A new classification of links and some calculations using it

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Abstract A new classification theorem for links by the authors and Roger Fenn leads to computable link invariants. As an illustration we distinguish the left and right trefoils and recover the result of Carter et al that the 2-twist-spun trefoil is not isotopic to its orientation reverse. We sketch the proof the classification theorem. Full details will appear elsewhere

AMS Classification 57Q45; 57M25, 57M27

Keywords Knot, link, invariant, 2–knot, twist-spinning, reversibility, chirality, rack, rack space, quandle

1 Introduction

Throughout the paper, links and link diagrams will be assumed to be framed, in other words to have trivialised normal bundle.

In [3, 4] Fenn, Rourke and Sanderson define a space BL associated to a link L in codimension 2 which is the *rack space* of the fundamental rack of the link. The link determines a *canonical class* in the homotopy of BL. For example, if the link is in S^3 then the canonical class $c(L)$ is in $\pi_2(BL)$.

Furthermore in [1] Carter, Jelsovsky, Kamada, Langford and Saito construct "state-sum" invariants of knots using a notion of quandle cohomology. As an application they prove that the twice twist-spun trefoil is not isotopic to its orientation reverse. Quandle cohomology groups are closely related to the cohomology groups of the rack space, and the state-sum invariant has a natural interpretation in terms of the canonical class. This suggests the importance of this canonical class.

In this paper we announce, with outline proof, a new result: the fundamental rack together with the canonical class classifies classical links.

Classification Theorem (Fenn–Rourke–Sanderson) *Suppose that L, M are two links in S^3 and suppose that there is an isomorphism of fundamental racks $\phi\colon \Gamma(L) \to \Gamma(M)$ such that $\phi_*(c(L)) = c(M)$ then L and M are isotopic.*

This result should be contrasted with existing classification results using the fundamental rack. The augmented fundamental rack classifies *irreducible* links in a 3–manifold up to homeomorphism [2; 5.2] and the unaugmented fundamental rack does the same for irreducible links in a homotopy 3–sphere (and in particular for classical links) [2; 5.3]. However because the fundamental rack of the *inverse mirror* of a link (meaning the mirror image with orientation change) is isomorphic to the original link, the fundamental rack alone cannot classify links up to isotopy. For example the fundamental racks of the left and right-handed trefoil knots are isomorphic. The situation is compounded for reducible links where the fundamental rack only determines the link up to inversion of the *blocks* (maximal irreducible sublinks).

The canonical class encodes the orientations of all the blocks and together with the fundamental rack enables a complete reconstruction of the link. It follows that all link invariants can theoretically be recovered from the rack and canonical class. Now the canonical class lies in π_2 of the rack space which is H_2 of the universal cover. Partial information can be obtained from H_2 of a cover intermediate between the rack space and the universal cover and by using a representation in a finite rack, this homology group becomes computable. Thus we have a simple way to construct computable link invariants which approximate in a systematic way to the full invariant of rack and canonical class.

In this paper we will demonstrate the practicality of this by constructing a computable invariant which distinguishes the left and right-handed trefoils. The calculation needed is of the third homology group of the three-colour rack and for this we rely on a Maple worksheet [8]. The same calculation allows us to recover the theorem of Carter et al [1].

The Classification Theorem is joint work with Roger Fenn. A full treatment will be given in [6]. Here we prove the result using results from [4].

Here is an outline of the paper. In section 2 we give some background material on racks and the rack space and prove the Classification Theorem. In section 3 we start our investigation of the use of the canonical class by using it together with a short Maple calculation to prove that the 2–twist-spun trefoil is not isotopic to its orientation reverse preserving orientations. In section 4 we extend the notion of the rack space and use the same calculation to prove that the left and right trefoils are different and we conclude in section 5 with some remarks about the proofs.

2 Racks and the Classification Theorem

The rack space and the canonical class of a diagram

Full details of all the material in this section can be found in [4]. For background material on racks see [2]. Here we shall summarise the results from [4] which are needed for the Classification Theorem and the later calculations.

Let R be a rack. The *rack space* BR of R is a cubical set with the set of $n-$cubes in bijection with R^n. In this paper we shall only need to consider cubes of dimension ≤ 3. The interpretation of such cubes is given by the pictures in figure 1.

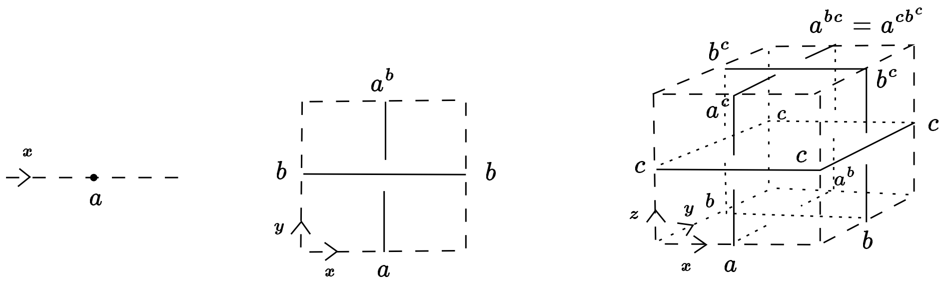

Figure 1

The figures make the boundary maps clear. Thus the back face of the 3–cube drawn (a, b, c) is glued to the 2–cube (a^b, c) and the top face is glued to (a^c, b^c).

There are similar descriptions of boundary maps from n–cubes to $(n-1)$–cubes and the general formula is:

$$\begin{aligned} \partial_i^0(x_1, \ldots, x_n) &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\ \partial_i^1(x_1, \ldots, x_n) &= (x_1^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \cdots, x_n) \quad \text{for} \quad 1 \leq i \leq n. \end{aligned}$$

The rack space determines a topological space (also denoted BR) by gluing real cubes together using these boundary maps.

Now suppose that we are given a diagram in $\mathbb{R}^2$ representing a link L in $\mathbb{R}^3$. A *labelling* of the diagram in a rack R means a labelling of arcs by elements of R so that at double points the rule indicated in figure 2 holds.

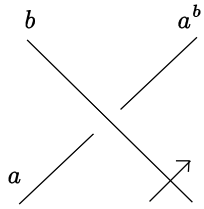

Figure 2

A labelling by R is precisely the same as a homomorphism of the fundamental rack of L to R, see [2; page 384].

Now L is canonically framed by taking the first framing vector parallel to the vertical and second vector normal to the image in $\mathbb{R}^2$. Using this second vector we can provide the diagram with a bicollar with collar lines identified with I (so that the submanifold corresponds to $\frac{1}{2} \in I$ and the vector points in the positive direction along I). We can assume that the collar lines determine little squares at double points. The squares are canonically identified with I^2 using the vertical ordering of arcs. Now suppose the diagram is labelled in R then using the labelling these copies of I and I^2 can be identified with cubes in BR. The diagram now determines a map of $\mathbb{R}^2$ to BR by mapping the collar lines to the 1–cubes of BR with which they are identified and the little squares at double points to the appropriate 2–cubes. Outside the collar all is mapped to the basepoint (the unique 0–cube) and hence we get a based map $S^2 \to BR$ by mapping infinity (the basepoint of S^2) to the basepoint of BR.

This map is almost canonical, depending only on the choice of collar, and we call it the *canonical map* determined by the diagram and the corresponding class in $\pi_2(BR)$ the *canonical class*. Given an isotopy of diagrams (or even a bordism) labelled in R, then the canonical map varies by a homotopy by applying a similar construction to the bordism.

There is a converse construction. Given a map $S^2 \to BR$ we can use transversality to make it transverse to the 2–skeleton of BR and this means that it pulls back a diagram in $\mathbb{R}^2$ (for some link in $\mathbb{R}^3$) labelled in R. Similarly a homotopy pulls back a bordism of diagrams, ie, a diagram for a bordism emdedded in $\mathbb{R}^3 \times I$.

These considerations are summarised in the following theorem:

Theorem 2.1 *There is a bijection between $\pi_2(BR)$ and the set of bordism classes of diagrams in $\mathbb{R}^2$ labelled in R.*

See [4] for the detailed proof of this result, for the interpretation in terms of James bundles and for more general results along the same lines.

Proof of the Classification Theorem

By theorem 2.1 the data in the theorem determines a bordism W between L and M with homomorphism Φ from $\Gamma(W)$ to $\Gamma(L)$ (ie a labelling in $\Gamma(L)$) which is the identity on L and on M is the given isomorphism ϕ.

We shall replace W by an isotopy (ie a level-preserving embedding of a link $\times I$) by induction on the number of blocks (ie maximal irreducible sublinks). Suppose that $L = L_1 \coprod L_2$ where L_1 is a block. Then the fundamental rack of L is the free product of the fundamental racks of L_1 and L_2 and likewise the fundamental group (see [2; page 357] for the notion of the free product of two racks). Now the fundamental group is the associated group of the fundamental rack (this is true for classical links by [2; 3.3] — it is not true for general links) and it follows that the fundamental group of M is also a free product. Then by Kneser's conjecture [5; pages 66–68] M is a split link and furthermore the proof shows that we can assume that the splitting induces the same free product decomposition of the fundamental group. We shall now relate the splittings of L as $L_1 \coprod L_2$ and of M, just constructed.

Think of the components of W as coloured red if they are labelled by elements of $\Gamma(L)$ in one of the orbits of L corresponding to L_1 (recall that orbits in a fundamental rack of a link are in bijection with components of the link) and coloured blue of they are likewise labelled in orbits corresponding to L_2. We obtain a decomposition of W as $W_1 \cup W_2$ where W_1 is red and W_2 is blue and an induced decomposition $M = M_1 \cup M_2$ of M. We claim that this is the same as the splitting of M constructed above.

To see this consider the augmentation ∂ which takes a element of the fundamental rack to the element of the fundamental group which is the "frying pan": down the rack element (represented as a path from the framing curve to the basepoint) round the meridian and back up the rack element. It follows that the augmentation takes an element in an orbit (ie component) to a fundamental group element which links the component once (and others zero times). But an element of $\Gamma(M_1)$ is mapped to an element in $\Gamma(L_1)$ and hence is augmented to an element in the free factor of the fundamental group corresponding to L_1, which was how we obtained the splitting of M using Kneser. Thus M splits as $M_1 \coprod M_2$ with M_1 coloured red and M_2 coloured blue. It follows that $\Gamma(M)$ is the free product $\Gamma(M_1) * \Gamma(M_2)$.

Now we claim that the isomorphism ϕ splits as an isomororphism respecting the two free product decompositions. To see this we observe that, given an element of a rack, its augmentation is non-trivial in the fundamental group (it links the appropriate arc once) and hence if we consider an element of $\Gamma(M_1)$

then its augmentation is non-trivial in $\pi_1(M_1)$ and maps to $\pi_1(L_1)$ since the splittings of the groups correspond. It follows that the image under ϕ is in $\Gamma(L_1)$. Similarly ϕ takes $\Gamma(M_2)$ to $\Gamma(L_2)$.

Now we perform a trick. Think of the splittings of L and M as into left and right halves (red to the left, blue to the right). W is not split, but we pull it apart by translating the red pieces to the left past the blue pieces and form in this way a new split bordism $W' = W_1 \coprod W_2$. We replace the labels on W_1 by the old labels on W composed with the homomorphism $\Gamma(L) = \Gamma(L_1) * \Gamma(L_2) \to \Gamma(L_1) \coprod \Gamma(L_2)$ given by ignoring the L_2 contributions to elements corresponding to L_1 and vice versa (recall the description of the free product on [2; page 357]). This gives a new labelling but observe that the labelling on M is undisturbed by this trick.

We now have new bordisms W_1 between L_1 and M_1 and W_2 between M_1 and M_2. We now use the proof of [4; 5.10]. There is no obstruction to constructing a map of $S^3 \times I$ to S^3 which is transverse to L_1 and such that the preimage of of L_1 is W_1 (essentially because $S^3 - L_1$ is a $K(\pi, 1)$). This map is the identity on the L_1 end and on the L_2 and deforms to a homeomorphism as in the proof of [2; 5.2] (essentially this is Waldhausen's theorem). Thus we have a homotopy between the identity and a homeomorphism of L_1 to L_2 which can be replaced by an isotopy since homotopic homeomorphisms of spheres are isotopic.

Finally the other half of W', namely W_2 can also be replaced by an isotopy by induction and the theorem is proved.

3 The 2–twist spun trefoil

We start our calculations using the canonical class by considering 2–knots (ie knots of S^2 in $\mathbb{R}^4$). This is because distinguishing the left and right trefoils introduces extra technical details.

Framings and diagrams for 2–knots in 4–space

For classical knots, the oriented and framed theories differ, for example the writhe of a knot is an invariant of framed knots but not of oriented knots. For 2–knots the distinction disappears:

Let $K \subset \mathbb{R}^4$ be an oriented, possibly knotted, 2–sphere in $\mathbb{R}^4$. The normal bundle of K has a section since K bounds a 3–manifold (a generalised "Seifert surface") and furthermore the orientations now give a canonical second section, in other words a framing of K. Thus we can confuse sections of the normal bundle and framings. Now two sections of K differ by a map $K \to S^1$ (since the normal bundle is trivial) and since $\pi_2(S^1) = 0$ any two sections are isotopic. Thus the orientations determine a canonical framing of K in $\mathbb{R}^4$.

Now consider the canonical projection $\mathbb{R}^4 \to \mathbb{R}^3$ and think of $\mathbb{R}^3$ as horizontal and the remaining $\mathbb{R}^1$ as vertical. Suppose that we are given a 2–knot K such that the projection of K on $\mathbb{R}^3$ has multiple set comprising transverse double lines and triple points. The image of the projection together with the information of the vertical order of sheets at the multiple set is called a *knot diagram* for K. The diagram determines K up to isotopy and also determines a natural framing of K corresponding to the section which is vertically up. Conversely, by the Compression Theorem [7] a 2–knot K with a section of its normal bundle can be isotoped so that the section is vertically up and hence K projects to an immersion in $\mathbb{R}^3$. By making this immersion self-transverse we now have a knot diagram for K. Furthermore, by the 1–parameter version of the Compression Theorem, isotopic framed embeddings determine isotopic knot diagrams. Combining the two facts just outlined, namely that orientation determines framing and that framing determines diagram, we have:

Lemma 3.1 *There is a natural bijection between isotopy classes of oriented embeddings of a 2–sphere in $\mathbb{R}^4$ and isotopy classes of knot diagrams in $\mathbb{R}^3$.*

By a similar construction to that given in section 2 for diagrams in the plane, a diagram of a 2–knot whose fundamental rack is labelled in a rack R determines a canonical class in $\pi_3(BR)$. Using lemma 3.1 we have:

Lemma 3.2 *The canonical class in $\pi_3(BR)$ of a labelled 2–knot diagram is determined by*

(a) *the isotopy class of the oriented knot*

(b) *the labelling, or equivalently, the homomorphism of the fundamental rack to R.*

There is a natural extension of theorem 2.1 to 2–knots:

Theorem 3.3 *There is a bijection between $\pi_3(BR)$ and the set of bordism classes of diagrams in $\mathbb{R}^3$ of 2–knots labelled in R.*

We use a calculation involving the canonical class to prove:

Theorem 3.4 *The 2–twist spun trefoil is not (oriented) isotopic to its orientation reverse.*

Proof The proof occupies the remainder of this section. Let K be the 2–twist spun trefoil. We shall exhibit an explicit diagram for K which is labelled by the "three-colour rack" $T := \{0,1,2\}$ with $a^b := 2b - a \bmod 3$, ie, $a^b = c$ iff a,b,c are all the same or all different.

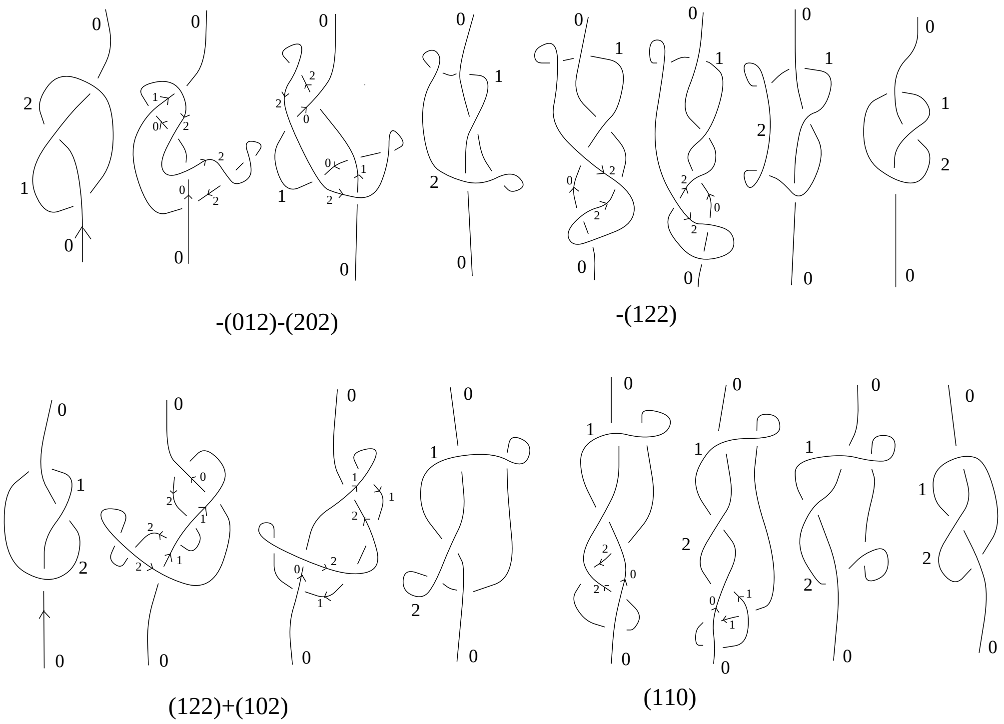

Figure 3

In figure 3 we give a series of slices of the 2–twist spun trefoil. The pictures are to be understood as follows. We are thinking of $\mathbb{R}^4$ as an "open-book decomposition" $\mathbb{R}^3_+ \times S^1$ with $x \times S^1$ identified with x for each $x \in \mathbb{R}^2 = \partial\mathbb{R}^3_+$, ie, $\mathbb{R}^2$ is the "spine" of the decomposition. Similarly we are thinking of $\mathbb{R}^3$ as an open-book decomposition with spine $\mathbb{R}^1$, ie, $\mathbb{R}^3$ is $\mathbb{R}^2_+ \times S^1$ with a similar identification for each $x \in \mathbb{R}^1 = \partial\mathbb{R}^2_+$. The projection $\mathbb{R}^4 \to \mathbb{R}^3$ is then given by the standard projection $\mathbb{R}^3_+ \to \mathbb{R}^2_+$ crossed with the identity on S^1. The figures are drawn as a sequence of diagrams in copies of $\mathbb{R}^2_+$ corresponding to embeddings in the corresponding copies of $\mathbb{R}^3_+$ and describe a diagram in $\mathbb{R}^3$

for an embedding in $\mathbb{R}^4$. The basepoint in $\mathbb{R}^4$ is chosen to lie on the spine $\mathbb{R}^2$ of the decomposition above the spine $\mathbb{R}^1$ on the decomposition of $\mathbb{R}^3$. Thus the basepoint lies above each of the diagrams drawn. The framing (not shown) is given by the left-hand rule in other words the framing vector is obtained from the orientation vector on arcs by turning to the left. Figure 3 shows one twist of the spun trefoil. To get the 2–twist spun trefoil, we repeat the sequence. The figure also shows the labelling in T. Observe that the finishing labels coincide with the starting ones with 1 and 2 interchanged, so to get the labels for the second twist, repeat the labels with 1 and 2 interchanged.

The moves from one diagram to the next should all be obvious except perhaps for the right-most pair in both rows. Here a ripple spins round in the surface to eliminate the two oppposite twists (creating no triple points in the projection).

Now let $\psi \in \pi_3(BT)$ be the canonical class of this labelled diagram for K. We shall compute $h(\psi)$ the Hurewicz image in $H_3(BT)$. Inspecting the definition of the canonical class given in section 2 above, we see that $h(\psi)$ is represented by a cycle C given as a sum of 3–cubes of BT one for each triple point of the diagram. In figure 3 we have written the 3–cubes (with sign) determined by the triple points (which appear as R3 moves in the sequence). Figure 4 gives more detail of the identification of these cubes. Figure 4 shows the cubes $-(012), -(122), (121)$ and (110) the other two ($-(202)$ and (102)) are the same as $-(012)$ and (121) respectively, with label changes.

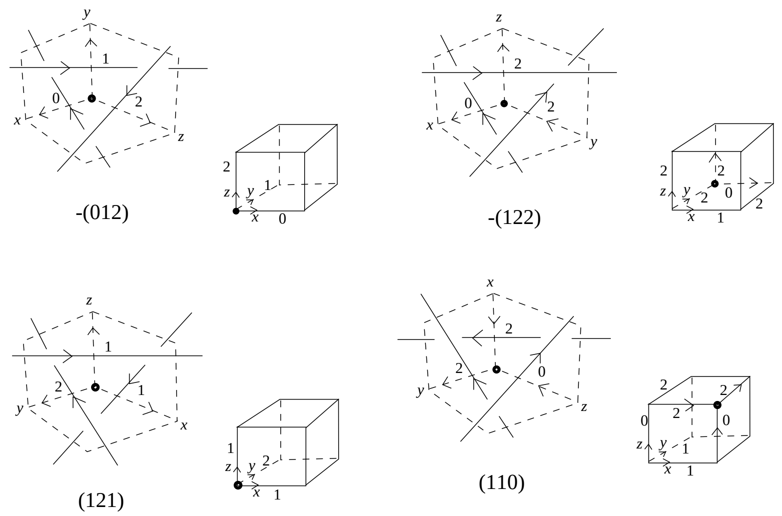

Figure 4

We can now read off the cycle which represents $h(\psi)$.

$$\begin{aligned} C = & -(012)-(202)-(122)+(121)+(102)+(110) \\ & -(021)-(101)-(211)+(212)+(201)+(220) \end{aligned}$$

We need two calculations for which details are to be found in the Maple worksheet [8].

Calculation 1 $H_3(BT) \cong \mathbb{Z} \times \mathbb{Z}_3$

Calculation 2 C represents a generator of the $\mathbb{Z}_3$–summand of $H_3(BT)$.

Lemma 3.5 *The $\mathbb{Z}_3$–summand of $H_3(BT)$ is fixed by all permutations of $T = \{0,1,2\}$.*

Proof Let Q denote the $\mathbb{Z}_3$–summand of $H_3(BT)$ and let S_3 denote the group of permutations of $\{0,1,2\}$. C represents a generator of Q and by construction is invariant under the interchange $(1,2) \in S_3$. Thus Q is fixed by $(1,2)$. Now notice that there are no symmetries of $H_3(BT)$ of order 3. It follows that the 3–cycle $(0,1,2) \in S_3$ also fixes Q. Since S_3 is generated by $(1,2)$ and $(0,1,2)$, Q is fixed by the whole of S_3. $\square$

Corollary 3.6 *$h(\psi)$ is independent of the choice of (non-constant) labelling in T.*

Proof The labelling is clearly determined by the initial labelling of the first diagram and it follows that two different (non-constant) labels are related by a permutation of T. $\square$

Now let K' denote K with opposite orientation. Notice that since we are labelling in an *involutory* rack T (ie one in which $a^{b^2} = a$ for all a, b) the original labels give a labelling after the orientation change. Let ψ' be the canonical class of K' using this labelling. To calculate $h(\psi')$ we observe that changing orientation reverses all arrows and from figure 5 we see that this corresponds to replacing each cube (a,b,c) by the cube (a^{bc}, b^c, c) with opposite orientation.

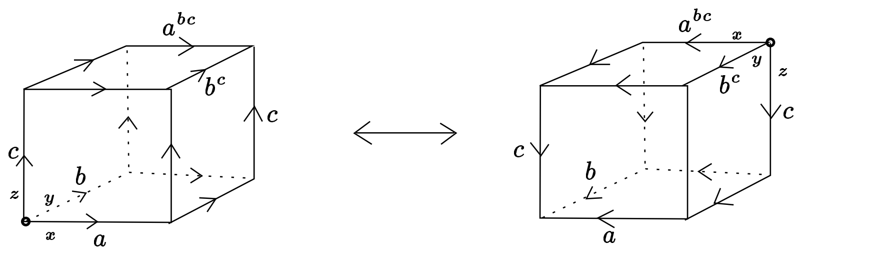

Figure 5

Performing this substitution for the cycle C given above we read off the cycle C' representing $h(\psi')$:

$$\begin{aligned} C' = {} & (202) + (012) + (122) - (201) - (212) - (220) \\ & (101) + (021) + (211) - (102) - (121) - (110) \end{aligned}$$

Serendipitously we notice:

Observation $C + C' = 0$

It follows that $h(\psi') = -h(\psi)$. Now suppose that K is isotopic to K' preserving orientation. Then since the fundamental racks of K, K' and the isotopy (thought of an an embedding of $S^2 \times I$ in $S^4 \times I$) are all isomorphic, the labelling of K in T induces a labelling through the isotopy which is non-trivial on K'. By corollary 3.6 we can use these labels to calculate $h(\psi)$ and $h(\psi')$. But by lemma 3.2 $\psi' = \psi$ and we have a contradiction, completing the proof of the main theorem. □

See the remarks in section 5 for explanations of some (but not all) of the concidences which occured in the above proof.

4 Trefoils left and right

We shall need a generalisation of the rack space. Let R be a rack. The *extended rack space* $B_R R$ has set of n–cubes in bijection with R^{n+1} (ie the $n+1$–cubes of BR) and the same formulae for face maps as in BR with an index shift (ie, ∂_i^ε in $B_R R$ is $\partial_{i+1}^\varepsilon$ in BR) cf [3, 1.3.2 and 3.1.2]. It follows that $H_n(B_R R) \cong H_{n+1}(BR)$ (cf [4, 5.14 and above]). Note that $B_R R$ is a covering space of BR.

The interpretation for diagrams is labelling of both arcs and regions. More precisely, suppose that D is a diagram in $\mathbb{R}^2$ for a knot K in $\mathbb{R}^3$. An *extended labelling* of D by a rack R is a labelling of arcs and regions of D by elements of R with the rules illustrated in figure 6 for labels of adjacent regions and at crossings:

Given an extended labelled diagram, there is a map $\mathbb{R}^2 \to B_R R$ constructed similarly to section 2: Choose bicollars, map little squares at crossings to the appropriate 2–cube of $B_R R$ (for example a square at the crossing in figure 6 would be mapped to the 2–cube (c, a, b)) collar lines across arcs to the appropriate 1–cube (eg, a collar line across the lower left arc in figure 6 would be mapped to the 1–cube (c, a)) and regions to the 0–cube given by the label.

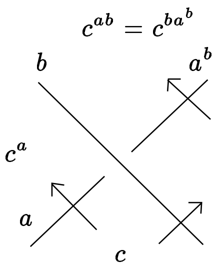

Figure 6

Thus the labelling determines a canonical class $\phi \in \pi_2(B_R R)$ (based at the vertex corresponding to the label of the infinite region). An isotopy of K in $\mathbb{R}^3$ corresponds to a diagram in $\mathbb{R}^2 \times I$ to which the labelling can be canonically extended. This in turn gives a map $\mathbb{R}^2 \times I \to B_R R$, ie a homotopy of the canonical class. Thus as in section 2, the canonical class is an invariant of the isotopy class of the labelled diagram.

We now exhibit extended labellings in the three colour rack T for diagrams of the right-hand trefoil K and the left-hand trefoil K', see figure 7.

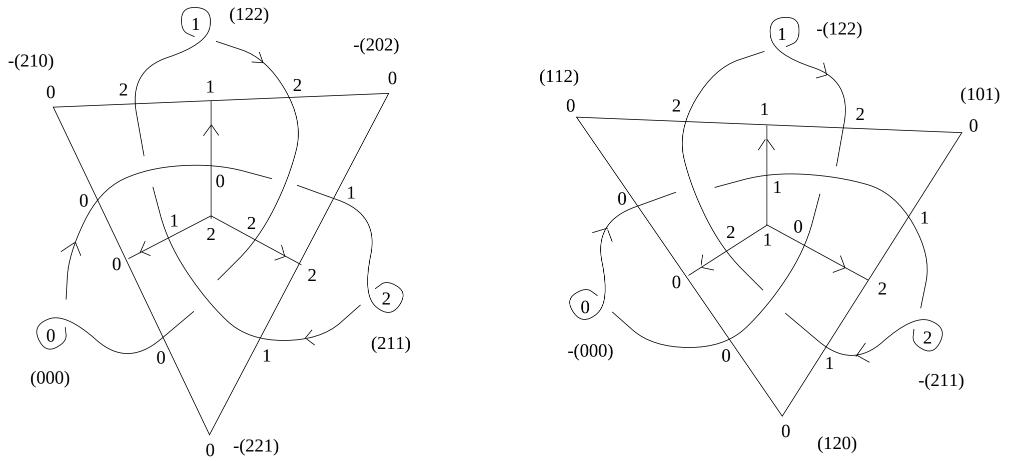

Figure 7

Let ϕ and ϕ' be the canonical classes determined by these diagrams and let $h(\phi)$ and $h(\phi')$ be their Hurewicz images in $H_2(B_T T) \cong H_3(BT) \cong \mathbb{Z} \times \mathbb{Z}_3$. We can read off representating cycles B and B' from the diagrams:

$$B = -(210) - (202) - (221) + (211) + (122) + (000)$$
$$B' = (221) + (202) + (210) - (122) - (000) - (211)$$

Observation $B = -B'$ and hence $h(\phi) = -h(\phi')$.

We need one final Maple calculation:

Calculation 3 B represents a generator of the $\mathbb{Z}_3$–summand of $H_2(B_T T)$.

Lemma 4.1 *The classes $h(\phi)$ and $h(\phi')$ are independent of the labels in T of arcs and regions in the diagrams provided that the knots themselves have non-constant labels.*

Proof We notice that both diagrams have rotational symmetry of order 3. But any two labellings which are non-constant on the knots are related by this symmetry followed by a permtuation of T. The result follows from lemma 3.5. □

We can now prove that right and left-hand trefoils are different.

Theorem 4.2 *There is no isotopy of $\mathbb{R}^3$ which carries the (unoriented) right-hand trefoil to the left-hand trefoil.*

Proof It is easy to contruct an isotopy carrying an oriented trefoil to its orientation reverse. Thus it suffices to prove that the oriented left and right trefoils are not isotopic. The diagrams drawn above both have zero writhe and if there is an isotopy between the oriented left and right trefoils then there would be a framed isotopy between the diagrams. Now the labelling on the right trefoil determines a labelling of the isotopy (thought of as a diagram of $S^1 \times I$ in $\mathbb{R}^3 \times I$) and hence a labelling of the left trefoil which is non-constant on the knot. By lemma 4.1 we can use these labels to calculate $h(\phi)$ and $h(\phi')$ and it follows that $h(\phi) = h(\phi')$ contradicting the observation made above. □

5 Some remarks

Fundamental quandles

The *involutive* fundamental quandle (ie the fundamental quandle with the operator relations $a^2 \equiv 1$ for each a) of the trefoil is the 3–colour rack/quandle T. This explains why there is a natural labelling in this rack. Furthermore using a Van Kampen argument it can be seen that the fundamental quandle of an n–twist-spun knot K is the fundamental quandle of K with the operator relations $a^n \equiv 1$ for each a. It follows that the fundamental quandle of the 2–twist-spun trefoil is T. So the labelling of the 2–twist-spun trefoil we used is the only non-trivial labelling (in a quandle).

Using the canonical class

By the Classification Theorem, the canonical class can be used to distinguish the left and right trefoils which have isomorphic fundamental racks. The canonical class lies in $\pi_2(B\Gamma)$ which is H_2 of the universal cover of $B\Gamma$ and therefore determines classes in every connected cover of $B\Gamma$. To distinguish the left and right trefoils we used $B_T T$ which is a 3–fold cover of BT and the canonical class we used was the image of the class in the 3–fold cover of $B\Gamma$ determined in this way. This is typical of the way in which invariants can be constructed to extract information from the Classification Theorem. For more examples see [6].

The $\mathbb{Z}$ factor

Notice that $H_3(BT) \cong H_2(B_T T)$ has a $\mathbb{Z}$ factor which played no role in the proofs. In fact every homology group of the rack space BR where R has a base element which acts trivially on itself (any element in a quandle will do) splits a $\mathbb{Z}$ factor which comes from the inclusion and natural projection $B* \subset BR \to B*$ where $B*$ denotes the rack space of the one element rack. Now $B*$ is a model for $\Omega(S^2)$ and has one cell in each dimension with all boundary maps trivial and hence has homology $\mathbb{Z}$ in each dimension [4; 3.3].

Now for a 2–knot the canonical class maps to $\pi_3(\Omega S^2) = \pi_4(S^3) = \mathbb{Z}_2$ which in turn maps to zero in $H_3(\Omega S^2) = \mathbb{Z}$. This explains why the $\mathbb{Z}$ factor was immaterial for the 2–twist spun trefoil. For a 1–knot the image in $H_2(\Omega S^2) = \mathbb{Z}$ is readily seen to be the writhe of the diagram. But we chose our diagrams for the left and right trefoils to have zero writhe, which explains why the $\mathbb{Z}$ factor was immaterial for this case as well.

Invariance under permutations of T

Rack spaces are *simple* in other words π_1 acts trivially on π_n for each n [4; proposition 5.2]. The proof is by diagram manipulation. Introduce a small sphere labelled by $x \in R$. Pull the sphere over the diagram (realising the change of labels corresponding to the action of x) and then pull the sphere back under the diagram and eliminate it. The same proof shows that any labelled diagram is bordant to the diagram with labels acted on by any element of the labelling rack. Thus the operator group of the rack acts trivially on the canonical class. Now the three colour rack T has operator group equal to its automorphism group, which is the symmetric group S_3. This explains why the canonical class determined by a labelling in T is invariant under choice of labels: any rack with operator group equal to its automorphism group would have the same property. Thus the labelling used for the 2–twist-spun trefoil was immaterial. For extended labels the same proof shows invariance under change of labelling with fixed label at infinity. This together with the obvious symmetry of the choice of label at infinity in T explains why the labelling was also immaterial for the left and right trefoils.

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