-1})^*(\nu_{M_1}) \oplus \xi$ is the stable normal bundle of M since its total space (which is the same as the total space of ν_{M_1}) is embedded in a sphere. We obtain

$$\begin{tikzcd} E(\nu_{M_1}) \arrow[rr, "g_1", "\cong"'] \arrow[dr, "\widehat{f}"'] & & E(\nu_M) \arrow[dl, "\widehat{h}"] \\ & E((f^{-1})^* \nu_{M_1}) & \end{tikzcd}$$

The pair $(f|_{M_1}, \widehat{f})$ is a normal map of M_1 to M which we can take to be $i(\sigma^0)$ (see definition of i in §3).

Now it is clear that g_1 commutes (up to homotopy) with the canonical reductions of $T(\nu_{M_1})$ and $T(\nu_M)$. Consequently $\widehat{h}$ can be taken to be the fibre homotopy equivalence $\nu_M \longrightarrow (f^{-1})^*(\nu_{M_1})$ determined by $\widehat{f}$. (Cf. diagram (3.8) et seq.) $\widehat{h}$ determines a fibre homotopy trivialization of $[\xi] = [\nu_M] - [(f^{-1})^*(\nu_{M_1})]$ which we may take to be h itself. So we may take $q_1(\sigma^0) = (\xi, h)$, as required.

§6. Weaker hypotheses

Here we sketch a proof that the condition (3) in the Main Theorem (1.1) can be weakened to

(3′) *$H^4(M; \mathbb{Z})$ has no elements of order 2.*

Full details of the proof are contained in chapter III of these notes.

The idea of the proof is this. We proved that the Sullivan obstruction $q_h : M_0 \longrightarrow G/PL$ corresponding to the homeomorphism $h : Q \longrightarrow M$ factored via the fibre $K(\mathbb{Z}_2, 3)$ of $\mu : G/PL \longrightarrow \Omega^{4n}(G/PL)$. This factoring is not unique, it can be altered by multiplication (in $K(\mathbb{Z}_2, 3)$) with any map of M_0 into the fibre of $K(\mathbb{Z}_2, 3) \longrightarrow G/PL$ which is $\Omega^m(G/PL)$, $m = 4n + 1$. We shall show that a suitable map of M_0 can be chosen so that the obstruction is changed by the mod 2 reduction of any class in $H^3(M, \mathbb{Z})$. Then consider the exact coefficient sequence:

$$H^3(M;\mathbb{Z}) \xrightarrow{\text{mod } 2} H^3(M;\mathbb{Z}_2) \xrightarrow{\beta} H^4(M;\mathbb{Z}) \xrightarrow{\times 2} H^4(M;\mathbb{Z})\,.$$

If condition (3′) holds, β is zero, and the entire obstruction can be killed.

To prove that a suitable map of M_0 into $\Omega^m(G/PL)$ can be found, it is necessary to examine the structure of G/PL for the prime 2.

Definition. Suppose X is an H-space and R is a subring of the rationals. $X \otimes R$ is a CW complex which classifies the generalized cohomology theory $[\ \ , X] \otimes R$ (see Brown [6]).

The ring $\mathbb{Z}_{(2)}$ of integers localized at 2 is the subring of the rationals generated by $\frac{1}{p_i}$ with p_i the odd primes. We write $X_{(2)} = X \otimes \mathbb{Z}_{(2)}$.

(6.1) Theorem. *The k-invariants of $(G/PL)_{(2)}$ are all trivial in dimension ≥ 5.*

Assume 6.1 for the moment. To prove our main assertion we deduce:

(6.2) Corollary. $\Omega^m(G/PL)_{(2)}$ $(m = 4n+1, n > 0)$ *is homotopy equivalent to the cartesian product of* $K(\mathbb{Z}_2, 4i+1)$ *and* $K(\mathbb{Z}_{(2)}, 4i-1)$, $i = 1, 2, \ldots$.

Next we assert that the composite

$$K(\mathbb{Z}_{(2)}, 3) \subset \Omega^m(G/PL)_{(2)} \longrightarrow K(\mathbb{Z}_2, 3)$$

is "reduction mod 2". This follows from the observation that, from the homotopy properties of μ, the map $\Omega^m(G/PL) \longrightarrow K(\mathbb{Z}_2, 3)$ is essential. Now let $\alpha \in H^3(M;\mathbb{Z})$ be any class and let $\alpha_1 \in H^3(M;\mathbb{Z}_{(2)})$ be the corresponding class. Let $\alpha_2 \in H^3(M;\mathbb{Z}_2)$ be the reduction mod 2 of α_1 (and α). α_1 is realized by a map $f : M_0 \longrightarrow K(\mathbb{Z}_{(2)}, 3) \subset \Omega^m(G/PL)_{(2)}$ and some odd multiple rf lifts to $\Omega^m(G/PL)$. But, on composition into $K(\mathbb{Z}_2, 3)$, rf also represents α_2, and so we can indeed alter the original obstruction by the mod 2 reduction of α, as asserted.

Proof of 6.1. The main step is the construction of cohomology classes in $H^{4*}(G/PL; \mathbb{Z}_{(2)})$ and $H^{4*+2}(G/PL;\mathbb{Z}_2)$ which determine the surgery obstructions:

(6.3) Theorem. *There are classes*

$$\mathcal{K} \;=\; k_2 + k_6 + \ldots \in H^{4*+2}(G/PL;\mathbb{Z}_2)$$

and

$$\mathcal{L} \;=\; \ell_4 + \ell_8 + \ldots \in H^{4*}(G/PL;\mathbb{Z}_{(2)})$$

with the following property. If $f : M^n \longrightarrow G/PL$ is a map then

$$s(f) \;=\; \begin{cases} \langle W(M) \cup f^*\mathcal{K}, [M]\rangle \in \mathbb{Z}_2 & \text{if } n = 4k+2 \\ 8\langle L(M) \cup f^*\mathcal{L}, [M]\rangle \in 8\mathbb{Z} & \text{if } n = 4k \end{cases}$$

where $W(M)$ is the total Stiefel-Whitney class and $L(M)$ is the Hirzebruch L-genus.

Remark $L(\ )$ is a rational class (obtained from the equivalence $BO \otimes \mathbb{Q} \simeq BPL \otimes \mathbb{Q}$, which follows from the finiteness of the exotic sphere groups $\Theta_i = \pi_i(PL/O)$ (Kervaire and Milnor [19]). The second formula must the therefore be interpreted in rational cohomology.

(6.4) Corollary *If $f : S^m \longrightarrow G/PL$ represents the generator of π_{4n} (resp. π_{4n+2}), $n > 1$, then*

$$\langle f^*(\ell_{4n}), [S^{4n}]\rangle \;=\; 1 \;\; (resp.\ \langle f^*(k_{4n+2}), [S^{4n+2}]\rangle \;=\; 1\ ) .$$

It follows from 6.4 that the Hurewicz map for $(G/PL)_{(2)}$ is indivisible in dimensions $4n$, $n > 1$, and that the mod 2 Hurewicz map is non-trivial in dimensions $4n+2$. From these facts, 6.1 follows by an exact sequence argument.

We now prove 6.3.

Definition of $\mathcal{K}$. Assuming that

$$\mathcal{K}^{r-1} \;=\; k_2 + k_6 + \ldots + k_{4r-2}$$

has already been defined we define k_{4r+2}. By Thom (see Conner and Floyd [8]) we have that

$$\Omega_{4r+2}(G/PL;\mathbb{Z}_2) \longrightarrow H_{4r+2}(G/PL;\mathbb{Z}_2)$$

is onto (where $\Omega_*(\ )$ denotes oriented bordism), with kernel generated by decomposables. Let $x \in H_{4r+2}(G/PL;\mathbb{Z}_2)$ and $f : M \longrightarrow G/PL$ represent x. Define

$$k(x) \;=\; K(f) - \langle f^*\mathcal{K}^{r-1} \cup W(M), [M]\rangle \in \mathbb{Z}_2 .$$

Then from the multiplicative formulae for the Kervaire obstruction (Rourke and Sullivan [36]) and the multiplicative property of $W(\ )$, it is easy to check that $k(\ )$ vanishes on decomposables and therefore defines a cohomology class k_{4r+2} with the required properties.

Definition of $\mathcal{L}$. The definition is very similar to $\mathcal{K}$. One uses instead the fact (also due to Thom) that

$$\Omega_{4r}(G/PL) \longrightarrow H_{4r}(G/PL; \mathbb{Z}_{(2)})$$

is onto, and the multiplicative properties of $L(\ )$ and the index obstruction.

This completes the proof of 6.3.

Remark. We have described as little of the homotopy type of G/PL as we needed. Sullivan has in fact completely determined the homotopy type of G/PL. We summarize these results:

At the prime 2. $(G/PL)_{(2)}$ has one non-zero k-invariant (in dimension 4) which is δSq^2 (this follows from 6.3 and the remarks below 4.4).

At odd primes. $(G/PL)_{(odd)}$ has the same homotopy type as $(BO)_{(odd)}$ (the proof of this is considerably deeper). [$X_{(odd)}$ means $X \otimes \mathbb{Z}[\frac{1}{2}]$.]

§7. Refinements of the Main Theorem

We consider three refinements:

(a) Relaxing the π_1-condition (2) in Theorem 1.1. No really satisfactory results are available here since one immediately meets the problem of topological invariance of Whitehead torsion.* However, if one is willing to bypass this problem and assume that h is a simple homotopy equivalence, then one can relax condition (2) considerably in the bounded case.

(b) and (c) Relaxing the condition that h is a homeomorphism. The two conditions we replace this by are:

(b) There is a topological h-cobordism between M and Q.

(c) h is a **cell-like** map (cf. Lacher [21]).

With both these replacements, the Main Theorem (1.1) holds good.

We first consider condition (a), assuming that M^n is connected with non-empty connected boundary, $n \geq 6$, and $\pi_1(\partial M) \longrightarrow \pi_1(M)$ (induced by inclusion) is an isomorphism.

Let $SHT(M)$ denote the Δ-set of simple homotopy triangulations of M, i.e. a typical k-simplex is a simple homotopy equivalence of pairs

$$(Q_{n,k}, \partial Q_{n,k}) \longrightarrow (M \times \Delta^k, \partial M \times \Delta^k)\ .$$

(7.1) Theorem. *$i : SHT(M) \longrightarrow NM(M)$ is a homotopy equivalence.*

Proof. The proof is the same as for 3.2, except that one deals with non-simply-

* Solved by Chapman in 1974.

connected surgery in the bounded case with the same fundamental group in the interior and on the boundary, so that the π-π theorem of Wall [44] applies.

Combining 7.1 with 3.1 we have:

(7.2) Corollary. *$q : SHT(M) \longrightarrow (G/PL)^M$ is a homotopy equivalence.*

Now define $Sht(M)$ to be the set of PL equivalence classes of simple homotopy triangulations of M. Then from the s-cobordism theorem we have

(7.3) Corollary. *$q_* : Sht(M) \longrightarrow [M, G/PL]$ is a bijection.*

Using 7.3 we now have precisely the same analysis as in the simply connected case and can deduce

(7.4) Theorem. *Suppose $h : Q \longrightarrow M$ is a homeomorphism* and *a simple homotopy equivalence. Suppose that $H_3(M; \mathbb{Z})$ has no 2-torsion. Then h is homotopic to a PL isomorphism.*

(7.5) Corollary. *If $h : Q \longrightarrow M^n$ is a topological homeomorphism, $n \geq 6$, and $M \searrow K^2$ then h is homotopic to a PL isomorphism.*

Proof. The dimension condition ensures that $\pi_1(\partial M) \longrightarrow \pi_1(M)$ is an isomorphism and Connell and Hollingsworth [7] show that h must be a simple homotopy equivalence.

We now move on to condition (b).

(7.6) Theorem. *Suppose M satisfies the conditions of the main theorem 1.1 with (2) replaced by the existence of a (topological) h-cobordism W between M and Q. Then the homotopy equivalence $h : Q \longrightarrow M$ determined by W is homotopic to a PL isomorphism.*

Proof. We only need to show that q_h factors via TOP/PL. By Gluck [10] we may assume that W is embedded properly in $S^N \times I$ with PL embeddings $M \subset S^N \times \{0\}$, $Q \subset S^N \times \{1\}$. By (stable) existence and uniqueness of normal bundles, we may assume that $W \subset S^N \times I$ has a normal bundle ξ which restricts to PL normal bundles ν_M and ν_Q on $M \subset S^N \times \{0\}, Q \subset S^N \times \{1\}$.

Since W deformation retracts on M and Q, ξ is determined by each of ν_M and ν_Q, therefore $(f^{-1})^* \nu_Q$ is topologically equivalent to ν_M. But this equivalence clearly commutes with the standard reductions of Thom spaces and hence (cf. §3) coincides, up to fibre homotopy, with the fibre homotopy equivalence which determines q_h. Thus q_h factors via TOP/PL, as required.

We now move on to condition (c).

Definition. *A map $f : Q \longrightarrow M$ of manifolds is* **cell-like** *(CL) if:*

(1) *f is* **proper**, *i.e. $f^{-1}(\partial M) = \partial Q$ and f^{-1} (compact)= compact; and one of the following holds :*

$(2)_1$ *$f^{-1}(x)$ has the Čech homotopy type of a point, for each $x \in M$*

$(2)_2$ *$f| : f^{-1}(U) \longrightarrow U$ is a proper homotopy equivalence, for each open set $U \subset M$ or ∂M.*

Lacher [21] proves equivalence of $(2)_1$ and $(2)_2$.

(7.7) Theorem. *Suppose M satisfies the conditions of the main theorem and $f : Q \longrightarrow M$ is a cell-like map. Then f is homotopic to a PL isomorphism.*

To prove 7.7 we shall define a Δ-monoid CL (analogous to TOP) and check that the same analysis holds.

Definition of CL_q. A typical k-simplex is a CL fibre map $f : \Delta^k \times \mathbb{R}^q \longrightarrow \Delta^k \times \mathbb{R}^q$, i.e. f commutes with projection on Δ^k and $f| : \{x\} \times \mathbb{R}^q \longrightarrow \{x\} \times \mathbb{R}^q$ is cell-like for each $x \in \Delta^k$.

The inclusion $CL_q \subset CL_{q+1}$ is defined by identifying f with $f \times \mathrm{id}$ and the stable limit is CL.

Now redefine G_q to consist of proper homotopy equivalences $\mathbb{R}^q \longrightarrow \mathbb{R}^q$ (clearly homotopy equivalent to our original definition) then we have $CL_q \subset G_q$ and $CL_q/PL_q \subset G_q/PL_q$.

Theorem 7.7 follows in the same way as the main theorem from the following three propositions:

(7.8) Proposition. *CL/PL classifies stable PL bundles with a CL-trivialization (a CL-trivialization of ξ/K is a fibre map $E(\xi) \longrightarrow K \times \mathbb{R}^q$ which is cell-like on fibres).*

(7.9) Proposition. *q_f factors via CL/PL.*

(7.10) Proposition. *There is a map $\lambda_1 : (CL/PL)_M \longrightarrow HT(M)$ (M closed,* *simply connected and $n \geq 6$) which makes*

$$\begin{tikzcd} (CL/PL)_M \arrow[rr, "j"] \arrow[dr, "\lambda_1"'] & & (G/PL)_M \\ & HT(M) \arrow[ur, "q_1"'] & \end{tikzcd}$$

homotopy commutative.

Proposition 7.8 is best proved directly (an easy argument) rather than as the fibre of $BPL \longrightarrow BCL$, since it is not clear what BCL classifies. We leave this to the reader. For 7.10 notice that the only fact used in defining λ_1 (cf. proof of 5.4) is that $h : E(\xi) \longrightarrow M \times \mathbb{R}^k$ is a proper homotopy equivalence on $h^{-1}(M \times T^{k-1} \times \mathbb{R})$, which is certainly implied if h is cell-like. It remains to prove 7.9. For this we associate to $f : Q \longrightarrow M$ a CL fibre map $\widehat{f} : \tau_Q \longrightarrow \tau_M$. Since the definition of $\widehat{f}$ is natural (induced by $f \times f : M \times M \longrightarrow Q \times Q$) it is easily proved that the CL trivialization of $\tau_M \oplus (f^{-1})^* \nu_Q$ which $\widehat{f}$ determines, coincides up to fibre homotopy, with the fibre homotopy trivialization determined by f as a homotopy equivalence (cf. §3 – one only needs to prove (3.8) commutative.).

Construct $\widehat{f}$ as follows : let

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

be the tangent microbundle of M. By Kuiper and Lashof [20], π_1 contains a PL fibre bundle τ_M with zero section Δ_M. Let

$$E_Q = (f \times f)^{-1} E(\tau_M) \subset Q \times Q \ .$$

We assert that E_Q is the total space of a PL fibre bundle, fibre $\mathbb{R}^n$, projection π_1 and zero-section Δ_Q. To see this observe that

$$V = E_Q \cap \pi_1^{-1}(x) \cong f^{-1}(U)$$

where $U = \pi_2(\pi_1^{-1}(f_x) \cap E(\tau_M)) \subset M$, which is $\cong \mathbb{R}^n$. Now $f| : V \longrightarrow U$ is a proper homotopy equivalence, since f is cell-like, so $V \cong \mathbb{R}^n$ by Stallings [39]. Hence E_Q is an $\mathbb{R}^n$-bundle; Δ_Q is a section since E_Q is a neighborhood of Δ_Q in $Q \times Q$ and we can take it as zero section. By the uniqueness part of [20], $E_Q \cong \tau_Q$ and $(f \times f)|_{E_Q}$ is the required CL bundle map $\widehat{f}$.

In the bounded case, one first extends f to a cell-like map of open manifolds (by adding an open collar to M and Q and extending f productwise), then the above procedure works on restricting again to Q.

§8. Block bundles and homotopy equivalences

We refer the reader to Rourke and Sanderson [30] for notions of block bundle etc. A **block homotopy equivalence** $h : E(\eta) \longrightarrow E(\xi)$, where $\xi^q, \eta^q/K$ are q-block bundles, is a map satisfying :

(1) $h|_K = \mathrm{id}$,
(2) h preserves blocks ,
(3) $h(E(\partial\eta)) \subset E(\partial\xi)$ ($\partial\xi, \partial\eta$ denote associated sphere block bundles)
(4) $h| : E(\partial\eta|_\sigma) \longrightarrow E(\partial\xi|_\sigma)$ has degree $\pm$ 1 each $\sigma \in K$.

Condition (3) makes sense since both sides have the homotopy type of S^{q-1}.

The proof of the following is left to the reader (cf. Dold [9]):

(8.1) Proposition. *There is an "inverse" block homotopy equivalence $g : E(\xi) \longrightarrow E(\eta)$ such that hg and gh are homotopic to the identity via block homotopy equivalences.*

The associated $\widetilde{G}_q/\widetilde{PL}_q$-bundle.

Define Δ-sets $\widetilde{G}_q, \widetilde{PL}_q$ to consist of b.h.e.'s $\Delta^k \times I^q \longrightarrow \Delta^k \times I^q$ and block bundle isomorphisms $\Delta^k \times I^q \longrightarrow \Delta^k \times I^q$. These sets have the same homotopy type as the sets defined in §2 (cf. Rourke and Sanderson [32]). We associate to ξ^q a Δ-fibration with base K and fibre $\widetilde{G}_q$ by taking as typical k-simplex a b.h.e. $f : \Delta^k \times I^q \longrightarrow E(\xi|_\sigma), \sigma^k \in K$, and a Δ-fibration fibre $\widetilde{G}_q/\widetilde{PL}_q$ by factoring by $\widetilde{PL}_q$ on the left. I.e. $f_1 \sim f_2$ if there is $g \in \widetilde{PL}_q^{(k)}$ such that $f_1 = f_2 \circ g$.

Now say b.h.e.'s $h_1 : E(\eta_1) \longrightarrow E(\xi)$, $h_2 : E(\eta_2) \longrightarrow E(\xi)$ are **isomorphic** (resp. **homotopic**) if there is a block bundle isomorphism $g : \eta_1 \longrightarrow \eta_2$ such that $h_2 g = h_1$ (resp. $h_2 g$ is homotopic to h_1 via b.h.e.'s). The following is easily proved (cf. Rourke and Sanderson [30]) :

(8.2) Proposition. *Isomorphism classes (resp. homotopy classes) of b.h.e.'s $E(\eta) \longrightarrow E(\xi)$ correspond bijectively to cross-sections (resp. homotopy classes of cross-sections) of the associated $\widetilde{G}_q/\widetilde{PL}_q$-bundle to ξ.*

Now write $Ht(\xi)$, "homotopy triangulations of ξ", for the set of homotopy classes of b.h.e.'s $\eta \longrightarrow \xi$.

(8.3) Corollary. *If $q \geq 3$, $Ht(\xi) \cong [K, G/PL]$.*

Proof. This follows from 8.2 and 2.1(d), from the fact that G/PL is an H-space, and from the existence of one cross-section (determined by id : $\xi \longrightarrow \xi$).

More generally define a Δ-set $HT(\xi)$ with $\pi_0(HT(\xi)) = Ht(\xi)$ by taking as typical k-simplex an isomorphism class of b.h.e.'s $\eta \longrightarrow \xi \times \Delta^k$ (see Rourke and

Sanderson [30] for the cartesian product of block bundles), then one has similarly:

(8.4) Corollary. *If* $q \geq 3$, $HT(\xi) \simeq (G/PL)^K$.

Now suppose $|K| = M^n$ then $E(\xi)$ is a manifold and a block homotopy equivalence $\eta \longrightarrow \xi$ gives a simple homotopy equivalence $(E(\eta), E(\partial\eta)) \longrightarrow (E(\xi), E(\partial\xi))$, so we have a Δ-map $j : HT(\xi) \longrightarrow SHT(E(\xi))$.

(8.5) Theorem. *If* $q \geq 3$, $n + q \geq 6$ *then* j *is homotopy equivalence.*

Proof. By 8.4 and 7.2, both sides have the homotopy type of $(G/PL)^M$, so one only needs to check that the diagram

$$\begin{tikzcd} HT(\xi) \arrow[rr, "j"] \arrow[dr, "8.4"'] & & SHT(E(\xi)) \arrow[dl, "q"] \\ & (G/PL)^M & \end{tikzcd}$$

commutes up to homotopy. Now 8.4 was defined by comparing ξ and η as (stable) block bundles and q was defined (cf. §3) by comparing $\tau(E(\xi))$ and $\tau(E(\eta))$ as stable bundles. But $\tau(E(\xi)) \sim \xi \oplus \tau_M$ and $\tau(E(\eta)) \sim \eta \oplus \tau_M$ (see Rourke and Sanderson [31]) and it follows that the diagram commutes up to inversion in G/PL.

Relative Sullivan theory.

Suppose $Q \subset M$ is a codimension 0 submanifold and consider homotopy triangulations $h : M_1 \longrightarrow M$ which are PL isomorphisms on $Q_1 = f^{-1}(Q) \subset M_1$. Denote the resulting Δ-set $HT(M/Q)$, cf. §2.

The following is proved exactly as 3.3 and 3.5:

(8.6) Theorem. *There is a homotopy equivalence*

$$HT(M/Q) \ \simeq \ (G/PL)^{M_0/Q}$$

if $n \geq 6, \pi_1(M\backslash Q) = \pi_1(\partial M\backslash\partial Q) = 0$ *and* $M_0 = M$ *if* $\partial M\backslash\partial Q$ *is non-empty, and* $M_0 = M\backslash\{pt \notin Q\}$ *if* $\partial M\backslash\partial Q = \emptyset$.

From 8.6 one has a Hauptvermutung relative to a codimension 0 submanifold, which we leave the reader to formulate precisely.

Now suppose $Q \subset M$ is a codimension q proper submanifold and ξ/Q a normal block bundle. Let $HT(M,\xi)$ denote the Δ-set of homotopy triangulations which are block homotopy equivalences on $E(\eta) = h^{-1}(E(\xi))$ (and hence in particular a PL isomorphism on Q_1 = zero-section of η).

(8.7) Corollary. *The natural inclusion defines a homotopy equivalence*

$$HT(M,\xi) \;\simeq\; HT(M)$$

if $n \geq 6, q \geq 3$ *and* $\pi_1(M) = \pi_1(\partial M) = 0$.

Proof. Consider the diagram

$$\begin{tikzcd}
HT(M|E(\xi)) \arrow[rr, "\simeq 8.6"] \arrow[d] & & (G/PL)^{M_0/Q} \arrow[d] \\
HT(M,\xi) \arrow[r, "\text{inc.}"] \arrow[d, "\text{restriction}"] & HT(M) \arrow[r, "\simeq 3.3\text{-}5"] & (G/PL)^{M_0} \arrow[d, "\text{restriction}"] \\
HT(\xi) \arrow[rr, "\simeq 8.4"] & & (G/PL)^{Q}
\end{tikzcd}$$

The outside vertical maps are fibrations, commutativity of the top square is clear and of the bottom square (up to sign) follows from the proof of 8.5. The result now follows from the 5-lemma.

Relative Hauptvermutung.

We apply 8.7 to give a Hauptvermutung relative to a submanifold of codimension ≥ 3.

(8.8) Theorem. *Suppose M satisfies the conditions of the Hauptvermutung and $Q \subset M$ is a proper codimension ≥ 3 submanifold. Then any homeomorphism $h : (M_1, Q_1) \longrightarrow (M, Q)$, which is a PL isomorphism on Q_1, is homotopic mod Q_1 to a PL isomorphism.*

(8.9) Theorem. *Suppose M and Q both satisfy the conditions of the Hauptvermutung and $Q \subset M$ is a proper codimension ≥ 3 submanifold. Then any homeomorphism of pairs $h : (M_1, Q_1) \longrightarrow (M, Q)$ is homotopic to a PL isomorphism of pairs.*

Proofs. In 8.8 it is easy to homotope h to be a b.h.e. on some block neighborhood η of Q_1 in M. $h \simeq PL$ isomorphism by the main theorem and it is homotopic via maps which are b.h.e.'s on η by Corollary 8.7.

In 8.9 one first homotopes $h|_{Q_1}$ to a PL isomorphism and extends to give a homotopy equivalence $h_1 : M_1 \longrightarrow Q_1$. It is again easy to make h_1 a b.h.e. on η and then the proof of 8.8 works.

The embedding theorem.

(8.10) Theorem. *Suppose $f : M^n \longrightarrow Q^{n+q}$ is a simple homotopy equivalence, M closed and $q \geq 3$. Then f is homotopic to a PL embedding.*

(8.11) Corollary. *Suppose $f : M^n \longrightarrow Q^{n+q}$ is $(n-q+1)$-connected, then f is homotopic to a PL embedding.*

Proof. This follows at once from 8.10 and Stallings [40].

Proof of 8.10. If $n + q < 6$ the theorem is trivial, so assume $n + q \geq 6$. Let $g : \partial Q \longrightarrow M$ be the homotopy inverse of f restricted to ∂Q.

Assertion 1. As a fibration, g is fibre homotopy equivalent to the projection of a sphere block bundle $g_1 : E(\partial\eta) \longrightarrow M$.

The theorem then follows by replacing g and g_1 by their mapping cylinders to obtain (up to homotopy type) :

$$\begin{tikzcd} Q \arrow[dr, "f^{-1}"] \arrow[d, "h"', "\simeq"] & \\ E(\eta) \arrow[r, "\pi = \text{projection}"'] & M \end{tikzcd}$$

where h is a homotopy equivalence $(Q, \partial Q) \longrightarrow (E(\eta), E(\partial\eta))$. But f^{-1} and π are both simple homotopy equivalences so h is a simple homotopy triangulation of $E(\eta)$ and hence by 8.5 homotopic to a b.h.e. Therefore Q is PL isomorphic to a block bundle over M and M is embedded in Q (by a map homotopic to f).

Instead of Assertion 1, we prove:

Assertion 2. Some large suspension (along the fibres) of $g : \partial Q \longrightarrow M$ is fibre homotopy equivalent to the projection of a sphere block bundle over M.

From this it follows that the fibre of g suspends to a homology sphere (and being simply connected) must be a homotopy sphere. Assertion 1 then follows at once from the classifying space version of 2.1(d) which asserts that

$$\begin{array}{ccc} B\widetilde{PL}_q & \longrightarrow & B\widetilde{G}_q \simeq BG_q \\ \downarrow & & \downarrow \\ B\widetilde{PL} & \longrightarrow & B\widetilde{G} \simeq BG \end{array}$$

is a pushout for $q \geq 3$, i.e. "a spherical fibre space stably equivalent to a sphere block bundle is already equivalent to one".

Now to prove Assertion 2 we only have to notice that $f \times \mathrm{id} : M \longrightarrow Q \times I^N$ (N large) is homotopic to the inclusion of the zero section of a block bundle. First shift to an embedding f_1, then choose a normal block bundle $\xi/f_1 M$ and observe (cf. Mazur [24]) that $c\ell(Q \times I^N \backslash E(\xi))$ is an s-cobordism and hence a product. So we may assume $E(\xi) = Q \times I^N$, as required.

Now $f_1^{-1}|_{\partial(Q \times I^N)}$ is the projection of a sphere block bundle and the suspension along the fibres of g.

Remarks. (1) There is a similarly proved relative version of 8.10 in case M and Q are bounded and $f|_{\partial M}$ is an embedding in ∂Q. Hence using Hudson [16] one has that any two embeddings homotopic to f are isotopic.
(2) 8.10 (and the above remark) reduce the embedding and knot problems to "homotopy theory" – one only has to embed up to homotopy type. The reduction of the problem to homotopy theory by Browder [3] follows easily from this one – for Browder's smooth theorems, one combines the PL theorems with smoothing theory using Haefliger [12] and Rourke and Sanderson [32].

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CHAPTER III

The Hauptvermutung according to Casson and Sullivan

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§1. Introduction

Chapter II contains a proof, due to Casson and Sullivan, of the Hauptvermutung for a PL manifold M which satisfies:

(1) M is either closed of dimension at least five, or bounded of dimension at least six;
(2) each component of both M and ∂M is simply connected;
(3) $H^3(M;\mathbb{Z}_2)=0$.

In this chapter we give a proof that condition (3) can be weakened to:

(3)′ $H^4(M;\mathbb{Z})$ has no 2-torsion.

This stronger result was stated in Sullivan [18]. An outline of the original proof may be found in Sullivan [17]. The argument presented here was sketched in §6 of Chapter II.

The main result is conveniently stated in terms of the obstructions to deforming a homotopy equivalence to a homeomorphism introduced by Sullivan in his thesis [16]. Assume that $h: Q \longrightarrow M$ is a topological homeomorphism. Write $M_0 = M$ if ∂M is non-empty, and $M_0 = M\backslash\{\mathrm{disc}\}$ when ∂M is empty. The i-th obstruction to deforming h to a PL homeomorphism is denoted by $o^i(h)$. It lies in $H^i(M_0;\pi_i(G/PL))$. We shall prove that:

(a) $o^4(h) \in H^4(M_0;\pi_4(G/PL)) = H^4(M_0;\mathbb{Z})$ is defined (all earlier obstructions are zero) and is an element of order at most two;
(b) if $o^4(h)=0$ then all higher obstructions vanish.

Our method is to use information about the bordism groups of G/PL to yield results on the k-invariants of G/PL. The following auxiliary result, which was originally stated in Sullivan [17] and which implies that the $4i$-th k-invariants are of odd order for $i>1$, is of independent interest:

(4.3) Theorem. *There exist classes $\ell_i \in H^{4i}(G/PL;\mathbb{Z}_{(2)})$ for each $i \geq 0$ such that if we write*

$$\mathcal{L} \;=\; \ell_0+\ell_1+\ell_2+\cdots \in H^{4*}(G/PL;\mathbb{Z}_{(2)})$$

then for any map $f : M^{4k} \longrightarrow G/PL$ of a smooth manifold the surgery obstruction of the map f (see §4 of Chapter II) is given by

$$s(f) \;=\; 8\langle L(M) \cup f^*\mathcal{L}, [M]\rangle \;,$$

where $L(M)$ is the Hirzebruch L-genus, and $\mathbb{Z}_{(2)}$ denotes the integers localized at 2.

We reproduce a proof given by Sullivan in his thesis that the $(4i+2)$-nd k-invariants of G/PL are zero, and explicitly calculate the 4th k-invariant (Theorem 4.4).

§2 contains two elementary results on principal fibrations. We follow the treatment of Spanier [14] so closely that proofs are unnecessary. In §3 various results on Postnikov systems are stated. The literature on Postnikov system is scattered. 2-stage Postnikov systems were discussed by Eilenberg and MacLane [4]; in particular, the notation k-invariant is due to them. Postnikov's fundamental papers appeared in 1951; see [10] for an English translation. Other basic references are Moore [8, 9]; recent treatments are in Spanier [14] and Thomas [20]. In §3 I give proofs of two well-known elementary results (3.7 and 3.8) which I was unable to find in the literature. I also quote a result of Kahn's [6] on Postnikov systems of H-spaces because of its general interest. In §4 the desired results on the homotopy properties of G/PL are obtained and applied to the Hauptvermutung.

I am happy to acknowledge the substantial help of several people in the preparation of this chapter. I wish to thank Colin Rourke for arousing my interest in the problem and for explaining much of the needed geometry such as Sullivan's thesis. I am grateful to Greg Brumfiel for outlining Sullivan's proof of the main result to me, and for showing me how to extend Theorem 4.3 to PL-manifolds. And I thank Bob Stong for patiently explaining the necessary results in cobordism theory – especially Theorem 4.5.

§2. Principal fibrations

Let B be a space with base point b_0. Let PB denote the space of paths in B starting at b_0. The evaluation map

$$p \;:\; PB \longrightarrow B \;;\; \lambda \longrightarrow \lambda(1)$$

is the projection of the **path-space fibration** (see Spanier [14]). The fibre of p is the space of loops in B based at b_0, which is denoted ΩB. If $f : X \longrightarrow B$ is a map, the induced fibration over X is called the principal ΩB-fibration induced by f. The total space E is defined by

$$E \;=\; \{(x, \lambda) \in X \times PB \mid f(x) = \lambda(1)\}$$

and the projection $E \xrightarrow{\pi} X$ is defined by $\pi(x, \lambda) = x$. (E is often called the **fibre** of the map f.) Suppose that $x_0 \in X$ is a base point and that $f(x_0) = b_0$. Then we stipulate that $c_0 = (x_0, \lambda_0)$ is the base point of E, where λ_0 is the constant path. We have an inclusion $\Omega B \xrightarrow{\pi} E$ given by $j(\lambda) = (x_0, \lambda)$. Let Y be a space with base point y_0. In the following theorem $[Y, \cdot]$ denotes the functor "homotopy-rel-base point classes of base-point preserving maps from Y to $\cdot$".

(2.1) Theorem. *The following sequence of pointed sets is exact:*

$$[Y, \Omega B] \xrightarrow{j_*} [Y, E] \xrightarrow{\pi_*} [Y, X] \xrightarrow{f_*} [Y, B] \ .$$

We define an action

$$m \ : \ \Omega B \times E \longrightarrow E \ ; \ (\lambda, (x, \lambda')) \longrightarrow (x, \lambda * \lambda') \ ,$$

where $\lambda * \lambda'$ denotes, as usual, the path

$$(\lambda * \lambda')(t) \ = \ \begin{cases} \lambda(2t) & t \leq \frac{1}{2} \\ \lambda'(2t-1) & t \geq \frac{1}{2} \ . \end{cases}$$

The action m is consistent with the inclusion $j : \Omega B \longrightarrow E$ and the multiplication in ΩB since

$$\begin{aligned} m(\lambda, j(\lambda')) \ &= \ m(\lambda, (x_0, \lambda')) \\ &= \ (x_0, \lambda * \lambda') \\ &= \ j(\lambda * \lambda') \ . \end{aligned}$$

The map m induces an action

$$m_* \ : \ [Y, \Omega B] \times [Y, E] \longrightarrow [Y, E]$$

where $[Y, \Omega B]$ inherits a group structure from the multiplication in ΩB.

(2.2) Theorem. *If u, v are elements of $[Y, E]$, then $\pi_* u = \pi_* v$ if and only if there exists w in $[Y, \Omega B]$ such that*

$$v \ = \ m_*(w, u) \ .$$

§3. Postnikov systems

Let X be a topological space. A **cofiltration** of X is a collection of spaces $\{X_i\}$ indexed on the non-negative integers and maps

$$f_i \ : \ X \longrightarrow X_i \quad , \quad g_i \ : \ X_i \longrightarrow X_{i-1}$$

such that the composition $g_i f_i$ is homotopic to f_{i-1}. A cofiltration is usually assembled in a diagram as below:

$$\begin{tikzcd}
& \vdots \arrow[d, "g_{i+1}"] \\
X \arrow[r, "f_i"] \arrow[dr, "f_{i-1}"] \arrow[dddr, "f_0"] & X_i \arrow[d, "g_i"] \\
& X_{i-1} \arrow[d, "g_{i-1}"] \\
& \vdots \arrow[d, "g_1"] \\
& X_0
\end{tikzcd}$$

A cofiltration is called **convergent** if for any integer N there is an M such that for all $m > M$, $f_m : X \longrightarrow X_m$ is N-connected. The notion of convergent cofiltration has many applications. For example, a very general problem in topology is to determine the homotopy classes of maps of a finite complex K to a space X (denoted $[K, X]$). If $\{X_i\}, \{f_i\}, \{g_i\}$ is a convergent cofiltration of X, then the problem of calculating $[K, X]$ is split up into a finite sequence of problems: given that $[K, X_i]$ is known, calculate $[K, X_{i+1}]$. Since the cofiltration is assumed convergent, $(f_i)_* : [K, X] \longrightarrow [K, X_i]$ is a bijection for large i and so the sets $[K, X_i]$ converge after a finite number of steps to a solution of the problem.

Naturally one would like to concentrate on cofiltrations where each step of the above process (calculate $[K, X_{i+1}]$ given $[K, X_i]$) is as simple as possible. Obstruction theory leads to the following requirement for simplifying the problem: that each map $g_i : X_i \longrightarrow X_{i-1}$ be a fibration with fibre an Eilenberg-MacLane space. The precise definition is given in the following way.

First note that the space of loops in a $K(\pi, n+1)$ is a $K(\pi, n)$, and so it makes sense to speak of a principal $K(\pi, n)$-fibration. Such a fibration is induced by a map of the base space into $K(\pi, n+1)$.

(3.1) Definition. A **Postnikov system** for a path-connected space X is a convergent cofiltration

$$\{X_i\} \quad , \quad \{f_i : X \longrightarrow X_i\} \quad , \quad \{g_i : X_i \longrightarrow X_{i-1}\}$$

of X such that X_0 is contractible, each X_i has the homotopy type of a CW complex, and $g_i : X_i \longrightarrow X_{i-1}$ $(i > 0)$ is a principal $K(\pi, i)$-fibration for some π.

Suppose that X is a path-connected space, and that $\{X_i\}$, $\{f_i : X \longrightarrow X_i\}$,

$\{g_i : X_i \longrightarrow X_{i-1}\}$ is a Postnikov system for X. Then it follows that:

(a) For each i, the map $f_i : X \longrightarrow X_i$ induces isomorphisms of homotopy groups in dimensions $\leq i$, and $\pi_j(X_i) = 0$ for $j > i$;

(b) for each $i > 0$, the map $g_i : X_i \longrightarrow X_{i-1}$ is actually a principal $K(\pi_i(X), i)$ fibration, induced by a map

$$k^i \ : \ X_{i-1} \longrightarrow K(\pi_i(X), i+1) \ .$$

The class $k^i \in H^{i+1}(X_{i-1}; \pi_i(X))$ is called the i-th k-invariant of the given Postnikov system. Furthermore, the inclusion of the fibre of g_i, $j : K(\pi_i(X), i) \longrightarrow X_i$, induces a homology homomorphism equivalent to the Hurewicz homomorphism h_i in the space X, in that the diagram below is commutative:

$$\begin{tikzcd}
H_i(K(\pi_i(X), i)) \arrow[r, "j_*"] \arrow[d, "\simeq"'] & H_i(X_i) \\
\pi_i(X) \arrow[r, "h_i"] & H_i(X) \arrow[u, "(f_i)_*"']
\end{tikzcd}$$

(c) X is a simple space; that is to say, $\pi_1(X)$ acts trivially on $\pi_n(X)$ for each n, or equivalently, any map of the wedge $S^1 \vee S^n \longrightarrow X$ extends over the product $S^1 \times S^n$. In particular, $\pi_1(X)$ is abelian.

On the other hand, if X is simple then there exists a Postnikov system for X. See Spanier [14, Corollary 8.3.1, p.444]. For a discussion of the uniqueness of Postnikov systems, see Barcus and Meyer [1].

The following facts about Postnikov systems are presented without proofs except in cases where I do not know of an appropriate reference:

(3.2) Maps of Postnikov systems. Let X and X' be spaces, with Postnikov systems $\{X_i, f_i, g_i\}$, $\{X'_i, f'_i, g'_i\}$. If $h : X \longrightarrow X'$ is a map, then there exists a map of Postnikov systems consistent with the map h; that is, there is a collection of maps $\{h_i : X_i \longrightarrow X'_i\}$ such that $h_i f_i \simeq f'_i h$ and $h_{i-1} g_i \simeq g'_i h_i$ for all i. (See Kahn [6].)

(3.3) Cohomology suspension. If B is a space with base point $*$, and $PB \xrightarrow{p} B$ is the path space fibration, then for $i > 1$ the composition

$$H^i(B; G) \xleftarrow{\approx} H^i(B, *; G) \xleftarrow{p^*} H^i(PB, \Omega B; G) \underset{\approx}{\xleftarrow{\delta}} H^{i-1}(\Omega B; G)$$
$$\longrightarrow H^{i-1}(\overline{\Omega} B; G)$$

(with G an arbitrary coefficient group) is called the **cohomology suspension** and is denoted by $\sigma : H^i(B; G) \longrightarrow H^{i-1}(\overline{\Omega} B; G)$. For $i \leq 1$ σ is set equal to zero.

(3.4) Postnikov system of a loop space. (See Suzuki [19] for a study of the case of 2 non-vanishing homotopy groups.) Let X be a path-connected space with base point. Let $\overline{\Omega}X$ denote the component of the loop space of X consisting of those loops which are homotopic to a constant. If $\{X_i\}, \{f_i : X \longrightarrow X_i\}$ $\{g_i : X_i \longrightarrow X_{i-1}\}$ is a Postnikov system for X, then a Postnikov system for $\overline{\Omega}X$ is obtained by applying the loop space functor. That is, set

$$\begin{aligned} Y_i &= \overline{\Omega}X_{i+1} \ , \\ f_i' &= \Omega f_{i+1} \ : \ \overline{\Omega}X \longrightarrow \overline{\Omega}X_{i+1} \ , \\ g_i' &= \Omega g_{i+1} \ : \ \overline{\Omega}X_{i+1} \longrightarrow \overline{\Omega}X_i \end{aligned}$$

and then $\{Y_i, f_i', g_i'\}$ is a Postnikov system for $\overline{\Omega}X$. The k-invariants of this Postnikov system for $\overline{\Omega}X$ are just the cohomology suspensions of the k-invariants of the Postnikov system $\{X_i, f_i, g_i\}$.

(3.5) Definition. For any space Y, the set $[Y, \overline{\Omega}B]$ inherits a group structure from the multiplication on $\overline{\Omega}B$. We shall often use the fact that if $u \in H^i(B; G)$ then $\sigma u \in H^{i-1}(\overline{\Omega}B; G)$ is **primitive** with respect to the multiplication on $\overline{\Omega}B$; this means that for any space Y and for any two maps $f, g \in [Y, \overline{\Omega}B]$,

$$(f \cdot g)^* \sigma u \ = \ f^* \sigma u + g^* \sigma u \ ,$$

where $f \cdot g$ denotes the product of f and g. (See Whitehead [22].)

(3.6) Postnikov system of an H-space. Let X be an H-space. Then X is equipped with a multiplication $h : X \times X \longrightarrow X$ such that the base point acts as a unit. If X and Y are H-spaces, then a map $f : X \longrightarrow Y$ is called an H-map if $f h_X \simeq h_Y (f \times f)$. It is proved by Kahn [6] that if X is an H-space and $\{X_i, f_i, g_i\}$ is a Postnikov system for X, then each X_i can be given an H-space structure in such a way that :

(a) for all i, f_i and g_i are H-maps,

(b) for all i, the k-invariant $k^i \in H^{i+1}(X_{i-1}; \pi_i(X))$ is primitive with respect to the multiplication on X_{i-1}.

(3.7) Vanishing of k-invariant. If X is a space and

$$\begin{array}{c} X_i \\ \Big\downarrow {\scriptstyle g_i} \\ X_{i-1} \xrightarrow{\ k^i\ } K(\pi_i(X), i+1) \end{array}$$

is the i-th stage of a Postnikov system for X, then $k^i = 0$ if and only if the Hurewicz map $h : \pi_i(X) \longrightarrow H_i(X)$ is a monomorphism onto a direct summand.

Proof. Serre [12] constructs for any fibre space such that the fundamental group of the base acts trivially on the homology of the fibre an exact sequence of homology groups. In the case of the fibration $g_i : X_i \longrightarrow X_{i-1}$, with coefficient group $\pi_i(X)$, we obtain

$$\ldots \longrightarrow H^i(X_i; \pi_i(X)) \xrightarrow{j^*} H^i(K(\pi_i(X), i); \pi_i(X)) \xrightarrow{\tau} H^{i+1}(X_{i-1}; \pi_i(X)) .$$

Here $j : K(\pi_i(X), i) \longrightarrow X_i$ is the inclusion of the fibre and τ is the transgression. The fundamental group of X_{i-1} acts trivially on the homology of the fibre because the fibre space is induced from the path-space fibration over $K(\pi_i(X), i+1)$. The sequence is exact in the range needed even if X_{i-1} is not simply-connected, as a simple argument using the Serre spectral sequence will show.

Let $\imath \in H^i(K(\pi_i(X), i); \pi_i(X))$ denote the fundamental class. The natural isomorphism

$$H^i(K(\pi_i(X), i); \pi_i(X)) \;\cong\; \mathrm{Hom}(\pi_i(X), \pi_i(X))$$

sends $\imath$ to the identity map. Consider the square below:

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d, "K(\pi_i(X), i)"'] & PK(\pi_i(X), i+1) \arrow[d, "K(\pi_i(X), i)"] \\
X_{i-1} \arrow[r] & K(\pi_i(X), i+1)
\end{tikzcd}$$

In the path-space fibration the fundamental classes of the fibre and base space correspond under transgression; the k-invariant $k^i \in H^{i+1}(X_{i-1}; \pi_i(X))$ is by definition the pull-back of the fundamental class of the base space $K(\pi_i(X), i+1)$. It follows that $\tau(\imath) = k^i$.

First suppose $k^i = 0$. Then by Serre's exact sequence there is a class $x \in H^i(X_i; \pi_i(X))$ such that $j^* x = \imath$. The action of x on the homology of X_i gives a map such that the diagram below is commutative:

$$\begin{tikzcd}
 & \pi_i(X) \arrow[d, "\cong"] \\
H_i(X_i) \arrow[ur, "x"] & \\
 & H_i(K(\pi_i(X), i)) \arrow[ul, "j_*"]
\end{tikzcd}$$

But j_* is essentially the Hurewicz homomorphism $h : \pi_i(X) \longrightarrow H_i(X)$ by remark (b) above, so x gives a splitting map and h is a monomorphism onto a direct summand.

Now assume that h is a monomorphism onto a direct summand. Then so is j_*, and we may choose a splitting map

$$p \;:\; H_i(X_i) \longrightarrow \pi_i(X)$$

such that the diagram above is commutative with p in place of x. The universal coefficient theorem implies that there is a class $x \in H^i(X_i; \pi_i(X))$ whose action on

$H_i(X_i)$ is the map p. It follows that $j^*x = \imath$, since $K(\pi_i(X), i)$ has no homology in dimension $i-1$. Thus $k^i = 0$ and the proof is complete.

(3.8) Order of k-invariant. Let X be a space such that $\pi_i(X) = \mathbb{Z}$ for some i. Then the i-th k-invariant k^i in any Postnikov system for X is of finite order if and only if there is a cohomology class in $H^i(X)$ which takes a non-zero value on the generator of $\pi_i(X)$. The order of k^i is equal to the least positive integer d such that there is a cohomology class in $H^i(X)$ which takes the value d on the generator of $\pi_i(X)$.

Proof. Let

$$\begin{tikzcd} X_i \arrow[d, "K(\mathbb{Z},i)"'] & \\ X_{i-1} \arrow[r] & K(\mathbb{Z}, i+1) \end{tikzcd}$$

be the i-th stage of a Postnikov system for X. We have as in 3.7 an exact sequence

$$\ldots \longrightarrow H^i(X_i) \xrightarrow{j^*} H^i(K(\mathbb{Z}, i)) \xrightarrow{\tau} H^{i+1}(X_{i-1})$$

and the fundamental class $\imath \in H^i(K(\mathbb{Z}, i))$ transgresses to k^i. It follows from remark (b) above that after identifying $H^i(X)$ with $H^i(X_i)$ the map j^* can be regarded as evaluation of $H^i(X)$ on $\pi_i(X)$. In other words

$$\begin{tikzcd} H^i(X_i) \arrow[r, "j^*"] \arrow[d, "(f_i)^*"'] & H^i(K(\mathbb{Z}, i)) \arrow[d, "\cong"] \\ H^i(X) \arrow[r, "\text{eval.}"] & \mathrm{Hom}(\pi_i(X), \mathbb{Z}) \end{tikzcd}$$

is commutative. Thus to prove the first part of (3.8) we have

$$\begin{aligned} k^i \text{ is of infinite order } &\iff \tau \text{ is a monomorphism} \\ &\iff j^* = 0 \\ &\iff \text{every cohomology class in } H^i(X) \\ &\qquad \text{takes the value } 0 \text{ on } \pi_i(X)\,. \end{aligned}$$

To prove the second statement of (3.8) we have

$$\begin{aligned} \text{the order of } k^i \text{ divides } d &\iff j^*x = d\imath \text{ for some } x \in H^i(X_i) \\ &\iff \text{some cohomology class in} \\ &\qquad H^i(X) \text{ takes the value } d \text{ on} \\ &\qquad \text{the generator of } \pi_i(X)\,. \end{aligned}$$

§4. Application to G/PL and the Hauptvermutung

Recall that in Chapter II it is proved that the periodicity map

$$\mu \ : \ G/PL \longrightarrow \overline{\Omega}^{4n}(G/PL)$$

has fibre an Eilenberg-MacLane space $K(\mathbb{Z}_2, 3)$. Furthermore, the composition

$$TOP/PL \xrightarrow{\ i\ } G/PL \xrightarrow{\ \mu\ } \overline{\Omega}^{4n}(G/PL)$$

is null-homotopic. Now let M and Q be PL manifolds, and assume that

$$h \ : \ Q \longrightarrow M$$

is a topological homeomorphism. Associated to h is a map

$$q_h \ : \ M_0 \longrightarrow G/PL$$

(where $M_0 = M\backslash\{\text{disc}\}$ if $\partial M = \emptyset$, $M_0 = M$ if $\partial M \neq \emptyset$) and, under certain conditions on M, h is homotopic to a PL homeomorphism if and only if q_h is homotopic to a constant. (For example, it is enough to assume:

(1) M and Q are closed of dimension ≥ 5 or bounded of dim ≥ 6

(2) $\pi_1(M) = \pi_1(\partial M) = 0$.)

In this section we study the question of whether the map q_h is null-homotopic. By Chapter II the map q_h factors through TOP/PL:

$$\begin{tikzcd} & TOP/PL \arrow[dr, "i"] & \\ M_0 \arrow[ur, dashed] \arrow[rr, "q_h"] & & G/PL \end{tikzcd}$$

and so the composition $\mu q_h : M_0 \longrightarrow \overline{\Omega}^{4n} G/PL$ is null-homotopic. By Theorems 2.1 and 2.2, this means that

(a) q_h lifts to a map into the total space of the $\Omega\overline{\Omega}^{4n}(G/PL)$ fibration induced by μ. This total space is just the fibre of the map μ and so we denote it by $K(\mathbb{Z}_2, 3)$,

(b) different liftings of q_h are related via the action of $\Omega\overline{\Omega}^{4n}(G/PL)$ on $K(\mathbb{Z}_2, 3)$.

Now any lifting of q_h to a map into $K(\mathbb{Z}_2, 3)$ defines a cohomology class in $H^3(M_0; \mathbb{Z}_2)$. We shall prove:

(4.1) Theorem. *The collection of cohomology classes defined by liftings of q_h is a coset of the subgroup of mod 2 reductions of integral classes in M_0 and so determines an element $V_h \in H^4(M_0; \mathbb{Z})$ of order 2. The map q_h is null-homotopic if and only if $V_h = 0$.*

The theorem above can be restated in terms of the obstruction theory defined by Sullivan [16]. Let

$$o^i(q_h) \in H^i(M_0; \pi_i(G/PL))$$

denote the i-th obstruction to deforming q_h to a constant. The theorem just stated implies that in our case (where h is a topological homeomorphism):

(a) $o^2(q_h) = 0$,
(b) $o^4(q_h) \in H^4(M_0; \mathbb{Z})$ is equal to V_h and is an element of order 2,
(c) if $o^4(q_h) = 0$ then all the higher obstructions vanish.

We begin with a study of the k-invariants of G/PL. Let

$$\{X_i\}\ ,\ \{f_i : G/PL \longrightarrow X_i\}\ ,\ \{g_i : X_i \longrightarrow X_{i-1}\}$$

be a Postnikov system for G/PL. Let

$$x^i \in H^{i+1}(X_{i-1}; \pi_i(G/PL))$$

denote the i-th k-invariant. Recall that $\pi_{4i+2}(G/PL) = \mathbb{Z}_2$, $\pi_{4i}(G/PL) = \mathbb{Z}$, and the odd groups are zero.

(4.2) Theorem. *For all i, $x^{4i+2} = 0$.*

Proof. This theorem was proved in Sullivan's thesis [16], and we reproduce the proof here. By 3.7 it is sufficient to show that the Hurewicz homomorphism

$$h\ :\ \mathbb{Z}_2\ =\ \pi_{4i+2}(G/PL) \longrightarrow H_{4i+2}(G/PL)$$

is a monomorphism onto a direct summand. But that is true if and only if the mod 2 Hurewicz homomorphism

$$h_2\ :\ \mathbb{Z}_2\ =\ \pi_{4i+2}(G/PL) \longrightarrow H_{4i+2}(G/PL) \xrightarrow{\text{mod 2 reduction}} H_{4i+2}(G/PL; \mathbb{Z}_2)$$

is a monomorphism.

Consider the following diagram:

$$\begin{tikzcd}
& \Omega_{4i+2}(G/PL) \arrow[d, "\times 2"'] & \\
& \Omega_{4i+2}(G/PL) \arrow[r] \arrow[d] & H_{4i+2}(G/PL) \arrow[d] \\
\pi_{4i+2}(G/PL) \arrow[ur, "h_0"] \arrow[r, "h_1"'] & \mathfrak{N}_{4i+2}(G/PL) \arrow[r] & H_{4i+2}(G/PL; \mathbb{Z}_2)
\end{tikzcd}$$

Here Ω and $\mathfrak{N}$ denote oriented and unoriented smooth bordism respectively. The column in the middle is exact by a result of Conner and Floyd [3]. The surgery obstruction gives a splitting

$$\begin{tikzcd} \pi_{4i+2}(G/PL) \arrow[rr] & & \Omega_{4i+2}(G/PL) \arrow[ll, bend right, dashed] \end{tikzcd}$$

and so h_0 is onto a direct summand $\mathbb{Z}_2$. By the exactness of the middle column, h_1 is non-zero. Thus the generator $f : S^{4i+2} \longrightarrow G/PL$ of π_{4i+2} does not bound a singular manifold in G/PL. According to Theorem 17.2 of [3], at least one of the Whitney numbers associated to the singular manifold $[S^{4i+2}, f]_2$ must be non-zero. Since all of the Stiefel-Whitney classes of S^{4i+2} vanish (except w_0 which is 1), this implies that $f_*[S^{4i+2}]$ is non-zero in $\mathbb{Z}_2$-homology. Thus the mod 2 Hurewicz homomorphism is a monomorphism and Theorem 4.2 is proved.

(4.3) Theorem. *For all $i > 1$, $x^{4i} \in H^{4i+1}(X_{4i-1}; \mathbb{Z})$ is of odd order.*

Proof. By 3.8 it is sufficient to find a cohomology class in $H^{4i}(G/PL)$ which takes an odd value on the generator of $\pi_{4i}(G/PL)$.

We shall prove that such cohomology classes exist as follows: we construct, for each i, a cohomology class

$$\ell_i \in H^{4i}(G/PL; \mathbb{Z}_{(2)}) ,$$

where $\mathbb{Z}_{(2)}$ = integers localized at 2 = the ring of rationals with odd denominators. The classes ℓ_i shall be constructed so that ℓ_i takes the value 1 on the generator of $\pi_{4i}(G/PL)$ for $i > 1$. Since the homology of G/PL is finitely generated, a sufficiently large odd multiple of ℓ_i is then the reduction of an integral class which takes an odd value on the generator of $\pi_{4i}(G/PL)$.

(4.4) Theorem. *There exist classes $\ell_i \in H^{4i}(G/PL; \mathbb{Z}_{(2)})$ for each $i \geq 0$ such that if we write*

$$\mathcal{L} \;=\; \ell_0 + \ell_1 + \ell_2 + \cdots \in H^{4*}(G/PL; \mathbb{Z}_{(2)})$$

then for any map $f : M^{4k} \longrightarrow G/PL$ of a smooth manifold the surgery obstruction of the map f (see Chapter II) is given by

$$(1) \qquad s(f) \;=\; 8\langle L(M) \cup f^*\mathcal{L}, [M]\rangle$$

where $L(M)$ is the L-genus of Hirzebruch [5, II§8] applied to the Pontrjagin classes of M.

Now if $\alpha_i \in \pi_{4i}(G/PL)$ is a generator then the surgery obstruction of the map $\alpha_i : S^{4i} \longrightarrow G/PL$ is 16 if $i = 1$ and 8 if $i > 1$. Since the Pontrjagin classes of S^{4i} are trivial, Theorem 4.4 implies that

$$\langle \ell_i, \alpha_i \rangle \;=\; \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i > 1 \end{cases}$$

and so Theorem 4.3 follows.

We now prove Theorem 4.1, postponing the proof of Theorem 4.4. Consider the diagram below

$$\begin{tikzcd} & K(\mathbb{Z}_2,3) \arrow[d, "\pi"] \\ M_0 \arrow[ur, dashed] \arrow[r, "q_h"] & G/PL \end{tikzcd}$$

Let $\imath_3 \in H^3(K(\mathbb{Z}_2,3);\mathbb{Z}_2)$ be the fundamental class. Choose a lifting $f_1 : M_0 \longrightarrow K(\mathbb{Z}_2,3)$. We prove first that given any other lifting $f_2 : M_0 \longrightarrow K(\mathbb{Z}_2,3)$ of q_h, we have

(a) $f_2^*\imath_3 - f_1^*\imath_3 =$ reduction of an integral cohomology class.

Proof of (a). By Theorem 2.2, there is a map $g : M_0 \longrightarrow \Omega\overline{\Omega}^{4n}(G/PL)$ such that the composition

$$\begin{tikzcd} M_0 \arrow[r, "g \times f_1"] & \Omega\overline{\Omega}^{4n}(G/PL) \times K(\mathbb{Z}_2,3) \arrow[d, "m"'] \\ & K(\mathbb{Z}_2,3) \end{tikzcd}$$

is homotopic to f_2. We have

$$m^*\imath_3 \;=\; j^*\imath_3 \times 1 + 1 \times \imath_3$$

(where $j : \Omega\overline{\Omega}^{4n}(G/PL) \longrightarrow K(\mathbb{Z}_2,3)$ is the inclusion of the fibre) for dimension reasons. We evaluate $j^*\imath_3$. Since

$$\pi_1(\Omega\overline{\Omega}^{4n}(G/PL)) \;=\; \mathbb{Z}_2 \;\;,\;\; \pi_3(\Omega\overline{\Omega}^{4n}(G/PL)) \;=\; \mathbb{Z}$$

and the even groups are zero, a Postnikov system for $\Omega\overline{\Omega}^{4n}(G/PL)$ looks like

$$\begin{tikzcd} \Omega\overline{\Omega}^{4n}(G/PL) \arrow[r] \arrow[dr] & E_3 \arrow[d, "K(\mathbb{Z},3)"] \\ & K(\mathbb{Z}_2,1) \end{tikzcd}$$

in low dimensions. The k-invariant $k^3 \in H^4(K(\mathbb{Z}_2,1);\mathbb{Z})$ is the $(4n+1)$-st suspension of the k-invariant x^{4n+4} of G/PL, by 3.4. Since x^{4n+4} is of odd order, so is k^3 (here we assume $n \geq 1$ and apply Theorem 4.3) and since k^3 lies in a 2-primary group it must be zero. Thus E_3 is a product. Hence

$$H^3(E_3;\mathbb{Z}_2) \;\cong\; \mathbb{Z}_2 \oplus \mathbb{Z}_2 \;,$$

generated by $\imath_1^3$ and $\bar{\imath}$, where $\imath_1 \in H^1(K(\mathbb{Z}_2,1);\mathbb{Z}_2)$ is the fundamental class and $\bar{\imath} \in H^3(K(\mathbb{Z},3);\mathbb{Z}_2)$ is the mod 2 reduction of the fundamental class $\imath \in H^3(K(\mathbb{Z},3);\mathbb{Z})$. Thus we may write

$$j^*\imath_3 \;=\; a\imath_1^3 + b\bar{\imath} \;\; (a,b \in \mathbb{Z}_2)\ .$$

Now $\pi_3(\Omega\overline{\Omega}^{4n}(G/PL)) \cong \mathbb{Z}$ maps onto $\pi_3(K(\mathbb{Z}_2,3)) \cong \mathbb{Z}_2$ since $\pi_3(G/PL) = 0$. It follows that $b = 1$. To evaluate a, we consider the fibre of the map j, which has the homotopy type of $\Omega(G/PL)$. We obtain a sequence of spaces:

$$\Omega(G/PL) \xrightarrow{j'} \Omega\overline{\Omega}^{4n}(G/PL) \xrightarrow{j} K(\mathbb{Z}_2,3) \xrightarrow{\pi} G/PL\ .$$

We shall show that $(j')^*\imath_1^3 \neq 0$, $(j')^*\bar{\imath} = 0$. Then, since $jj' \simeq *$, it follows that $a = 0$. Let

$$\begin{tikzcd}
\Omega(G/PL) \arrow[r] \arrow[dr] & E_3' \arrow[d, "K(\mathbb{Z},3)"] \\
& K(\mathbb{Z}_2,1)
\end{tikzcd}$$

be a section of a Postnikov system for $\Omega(G/PL)$ obtained by looping the corresponding section of a Postnikov system for G/PL. The k-invariant $k^3 \in H^4(K(\mathbb{Z}_2, 1);\mathbb{Z})$ is then the suspension σx^4 of the k-invariant $x^4 \in H^5(K(\mathbb{Z}_2,2);\mathbb{Z})$ for G/PL. We shall prove later (Theorem 4.6) that $x^4 = \delta Sq^2\imath_2$, where $\imath_2$ is the fundamental class and δ is the Bockstein operation associated to the coefficient sequence

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2\ .$$

Since δ and Sq^2 commute with suspension,

$$k^3 \;=\; \sigma x^4 \;=\; \delta Sq^2\imath_1 \;=\; 0\ .$$

Thus E_3' is a product. Now associated to the map j is a map $E_3' \longrightarrow E_3$ by 3.2. This map multiples by 2 in π_3 and is an isomorphism on π_1, so a calculation gives

$$(j')^*\imath_1 \;=\; \imath_1 \;\;,\;\; (j')^*\imath_1^3 \neq 0 \;\;,\;\; (j')^*\bar{\imath} \;=\; 0\ .$$

This completes the proof that $a = 0$, and we may write

$$m^*\imath_3 \;=\; \bar{\imath} \times 1 + 1 \times \imath_3\ .$$

Then since $f_2 \simeq m(g \times f_1)$ we have

$$\begin{aligned} f_2^*\imath_3 \;&=\; (g \times f_1)^*(\bar{\imath} \times 1 + 1 \times \imath_3) \\ &=\; g^*\bar{\imath} + f_1^*\imath_3\ . \end{aligned}$$

The difference $f_2^*\imath_3 - f_1^*\imath_3$ is thus the reduction of the integral class $g^*\imath$, and (a) is proved.

We complete the proof of Theorem 4.1 by showing that given any lifting f_1 of q_h and any class $u \in H^3(M_0;\mathbb{Z}_2)$ such that $u = \overline{v}$, $v \in H^3(M_0;\mathbb{Z})$, then

(b) there is a lifting f_2 of q_h such that

$$f_2^*\imath_3 - f_1^*\imath_3 \;=\; u\ .$$

Proof of (b). Since $m^*\imath_3 = \bar{\imath} \otimes 1 + 1 \otimes \imath_3$ we need only find a map

$$g \;:\; M_0 \longrightarrow \Omega\overline{\Omega}^{4n}(G/PL)$$

such that $g^*\imath = (2c+1)v$ for some integer c. For then the map $f_2 = m(g \times f_1)$ satisfies

$$\begin{aligned} f_2^*\imath_3 \;&=\; g^*\bar{\imath} + f_1^*\imath_3 \\ &=\; (2c+1)\overline{v} + f_1^*\imath_3 \\ &=\; u + f_1^*\imath_3\ . \end{aligned}$$

Construction of the map g. Let $\{E_i\}$ denote the stages of a Postnikov system for $\Omega\overline{\Omega}^{4n}(G/PL)$. It was shown above that E_3 is a product, and so there exists a map

$$g_3 \;:\; M_0 \longrightarrow E_3$$

such that $g_3^*\imath = v$. Now each E_i is assumed to be a loop space, and so for any i, $[M_0, E_i]$ is a group. The map g_3 is constructed by lifting odd multiples of g_3 to successively higher stages E_i. Suppose we have obtained a map $g_i : M_0 \longrightarrow E_i$. The obstruction to lifting g_i

$$\begin{tikzcd} & E_{i+1} \arrow[d] \\ M_0 \arrow[ur, dashed] \arrow[r, "g_i"] & E_i \end{tikzcd}$$

is equal to $g_i^*k^i$, where k^i is the i-th k-invariant. Now the k-invariants are either zero or of odd order. In a case where $k^i = 0$, there is no obstruction and g_i lifts. If $k^i \neq 0$ and is of odd order $2d+1$, then the map $(2d+1)g_i$ obtained by multiplying g_i with itself $(2d+1)$ times in the group $[M_0, E_i]$ satisfies

$$((2d+1)g_i)^*k^i \;=\; (2d+1)g_i^*k^i \;=\; 0$$

since k^i is primitive (see 3.5). Since M_0 is finite dimensional the obstructions vanish after a finite number of iterations of this procedure. It follows that an odd multiple of g_3, say $(2c+1)g_3$, lifts to a map

$$g \;:\; M_0 \longrightarrow \Omega\overline{\Omega}^{4n}(G/PL)$$

Now the class $\imath \in H^3(\Omega\overline{\Omega}^{4n}(G/PL))$ is primitive. This is true because it is actually a suspension; we argued previously that the third k-invariant $k^3 \in H^4(K(\mathbb{Z}_2,1);\mathbb{Z})$

is trivial because it is of odd order and 2-primary. The same is true of the fourth k-invariant in $H^5(K(\mathbb{Z}_2,2);\mathbb{Z})$ of $\overline{\Omega}^{4n}(G/PL)$. Thus the fourth stage of a Postnikov system for $\overline{\Omega}^{4n}(G/PL)$ splits as a product $K(\mathbb{Z}_2,2)\times K(\mathbb{Z},4)$, and the fundamental class of $K(\mathbb{Z},4)$ suspends to $\imath \in H^3(E_3)$. Since $\imath$ is primitive we have

$$g^*\imath \;=\; (2c+1)g_3^*\imath \;=\; (2c+1)v$$

and the proof of (b) is complete.

Proof of Theorem 4.4. We first recall some results on the smooth oriented bordism groups $\Omega_*(X)$ of a space X. The reader is referred to Conner and Floyd [3] or Stong [15, Chapter IX] for definitions.

Let MSO_k denote the Thom complex of the universal oriented k-plane bundle over BSO_k. The spectrum $MSO = \{MSO_k\}$ classifies the bordism groups of a space X in that

$$\Omega_*(X) \;=\; \pi_*(X_+ \wedge MSO)\ ,$$

with $X_+ = X \cup \{\text{pt.}\}$. Let $\mathbb{K}(\mathbb{Z},0)$ denote the Eilenberg-MacLane spectrum with k-th space $K(\mathbb{Z},k)$. For any connected space X we have

$$\pi_*(X_+ \wedge \mathbb{K}(\mathbb{Z},0)) \;=\; H_*(X)\ .$$

The Thom class $U \in H^0(MSO)$ induces a map $MSO \longrightarrow \mathbb{K}(\mathbb{Z},0)$ which on any space X yields the Hurewicz homomorphism $h : \Omega_*(X) \longrightarrow H_*(X)$. Now $\Omega_*(X)$ is an $\Omega_*(pt)$-module. An element in $\Omega_*(X)$ is **decomposable** if it is a linear combination of elements of the form

$$N_1 \times N_2 \xrightarrow{\ p_2\ } N_2 \xrightarrow{\ g\ } X$$

where $\dim N_1 > 0$. We need the following result, writing

$$\underline{G} \;=\; G/\text{torsion} \otimes \mathbb{Z}_{(2)}$$

for any group G.

Theorem 4.5. *For any space X the Hurewicz homomorphism $h : \Omega_*(X) \longrightarrow H_*(X)$ induces an epimorphism*

$$\underline{h} \;:\; \underline{\Omega_*(X)} \longrightarrow \underline{H_*(X)}$$

with kernel generated by decomposables.

Proof. According to Stong [15, p.209] the Hurewicz homomorphism in MSO induces a monomorphic map of graded rings

$$h \;:\; \pi_*(MSO)/\text{torsion} \longrightarrow H_*(MSO)/\text{torsion}$$

with finite odd order cokernel in each dimension, so that

$$\underline{h} \;:\; \underline{\pi_*(MSO)} \longrightarrow \underline{H_*(MSO)}$$

is an isomorphism of graded rings. For any space X we have a commutative diagram:

$$
\begin{tikzcd}
\underline{\Omega_*(X)} \arrow[r, "h"] \arrow[d, "\approx"] & \underline{H_*(X)} \arrow[ddd, "\approx"] \\
\underline{\pi_*(X_+ \wedge MSO)} \arrow[d, "\approx"] & \\
\underline{H_*(X_+ \wedge MSO)} \arrow[d, "\approx"] & \\
\underline{H_*(X)} \otimes \underline{H_*(MSO)} \arrow[r, "1 \otimes U"] & \underline{H_*(X)} \otimes \underline{H_*(\mathbb{K}(\mathbb{Z},0))}
\end{tikzcd}
$$

The vertical maps on the left are isomorphisms of $\underline{\pi_*(MSO)}$ (or $\underline{H_*(MSO)}$)-modules. The kernel of $1 \otimes U$ consists of decomposables, and so the same is true of the kernel of h.

The classes $\ell_i \in H^{4i}(G/PL; \mathbb{Z}_{(2)})$ are constructed inductively. Set $\ell_0 = 0$. Then the conclusion (1) of Theorem 4.4 holds for manifolds of dimension zero. Suppose that $\ell_0, \cdots, \ell_{i-1}$ have been defined in such a way that the conclusion of Theorem 4.4 holds for manifolds of dimension $4k$, $k < i$. We define the cohomology class ℓ_i as follows. The formula (1) forces the action of ℓ_i on the $4i$-th bordism group of G/PL; ℓ_i must map $\Omega_{4i}(G/PL)$ to $\mathbb{Z}_{(2)}$ by the homomorphism ℓ' which is defined by

$$\ell'[M^{4i}, f] \;=\; \frac{s(f)}{8} - \langle L(M) \cup f^* \sum_{j<i} \ell_j, [M] \rangle$$

for any $[M^{4i}, f] \in \Omega_{4i}(G/PL)$. The values taken by ℓ' lie in $\mathbb{Z}_{(2)}$ because the Hirzebruch polynomials have coefficients in $\mathbb{Z}_{(2)}$. Now suppose that $[M^{4i}, f]$ is a boundary. Then there is a smooth manifold W^{4i+1} with boundary $\partial W^{4i+1} = M^{4i}$ and a map $F : W^{4i+1} \longrightarrow G/PL$ such that $F|_{\partial W} = f$. Now the surgery obstruction $s(f)$ vanishes because it is a cobordism invariant. Let $i : M \subseteq W$ denote the inclusion. Then

$$
\begin{aligned}
\langle L(M) \cup f^* \sum_{j<i} \ell_j, [M] \rangle \;&=\; \langle i^* L(W) \cup i^* F^* \sum_{j<i} \ell_j, [M] \rangle \\
&=\; \langle L(W) \cup F^* \sum_{j<i} \ell_j, i_*[M] \rangle \\
&=\; 0 \;\; (\text{since } i_*[M] = 0) \; .
\end{aligned}
$$

Thus ℓ' vanishes on boundaries and is well-defined on $\Omega_{4i}(G/PL)$. Since ℓ' must map torsion to zero it induces a homomorphism

$$\underline{\ell'} \ : \ \underline{\Omega_{4i}(G/PL)} \longrightarrow \mathbb{Z}_{(2)} \ .$$

Theorem 4.5 states that $\underline{\ell'}$ induces a map

$$\ell \ : \ \underline{H_{4i}(G/PL)} \longrightarrow \mathbb{Z}_{(2)}$$

if and only if ℓ' vanishes on decomposables. We assume for the moment that ℓ' vanishes on decomposables. The universal coefficient theorem states that the evaluation map

$$H^{4i}(G/PL;\mathbb{Z}_{(2)}) \longrightarrow \mathrm{Hom}(H_{4i}(G/PL);\mathbb{Z}_{(2)}) \tag{2}$$

is onto. Thus there exists a cohomology class ℓ_i whose action on $H_{4i}(G/PL)$ is the composition

$$H_{4i}(G/PL) \longrightarrow \underline{H_{4i}(G/PL)} \xrightarrow{\ \ell\ } \mathbb{Z}_{(2)} \ .$$

The action of ℓ_i on $\Omega_{4i}(G/PL)$ is then exactly what is needed to satisfy (1) for manifolds of dimension $4k$, $k \le i$.

Proof that ℓ' vanishes on decomposables. The decomposables of $\Omega_{4i}(G/PL;\mathbb{Z}_{(2)})$ are linear combinations of elements of the form

$$N_1^{4i-n} \times N_2^n \xrightarrow{\ p_2\ } N_2^n \xrightarrow{\ g\ } G/PL$$

where $n < 4i$ and p_2 is projection onto the second coordinate. To evaluate ℓ' on $[N_1 \times N_2, gp_2]$, we note that the L-genus is multiplicative and the Pontrjagin classes satisfy a Whitney sum formula modulo 2-torsion and so

$$L(N_1 \times N_2) \ = \ L(N_1) \times L(N_2)$$

modulo 2-torsion. Thus

$$\begin{aligned} \ell'[N_1 \times N_2, gp_2] \ &= \ \frac{s(gp_2)}{8} - \langle L(N_1) \times (L(N_2) \cup g^* \sum_{j<i} \ell_j), [N_1] \times [N_2]\rangle \\ &= \ \frac{s(gp_2)}{8} - \langle L(N_1), [N_1]\rangle \cdot \langle L(N_2) \cup g^* \sum_{j<i} \ell_j, [N_2]\rangle \ . \end{aligned} \tag{3}$$

First assume that $n \not\equiv 0 (\mathrm{mod}\, 4)$. Then $s(gp_2) = 0$ by the product formula for the index surgery obstruction of Rourke and Sullivan [11, Theorem 2.1]. Also, $\langle L(N_1), [N_1]\rangle = I(N_1) = 0$, so that both terms of (3) vanish and $\ell'[N_1 \times N_2, gp_2] = 0$.

Next assume $n \equiv 0 (\mathrm{mod}\, 4)$. If $n = 0$ then both terms of (3) are obviously zero. If $n > 0$ then

$$\frac{s(gp_2)}{8} \ = \ I(N_1) \cdot \frac{s(g)}{8}$$

by the product formula and

$$\langle L(N_1),[N_1]\rangle \cdot \langle L(N_2)\cup g^*\sum_{j<i}\ell_j,[N_2]\rangle \;=\; I(N_1)\cdot\frac{s(g)}{8}$$

by the inductive hypothesis (N_2 is a manifold of dimension $4j$ for some $j<i$). Thus $\ell'[N_1\times N_2, gp_2]=0$, and ℓ' vanishes on decomposables. The proof of Theorem 4.4 is complete.

Remarks on Theorem 4.4. (i) Since the evaluation map (2) has kernel a torsion group, the ℓ_i are unique up to the addition of torsion elements.
(ii) There are classes $L_i^{PL}\in H^{4i}(BPL;\mathbb{Q})$ which pull back to the L-genus in BO. (See Milnor and Stasheff [7].) The natural map $\pi: G/PL\longrightarrow BPL$ then satisfies

(4) $$\pi^*(L^{PL}-1) \;=\; 8\overline{\mathcal{L}} \qquad \text{(Sullivan [17], p.29)}$$

where $\overline{\mathcal{L}}$ denotes the image of $\mathcal{L}$ in $H^*(G/PL;\mathbb{Q})$.

Proof of (4). By our first remark we only need to verify that $\pi^*(L^{PL}-1)$ can be used to calculate the surgery obstruction for smooth manifolds. Let M^{4k} be a PL manifold, $f: M^{4k}\longrightarrow G/PL$ a map. The composition πf is a stable PL bundle over M. Then let ν_M be the stable normal bundle. We obtain a stable bundle $\nu_M-\pi f$ over M and the fibre homotopy trivialization of πf determines a normal invariant in $\pi_*(T(\nu_M-\pi f))$. The resulting surgery problem is the normal map associated to the map f. (See Chapter II.) The surgery obstruction of the map f is thus equal to $[I(\nu_M-\pi f)-I(M)]$. The "index" of a stable bundle ξ over M^{4k} is defined by

$$I(\xi) \;=\; \langle L_k^{PL}(-\xi),[M^{4k}]\rangle\ .$$

Thus

$$\begin{aligned} s(f) &= \langle L_k^{PL}(\tau_M+\pi f),[M^{4k}]\rangle - I(M)\\ &= \langle L^{PL}(\tau_M)\cup L^{PL}(\pi f),[M^{4k}]\rangle - I(M)\\ &= \langle L^{PL}(M)\cup(L^{PL}(\pi f)-1),[M^{4k}]\rangle \end{aligned}$$

since $\langle L^{PL}(\tau_M),[M]\rangle = I(M)$. But $L^{PL}(\pi f)-1 = f^*\pi^*(L^{PL}-1)$, so we have proved the desired formula for the surgery obstruction. We have also proved that Theorem 4.4 holds for PL manifolds.
(iii) Let M^{4k} be a manifold, smooth or PL. Then $[M,G/PL]$ forms a group via Whitney sum, and so it is natural to ask whether the surgery obstruction

$$s \;:\; [M,G/PL] \longrightarrow \mathbb{Z}$$

is a homomorphism. The answer is in general no. Since $L^{PL}\in H^*(BPL;\mathbb{Q})$ is multiplicative it follows from (4) that

$$h^*(\overline{\mathcal{L}}) \;=\; \overline{\mathcal{L}}\times 1 + 1\times\overline{\mathcal{L}} + \overline{\mathcal{L}}\times\overline{\mathcal{L}}$$

where $h: G/PL\times G/PL\longrightarrow G/PL$ is the multiplication in G/PL induced by

Whitney sum. Thus if $fg \in [M, G/PL]$,

$$s(f \cdot g) \;=\; s(f) + s(g) + 8\langle L(M) \cup f^*\overline{\mathcal{L}} \cup g^*\overline{\mathcal{L}}, [M]\rangle \ .$$

(iv) By 3.8 the order of $x^{4i} \in H^{4i+1}(X_{4i-1}; Z)$ divides $\langle u, \alpha_i\rangle$ for any integral class $u \in H^{4i}(G/PL)$. Let ν_i denote the least positive integer such that $\nu_i \ell_i$ is integral. Then ν_i is of course always odd. We obtain a bound on ν_i as follows. By (ii) above $\pi^*(L^{PL} - 1) = 8\overline{\mathcal{L}}$. Let μ_i denote the least positive integer such that $\mu_i L_i^{PL}$ is integral. Brumfiel [2] has proved that

$$\mu_i \;=\; \prod_p \,[\frac{4i}{2(p-1)}]$$

where the product is taken over all odd primes $p \leq 2i+1$. Since ν_i divides μ_i we have: the order of x^{4i} is a divisor of μ_i. (The precise order of x^{4i} can be computed using a result due to Dennis Sullivan, that G/PL and BO have the same homotopy type in the world of odd primes. It follows that the order of x^{4i} is the odd part of $(2i-1)!$, for $i > 1$.)

We conclude with a calculation of the fourth k-invariant $x^4 \in H^5(K(\mathbb{Z}_2, 2); \mathbb{Z})$ of G/PL. The following theorem is due to Sullivan [17].

(4.6) Theorem. *$x^4 = \delta Sq^2 \imath_2$, where $\imath_2 \in H^2(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$ is the fundamental class and δ is the Bockstein operation associated to the coefficient sequence*

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \ .$$

Proof. Consider a section of the Postnikov system for G/PL

$$\begin{array}{ccc} G/PL & \longrightarrow & X_4 \\ & & \Big\downarrow K(\mathbb{Z},4) \\ & & X_2 = K(\mathbb{Z}_2, 2) \end{array}$$

By 3.8 the order of x^4 is the smallest positive integer d such that there exists a cohomology class $u \in H^4(G/PL)$ satisfying $\langle u, \alpha_1\rangle = d$, where $\alpha_1 \in \pi_4(G/PL)$ is a generator. By Theorem 4.4 there is a class $\ell_1 \in H^4(G/PL; \mathbb{Z}_{(2)})$ such that $\langle \ell_1, \alpha_1\rangle = 2$. Since there is an odd multiple of ℓ_1 which is the reduction of an integral class, the order of x^4 divides an odd multiple of 2. But x^4 is in a 2-primary group. Thus $2x^4 = 0$. By the exactness of the sequence

$$H^4(K(\mathbb{Z}_2, 2); \mathbb{Z}_2) \xrightarrow{\delta} H^5(K(\mathbb{Z}_2, 2); \mathbb{Z}) \xrightarrow{\times 2} H^5(K(\mathbb{Z}_2, 2); \mathbb{Z})$$

there is a class $y \in H^4(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$ such that $\delta y = x^4$. But $H^4(K(\mathbb{Z}_2, 2); \mathbb{Z}_2) \cong \mathbb{Z}_2$ generated by $\imath_2^2 = Sq^2 \imath_2$ (Serre [13]), and so $x^4 = a\delta Sq^2 \imath_2$ for some $a \in \mathbb{Z}_2$. We complete the proof of Theorem 4.6 by showing that $x^4 \neq 0$.

The inclusion of the base point $i : * \subseteq G/PL$ induces a monomorphism

$$i_* \;:\; \Omega_4(*) \longrightarrow \Omega_4(G/PL)$$

and since the image of i_* is in the kernel of the Hurewicz homomorphism, there is a diagram

$$\begin{tikzcd}
\pi_4(G/PL) \arrow[r] \arrow[dr, "h'"'] & \Omega_4(G/PL) \arrow[r] \arrow[d] & H_4(G/PL) \\
& \operatorname{cok} i_* \arrow[ur] &
\end{tikzcd}$$

We have $\Omega_4(G/PL)/\text{torsion} = \mathbb{Z} \oplus \mathbb{Z}$, and so $\operatorname{cok} i_* \cong \mathbb{Z} \oplus$ finite group. Now the surgery obstruction $s : \Omega_4(G/PL) \longrightarrow \mathbb{Z}$ vanishes on $\operatorname{im} i_*$ and so induces a map $s' : \operatorname{cok} i_* \longrightarrow \mathbb{Z}$ such that

$$\begin{tikzcd}
\pi_4(G/PL) \arrow[rr, "h'"] \arrow[dr, "s"'] & & \operatorname{cok} i_* \arrow[dl, "s'"] \\
& \mathbb{Z} &
\end{tikzcd}$$

In order to prove that $x^4 \neq 0$, we show that $h' : \pi_4(G/PL) \longrightarrow \operatorname{cok} i_*$ is not an isomorphism onto a direct summand and apply 3.7. Since $\operatorname{cok} i_* \neq \mathbb{Z} \oplus$ finite group, we need only show that $\operatorname{im} s'$ properly contains $\operatorname{im} s$. Now $s(\alpha_1) = 16$, so that $\operatorname{im} s$ consists of multiples of 16. Thus it suffices to show

$(*)$ there exists a map $f : \mathbb{CP}^2 \longrightarrow G/PL$ such that $s(f) = -8$.

Proof of $(*)$. Let γ denote the canonical complex line bundle over $\mathbb{CP}^2$. The total Chern class of γ is $1 + x$, x a generator of $H^2(\mathbb{CP}^2)$, and so the first Pontrjagin class $p^1(r\gamma)$ of the realification of γ is $-x^2$. (The reader is referred to Milnor and Stasheff [7] for details.)

We show first that $24r\gamma$ is fibre homotopically trivial. The cofibration sequence

$$S^3 \xrightarrow{\ \eta\ } S^2 \longrightarrow \mathbb{CP}^2 \longrightarrow S^4 \xrightarrow{\ \Sigma\eta\ } S^3$$

induces an exact sequence

$$[S^3, BG] \xrightarrow{(\Sigma\eta)^*} [S^4, BG] \longrightarrow [\mathbb{CP}^2, BG] \longrightarrow [S^2, BG] \xrightarrow{\ \eta^*\ } [S^3, BG]\ .$$

We have

$$[S^3, BG] \;\cong\; \pi_2(G) \;\cong\; \pi_2^S \;=\; \mathbb{Z}_2 \;\; (\text{generated by } \eta^2)\ ,$$
$$[S^4, BG] \;\cong\; \pi_3(G) \;\cong\; \pi_3^S \;=\; \mathbb{Z}_{24} \;\; (\text{generated by } \nu)\ .$$

Since $\eta^3 = 12\nu$ (for example, see Toda [21]) the cokernel of $(\Sigma\eta)^*$ is isomorphic to $\mathbb{Z}_{12}$. We also have

$$[S^2, BG] \;\cong\; \pi_1(G) \;\cong\; \pi_1^S \;=\; \mathbb{Z}_2 \;\; \text{generated by } \eta\ ,$$

so that $[S^2, BG]$ is generated by the Hopf bundle. Since the pullback of the Hopf bundle

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\eta^*(\eta)"'] & S^3 \arrow[d, "\eta"] \\ S^3 \arrow[r, "\eta"] & S^2 \end{tikzcd}$$

is trivial, $\eta^* : [S^2, BG] \longrightarrow [S^3, BG]$ is the zero map and there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_{12} \longrightarrow [\mathbb{CP}^2, BG] \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \ .$$

Thus $[\mathbb{CP}^2, BG]$ is a group of 24 elements and $24r\gamma$ is fibre homotopically trivial.

The composite

$$\mathbb{CP}^2 \xrightarrow{24r\gamma} BO \longrightarrow BPL \longrightarrow BG$$

is trivial and so the associated PL bundle

$$\xi \ : \ \mathbb{CP}^2 \longrightarrow BPL$$

factors through G/PL:

$$\begin{tikzcd} \mathbb{CP}^2 \arrow[r, "24r\gamma"] \arrow[d, "f"'] & BO \arrow[d] \\ G/PL \arrow[r, "\pi"] & BPL \end{tikzcd}$$

We calculate $s(f)$ using the remarks following Theorem 4.4

$$\begin{aligned} s(f) &= \langle L(\mathbb{CP}^2) \cup (L^{PL}(\pi f) - 1), [\mathbb{CP}^2] \rangle \\ &= \langle (1 + L_1(\mathbb{CP}^2)) \cup L_1^{PL}(\pi f), [\mathbb{CP}^2] \rangle \\ &= \langle L_1^{PL}(\pi f), [\mathbb{CP}^2] \rangle \\ &= \langle \frac{p_1(24r\gamma)}{3}, [\mathbb{CP}^2] \rangle \\ &= \langle \frac{-24x^2}{3}, [\mathbb{CP}^2] \rangle \\ &= -8 \ . \end{aligned}$$

This completes the proof of $(*)$ and Theorem 4.6 follows.

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Coda: connection with the results of Kirby and Siebenmann

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We have shown that $TOP/PL \longrightarrow G/PL$ factors via

$$K(\mathbb{Z}_2, 3) \;=\; \text{fibre}(G/PL \longrightarrow \Omega^{4n}(G/PL)) \;.$$

Now Kirby and Siebenmann have shown [2, 3, 4] that TOP/PL is also a $K(\mathbb{Z}_2, 3)$, and that the map $q_h : M \longrightarrow TOP/PL$ is the obstruction to an isotopy from h to a PL homeomorphism. There are two possibilities:

(1) $TOP/PL \longrightarrow K(\mathbb{Z}_2, 3)$ is null-homotopic;

(2) $TOP/PL \longrightarrow K(\mathbb{Z}_2, 3)$ is a homotopy equivalence.

We shall eliminate (1). Both the fibration and the theorem mentioned in the introduction then follow if we combine the Kirby-Siebenmann result with the main theorem of Chapter III. In order to eliminate (1) it is necessary to consider the structure sequence for the torus T^r. There is a fibration onto its image

$$HT(T^r) \longrightarrow (G/PL)^{T^r} \longrightarrow \mathbb{L}_{r+4n}(\mathbb{Z}^r)$$

due to Casson and Quinn [5]. Now $\mathbb{L}_{r+4n}(\mathbb{Z}^r)$ consists of $(\Delta^k, r + 4n)$-oriented normal maps (which are homotopy equivalences on boundaries) together with a reference map to a $K(\mathbb{Z}^r, 1)$, which we can take to be T^r itself. Consequently there is a map

$$\alpha \;:\; \mathbb{L}_{4n}(\{1\})^{T^r} \longrightarrow \mathbb{L}_{r+4n}(\mathbb{Z}^r)$$

defined as follows. Let $f : T^r \longrightarrow \mathbb{L}_{4n}(\{1\})$ be given; then f determines an $(i+4n)$-normal map for each i-simplex of T^r and, glueing together, we obtain an $(r+4n)$-normal map over T^r, in other words a simplex of $\mathbb{L}_{r+4n}(\mathbb{Z}^r)$. Using the Splitting Theorem of Farrell [1] we can convert any normal map (homotopy equivalence on boundary) over T^r into an assemblage of normal maps (homotopy equivalences on boundaries) over simplexes of T^r. This argument generalizes to show that α is a homotopy equivalence. Now $\mathbb{L}_{4n}(\{1\})$ and $\Omega^{4n}(G/PL)$ have the same homotopy type, by considering the structure sequence for D^{4n} rel ∂, and we can rewrite our fibration as

$$HT(T^r) \longrightarrow (G/PL)^{T^r} \longrightarrow (\Omega^{4n}(G/PL))^{T^r} \;.$$

It follows that $HT(T^r)$ and $(K(\mathbb{Z}_2, 3))^{T^r}$ have the same homotopy type. Now if the map $TOP/PL \longrightarrow K(\mathbb{Z}_2, 3)$ is null-homotopic, then any self-homeomorphism of T^r is homotopic to a PL homeomorphism. However Siebenmann [2, 3, 4] has constructed a self-homeomorphism of T^6 which is **not** homotopic to a PL homeomorphism. Hence

$$TOP/PL \longrightarrow K(\mathbb{Z}_2, 3)$$

must be a homotopy equivalence.

It is clear from the above discussion that any homotopy equivalence onto T^r

is homotopic to a homeomorphism, and that the obstructions to the homotopy and isotopy Hauptvermutung coincide for $Q = T^r$. This contrasts with the simply connected case and shows that the general solution is bound to be somewhat complicated.

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