# Markov's theorem in 3–manifolds

Sofia Lambropoulou and Colin Rourke

**Abstract**

In this paper we first give a one-move version of Markov's braid theorem for knot isotopy in $S^3$ that sharpens the classical theorem. Then a relative version of Markov's theorem concerning a fixed braided portion in the knot. We also prove an analogue of Markov's theorem for knot isotopy in knot complements. Finally we extend this last result to prove a Markov theorem for links in an arbitrary orientable 3–manifold.

## 1 Overview

According to Birman [3], Markov [13] originally stated his braid equivalence theorem using *three* braid moves; later there was another brief announcement of an improved version of Markov's theorem by Weinberg [23] consisting of the *two* well-known braid-equivalence moves: conjugation in the braid groups and the 'stabilizing' or 'Markov' moves ($M$–moves). Our first main result is a *one-move* Markov theorem which we now state.

An $L$*–move* on a braid consists of cutting one arc of the braid open and splicing into the broken strand new strands to the top and bottom, both either *under* or *over* the rest of the braid:

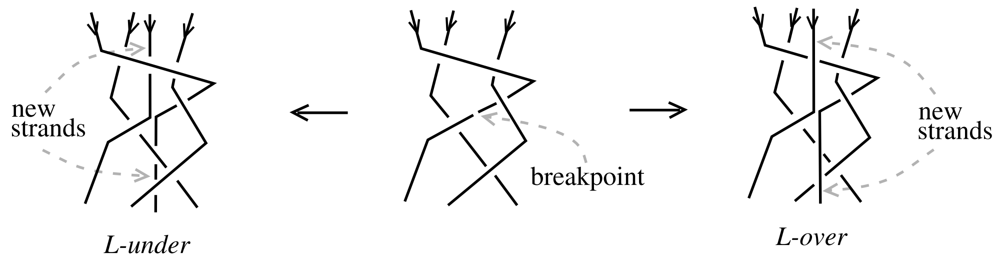


figure 1

$L$–moves and isotopy generate an equivalence relation on braids called *$L$–equivalence*. In §4 we prove that $L$–equivalence classes of braids are in bijective correspondence with isotopy classes of oriented links in $S^3$, where the bijection is induced by 'closing' the braid to form a link. As a consequence, $L$–equivalence is the same as the usual Markov equivalence and thus the classical Markov theorem is sharpened. The proofs are based on a canonical process for turning a combinatorial oriented link diagram in the plane (with a little extra structure) into an open braid. Our braiding as well as the $L$–moves are based on the building blocks of combinatorial isotopy, the *triangle moves* or *$\Delta$–moves*. This makes the proof conceptually very simple[1]. Our braiding operation is essentially the same as the operation using a "saw-tooth" given by J. Birman [3]. We use the point at infinity as the reference point for braiding and so a saw-tooth becomes a set of vertical lines that meet at infinity. The change of reference point to infinity makes the proof of Markov's theorem easier because there are very few ways that saw-teeth can obstruct each other.

Moreover, the local nature of our proof allows us to formulate and prove first the relative version of Markov theorem, another new result stating that, if two isotopic links contain the same braided portion then any two corresponding braids which both contain that braided portion differ by a sequence of $L$–moves that do not affect the braided portion. We then prove a second relative version of the theorem for links which contain a common closed braid and deduce an analogue of Markov theorem for knot isotopy in complements of knots/links, which we finally extend to an analogue of Markov theorem for links in arbitrary closed 3–manifolds.

More precisely, let $M$ be a closed connected orientable 3–manifold. We can assume that $M$ is given by surgery on the closure of a braid $B$ in $S^3$, thus $M$ is represented by $B$. Furthermore, any oriented link in $M$ can be represented as the closure of a further braid $\beta$ such that $B \cup \beta$ is also a braid and we shall call it *mixed braid*. Then the mixed braid equivalence in $S^3$ that reflects isotopy in $M$ is generated firstly by $L$–moves, as above, but performed only on strings of $\beta$ and secondly by *braid band moves* or *b.b.–moves*. These are moves that reflect the sliding of a part of a link across the 2–disc bounded by the specified longitude of a surgery component of the closure of $B$. I.e. suppose a string of $\beta$ is adjacent to one of $B$, then a slide of the first over the second (with a

[1] Our braiding was first used in [7] and then in [8] and in [11] (preliminary version of our results). The first complete proof of the classical theorem has been given by J. Birman [3], and other proofs by H. Morton [14], D. Bennequin [2] and P. Traczyk [20].

half-twisted band) replaces $\beta$ by a braid with one or more extra strings parallel to the strings of $B$ which form the appropriate component in the closure:

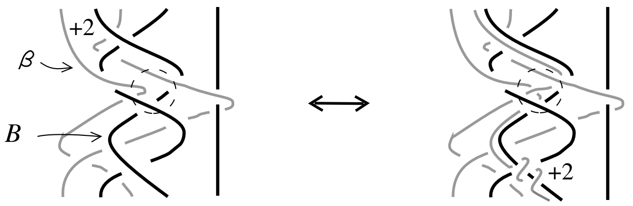


figure 2

Most results are based on material in [8] worked at Warwick University under the advice of the second author. However, they are sharpened versions and some parts of the proofs are improved considerably.

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## 2 Diagrams, isotopy and the $L$-moves

A knot is a special case of a link, so from now on we shall be referring to both knots and links as 'links'. Throughout the first part of this paper we shall be working in a combinatorial setting. In particular, we consider oriented *links/geometric braids* in 3–space each of the components/strands of which is made of a finite number of straight arcs endowed with matching orientation; also their *diagrams*, that is, regular projections on the plane (with only finitely many crossings), where in addition no vertex of the link/braid should be mapped onto a double point.

A *geometric braid diagram* or simply a *braid* has a top-to-bottom direction and, in addition, we require that no two crossings are on the same horizontal level. If we slice up in general position (i.e. without cutting through crossings) a braid diagram on $n$ strands, it may be seen as a word on the (well-known) basic crossings $\sigma_i$ and ${\sigma_i}^{-1}$ for $i = 1, \ldots, n-1$ (we refer the reader to figure

9 for an example). The set of braids on $n$ strands modulo isotopy gives rise to the braid group $B_n$ with presentation:

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \ \text{ for } \ |i-j|>1, \ \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle.$$

The operation in the group is *concatenation* (we place one braid on top of the other), and the identity element is the braid on $n$ strands that does no braiding. Note that the elements of $B_n$ are also called 'braids', but in here there will be no ambiguity, since we shall be working with geometric braid diagrams.

There are two combinatorial moves on diagrams which we shall consider.

(1) **$\Delta$–move:** An arc is replaced by two arcs forming a triangle (and its inverse), respecting orientation and crossings. 'Respecting crossings' means that, if we lift the diagram to an embedding in 3–space, then the $\Delta$–move lifts to an elementary isotopy (see figure 3 for examples).

(2) **Subdivision:** A vertex is introduced/deleted in an arc of the diagram.

Subdivision moves may be viewed as special cases of $\Delta$–moves. We shall call the equivalence relation generated by these two moves a *combinatorial isotopy* or just an *isotopy*. It is a classical result of combinatorial topology that this notion of isotopy is equivalent to the standard definition of combinatorial (or PL) isotopy of the embedding obtained by lifting to 3–space, and this in turn is equivalent to the notion of isotopy in the smooth category. Moreover, Reidemeister [15] (and Alexander, Briggs [1]) proved that a $\Delta$–move can break into a finite sequence of planar $\Delta$–moves and the three local $\Delta$–moves illustrated below (known as 'Reidemeister moves') with their obvious symmetries (for a detailed account see [5]).

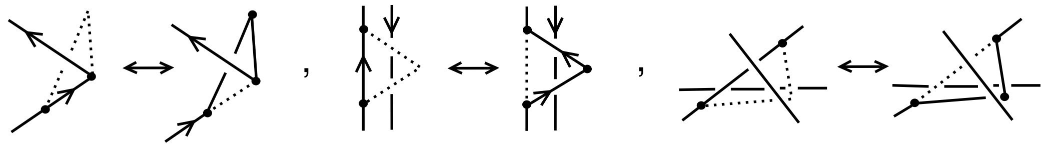


figure 3

**Definition 2.1** (**$L$–moves**) Let $D$ be a link diagram/braid and $P$ a point of an arc of $D$ such that $P$ is not vertically aligned with any of the crossings or (other) vertices of $D$ (note that $P$ itself may be a vertex). Then we can perform the following operation: Cut the arc at $P$, bend the two resulting smaller arcs apart slightly by a small isotopy and introduce two new vertical arcs to new top and bottom end-points in the same vertical line as $P$. The new arcs are both oriented downwards and they run either both *under* or both *over* all other arcs of the diagram. Thus there are two types of $L$–moves, an *under* $L$*–move* or $L_u$*–move* and an *over* $L$*–move* or $L_o$*–move.* (Recall figure 1 above for an example of the two braid $L$–moves.)

Below we illustrate $L_o$–moves applied to an edge $QR$ of an oriented link diagram such that $QR$ moves via isotopy from a downward arc to an upward arc through the horizontal position. The thick circle will represent 'the rest of the diagram', while the region inside the circle shall be called 'the magnified region'. Note that, if we join freely (using the dotted arcs) the two new arcs we obtain a link diagram isotopic to the one on which the $L$–move was applied.

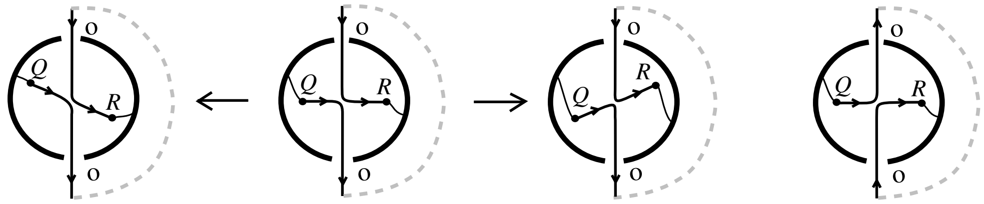



figure 4a figure 4b

Definition 2.1 could be given even more generally so as to accommodate the possibility of the new vertical strands running *upwards* (figure 4b). Although, for the purposes of this paper we shall only consider $L$–moves with the new vertical arcs oriented *downwards*.

**Remark 2.2** Using a small braid isotopy, a braid $L$–move can be equivalently seen with a crossing (positive or negative) formed:

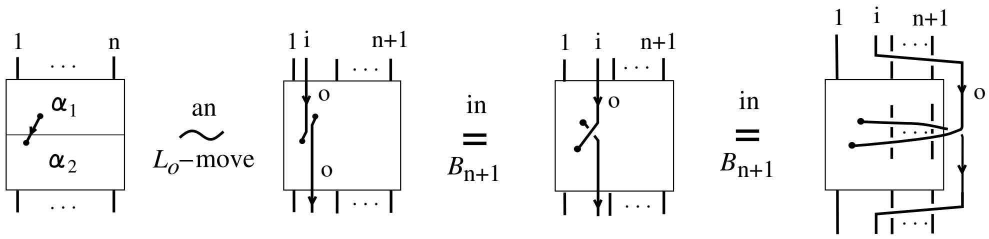



figure 5

This gives the following algebraic expression for an $L_o$–move and an $L_u$–move respectively.

$$\alpha = \alpha_1\alpha_2 \sim \sigma_i^{-1} \ldots \sigma_n^{-1} \widetilde{\alpha_1} \sigma_{i-1}^{-1} \ldots \sigma_{n-1}^{-1} \sigma_n^{\pm 1} \sigma_{n-1} \ldots \sigma_i \widetilde{\alpha_2} \sigma_n \ldots \sigma_i$$

$$\alpha = \alpha_1\alpha_2 \sim \sigma_i \ldots \sigma_n \widetilde{\alpha_1} \sigma_{i-1} \ldots \sigma_{n-1} \sigma_n^{\pm 1} \sigma_{n-1}^{-1} \ldots \sigma_i^{-1} \widetilde{\alpha_2} \sigma_n^{-1} \ldots \sigma_i^{-1}$$

where $\alpha_1$, $\alpha_2$ are elements of $B_n$ and $\widetilde{\alpha_1}$, $\widetilde{\alpha_2} \in B_{n+1}$ are obtained from $\alpha_1$, $\alpha_2$ by replacing each $\sigma_j$ by $\sigma_{j+1}$ for $j = i, \ldots, n-1$.

$L$–moves and isotopy generate an equivalence relation in the set of braids, which we shall call *$L$–equivalence*. Let now $B$ be a braid. The *closure* of $B$ is the oriented link diagram $\mathcal{C}(B)$ obtained by joining each top end-point to the corresponding bottom end-point by almost vertical arcs as illustrated in figure 6, where $B$ is contained in a 'box'. (Note that we draw some smooth arcs for convenience.)

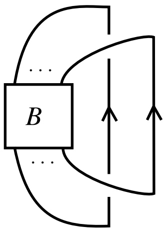



figure 6

Our aim is to prove the following theorem.

**Theorem 2.3 (One-move Markov theorem)** *$\mathcal{C}$ induces a bijection between the set of $L$–equivalence classes of braids and the set of isotopy types of (oriented) link diagrams.*

# 3 The braiding process

We shall define an inverse bijection to $\mathcal{C}$ by means of a canonical *braiding process* which turns an oriented link diagram (with a little extra structure) into a braid. Note that we only work with oriented diagrams, so in the sequel we shall drop the adjective. Let $D$ be a link diagram with no horizontal arcs, and consider the arcs in $D$ which slope upwards with respect to their orientations; call these arcs *opposite arcs.* In order to obtain a braid from that diagram we want:

1) to keep the arcs that go downwards.
2) to eliminate the opposite arcs and produce instead braid strands.

If we run along an opposite arc we are likely to meet a succession of overcrossings and undercrossings. We subdivide (marking with points) every opposite arc into smaller – if necessary – pieces, each containing crossings of only *one* type; i.e. we may have:

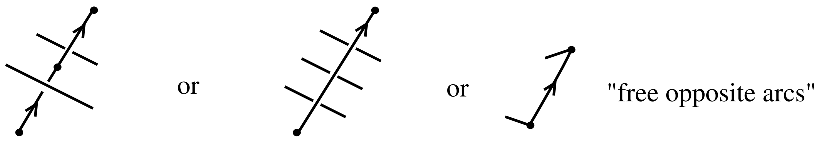



figure 7

We call the resulting pieces *up–arcs*, and we label every up–arc with an *'o'/'u'* according as it is the *over/under* arc of a crossing (or some crossings). If it is a *free up–arc* (and therefore it contains no crossings), then we have a choice whether to label it *'o'* or *'u'*. The idea is to eliminate the opposite arcs by eliminating their up–arcs one by one and create braid strands instead. Let now $P_1, P_2, \ldots, P_n$ be the top vertices of the up–arcs; fix attention on one particular top vertex $P = P_i$ and suppose that $P$ is the top vertex of the up–arc $QP$.

Associated to $QP$ is the *sliding triangle* $T(P)$, which is a special case of a triangle needed for a $\Delta$–move; it is right-angled with hypotenuse $QP$ and with the right angle lying below the up–arc. Note that, if $QP$ is itself vertical, then $T(P)$ degenerates into the arc $QP$. We say that a sliding triangle is of type *over* or *under* according to the label of the up–arc it is associated with. (This implies that there may be triangles of the same type lying one on top of the other.)

The germ of our braiding process is this. Suppose for definiteness that $QP$ is of type *over*. Then perform an $L_o$–move at $P$ followed by a $\Delta$–move across the sliding triangle $T(P)$ (see figure 8). By general position the resulting diagram will be regular and $QQ'$ may be assumed to slope slightly downwards. If $QP$ were *under* then the $L_o$–move would be replaced by an $L_u$–move. Note that the effect of these two moves has been to replace the up–arc $QP$ by three arcs none of which are up–arcs, and therefore we now have fewer up–arcs. If we repeat this process for each up–arc in turn, then the result will be a braid.

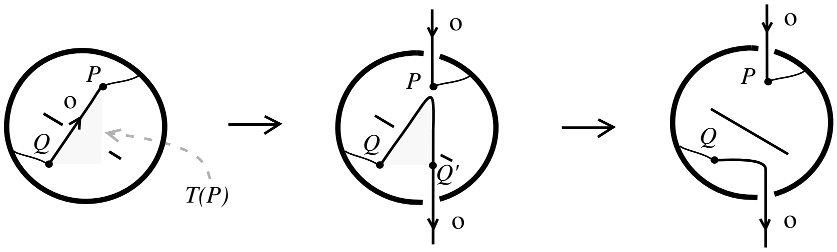
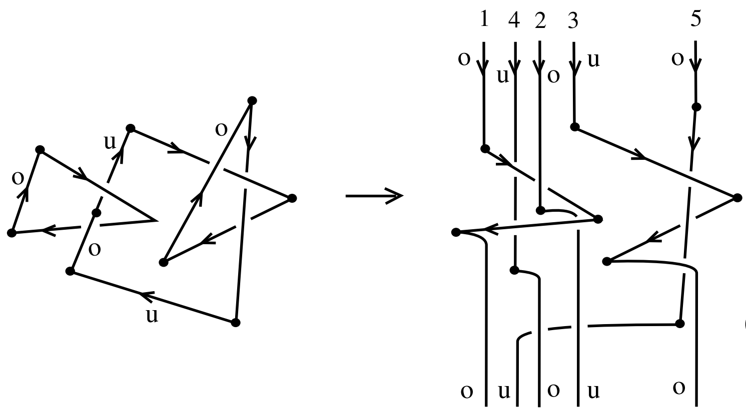



figure 8

In the following example we illustrate the braiding process applied to a particular link diagram (we do not draw the sliding triangles). The numbering of the strands indicates the order in which we eliminated the up–arcs. Obviously, for different orders we may obtain different braids.




$(\sigma_1^{-1}\sigma_4^{-1}\sigma_2\,\sigma_4^{-1}\sigma_2\,\sigma_1\,\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1})$

figure 9

It is clear from the above that different choices when doing the braiding (e.g. when choosing labels for the free up–arcs or when choosing an order for layering sliding triangles of the same type) as well as slight isotopy changes on the diagram level may result in important changes in the braid picture. For the proof of Theorem 2.3 we would like to control such changes as much as possible. For this we shall need a more rigorous setting.

Sliding triangles are said to be *adjacent* if the corresponding up–arcs have a common vertex (then the sliding triangles will have a common corner).

**Triangle condition** *Non-adjacent sliding triangles are only allowed to meet if they are of opposite types (i.e. one over and the other under).*

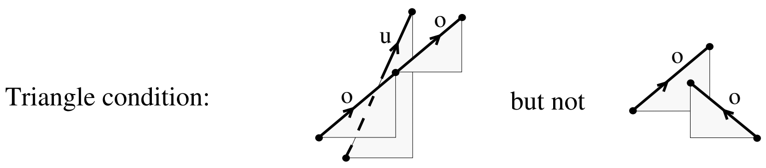



figure 10

**Lemma 3.1** *Given a link diagram $D$, there is a subdivision $D'$ of $D$ such that (for appropriate choices of under/over for free up–arcs) the triangle condition is satisfied.*

**Proof.** Let $d$ = minimum distance between any two crossings of $D$. Let $0 < r < d/2$ be such that any circle of radius $r$ centred at a crossing point does not intersect with any other arc of the diagram. Let $s$ be the minimum distance between any two points in $D$ further than $r$ away from any crossing point. Now let $\varepsilon = \frac{1}{2}\min\{s,r\}$ and $D'$ be a subdivision of the diagram such that the length of every up–arc is less than $\varepsilon$. Then the triangle condition is satisfied, provided we make the right choices (under/over) for sliding triangles of free up–arcs near crossings (see picture above). $\square$

**Remarks 3.2** (1) The triangle condition implies that the eliminating moves do not interfere with each other, so there are no pairs of sliding triangles that need layering and therefore it does not matter in what order we eliminate the up–arcs. In fact we can eliminate all of them *simultaneously*.

(2) The choice of $\varepsilon$ in the last proof is far smaller than necessary. In fact we only need to consider the subdiagram consisting of up–arcs and crossings on up–arcs.

(3) Notice that the choice of under/over for some of the free up–arcs may be forced by the labelling of other up–arcs (see for example the top *over* up–arc in figure 10). We can – if we wish – further subdivide the diagram (here the under up–arc) to make such pairs disjoint.

(4) If a diagram satisfies the triangle condition, then so does any subdivision (where the smaller triangles may need to retain the same labels).

**Definition 3.3** A *generic diagram* is a link diagram with subdividing points and sliding triangles put in general position with respect to the height function, such that the following conditions hold:

1) there are no horizontal arcs,
2) no two disjoint subdividing points are in vertical alignment, where by 'disjoint' we mean subdividing points that do not share a common edge.
3) any two non-adjacent sliding triangles satisfy the triangle condition and if they intersect, this should be along a common interior (and not a single point).

Conditions 1 and 3 are related to the braiding, whilst condition 2 ensures that no pair of strands in the resulting braid will be in the same vertical line.

**Definition 3.4** A *generic* $\Delta$*–move* is a $\Delta$–move between generic diagrams.

In the sequel, by '$\Delta$–moves' we shall always refer to the local planar $\Delta$–moves together with the Reidemeister moves.

**Lemma 3.5** *An isotopy between generic link diagrams can be realized using only generic* $\Delta$*–moves.*

**Proof.** If after some $\Delta$–move during the isotopy appears a horizontal arc or vertical alignment of vertices we remove it by replacing one of the participating vertices by a point arbitrarily close to it, so that the new point will not cause such a singularity or violation of condition 3 in *all* the diagrams of the isotopy chain. This is possible by a general position argument. If two non-adjacent sliding triangles touch so that condition 3 is violated (figure 11a) we argue as above. Also, by definition of regular isotopy, two non-adjacent triangles cannot touch on two subdividing points or on a point with a hypotenuse or along their hypotenuses. Therefore, the remaining possibilities are the ones illustrated in figure 11b.

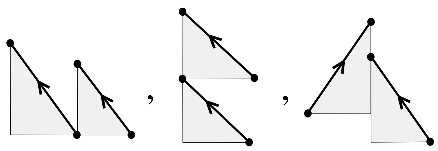


figure 11a

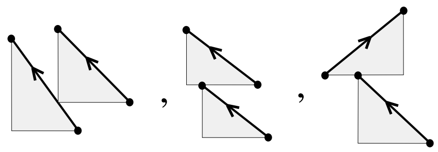


figure 11b

As before, by general position we can replace one of the participating vertices by a point arbitrarily close to it, so that the new point will not violate conditions 1, 2 and 3 in all the diagrams of the isotopy chain. Finally, if two triangles happen to intersect after a $\Delta$–move and they are of opposite type, then the move is generic. If they are of the same type and the intersection is not essential (see figure 12) we simply subdivide further and we carry the subdivision through in the whole chain of the isotopic diagrams, taking care that the subdividing point will not violate condition 2 in the whole chain. (Note that, by Remark 3.2 (4), condition 3 will not be violated.)

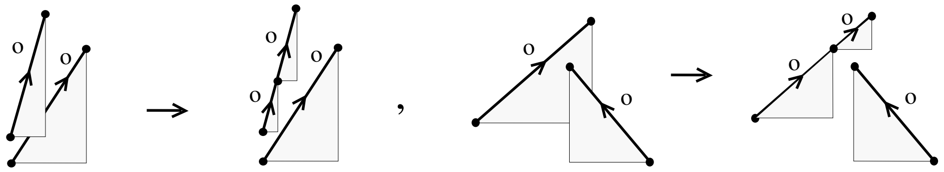



figure 12

If the intersection is essential (for example in figure 13 a free up–arc labelled 'u' moves by a $\Delta$–move over another arc), then we introduce an appropriate extra subdividing point so as to create a smaller free up–arc to which we attach the opposite label. This is always possible by Lemma 3.1.

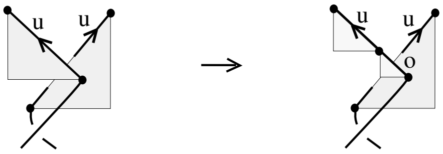



figure 13

□

It is clear from Lemma 3.5 that generic link diagrams are dense in the space of all diagrams; indeed, a non-generic link diagram may be seen as a middle stage of the isotopy between generic diagrams. Thus w.l.o.g. we shall assume from now on that all diagrams are generic; also, by virtue of Lemma 3.5 that all isotopy moves are also generic.

We are now ready to give a rigorous braiding process. Namely, take a link diagram and eliminate one by one the up–arcs in the way described above. By Remark 3.2 (1) the order of the eliminating moves is now irrelevant.

**Corollary 3.6 (Alexander's theorem)** *Any (oriented) link diagram is iso-*

*topic to the closure of a braid* [2].

**Proof.** The braiding process comprises $L$–moves and then isotopies. But the effect of an $L$–move after closure is by definition an isotopy. □

# 4 Proof of Theorem 2.3

By the local nature of the $\Delta$–moves we may assume that for a given link diagram we have done the braiding for all up–arcs except for the ones that we are interested in every time; these will be lying in the magnified region placed inside the braid. By Remark 3.2 (1) this choice does not affect the final braid.

Now, two braids that differ by a finite sequence of $L$–moves have isotopic closures. Therefore the function $\mathcal{C}$ from $L$–equivalence classes of braids to isotopy types of link diagrams is well-defined. To show that $\mathcal{C}$ is a bijection we shall use our braiding process to define an inverse function $\mathcal{B}$. Namely, for a diagram $D$ let $\mathcal{B}(D)$ be the braid resulting from the braiding algorithm applied to it. We have to show that $\mathcal{B}$ is a well-defined function from link-diagram types to $L$–equivalence classes of braids, therefore we have to check that $\mathcal{B}(D)$ does not depend up to $L$–equivalence on the choices made before the braiding and on $\Delta$–moves between link diagrams.

The choices made before the braiding consist of the subdividing points we choose and the labelling we may have to choose for some free up–arcs. Finally, we have to show that $\mathcal{C}$ and $\mathcal{B}$ are mutually inverse. This is easy: Closing the result of the braiding process yields a link isotopic to the original one, therefore $\mathcal{C} \circ \mathcal{B} = \mathrm{id}$. Moreover, applying the braiding process to the closure of a braid yields precisely the same braid back again (from the way we defined closure), so $\mathcal{B} \circ \mathcal{C} = \mathrm{id}$.

The proof relies entirely on Remark 3.2 (1) and on the following two lemmas.

**Lemma 4.1** *If we add on an up–arc, $\alpha$, an extra subdividing point $P$ and label the two new up–arcs, $\alpha_1$ and $\alpha_2$, the same as $\alpha$, the corresponding braids are $L$–equivalent.*

[2] H. Brunn [4] in 1897 proved that any link has a projection with a single multiple point; from which it follows immediately (by appropriate perturbations) that we can braid any link diagram. Other proofs of Alexander's theorem have been given by H.R. Morton [14], S. Yamada [24], P. Vogel [21].

**Proof.** For definiteness we assume that $\alpha$ is labelled with an 'o'. We complete the braiding of the original diagram by eliminating $\alpha$ (see picture below). Then, on the new horizontal piece of string, we take an arbitrarily small neighbourhood $N'$ around $P'$, the projection of $P$ (see picture below). By general position $N'$ slopes slightly downwards. We then perform an *over* L-move at $P'$. Finally, sliding an appropriate piece of string using braid planar isotopy, we obtain the braid that would result from the original diagram with the subdividing point $P$ included (see figure 14). (Note that new vertical strands coming from the braiding that may run over or under $\alpha$ do not affect the result.)

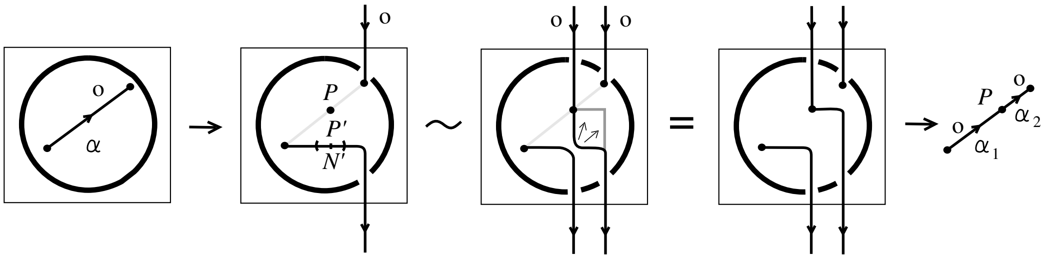



figure 14

□

**Lemma 4.2** *When we meet a free up–arc, which we have the choice of labelling 'u' or 'o', the resulting braid does not depend – up to L–equivalence – on this choice.*

**Proof.** First, we shall assume for simplicity that the sliding triangle of the up–arc does not lie over or under any other arcs of the original diagram. Also we assume for definiteness that the up–arc is originally labelled 'o'. We complete the braiding by eliminating it. Then on the new almost-horizontal piece of string we take an arbitrarily small neighbourhood $N'$ of a point $P'$ that is a projection of an arbitrarily small neighbourhood $N$ on the original up–arc, and such that there is no other vertical line between the vertical line of $P$ and the one of $P'$ (figure 15). We then perform an $L_u$–move at $P'$.

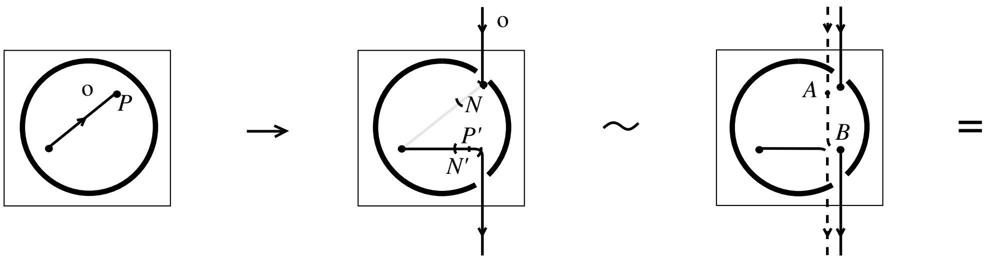



figure 15

The fact that the original up–arc is free and small enough implies that only vertical strands can pass over or under its sliding triangle. Therefore – as $N$ is arbitrarily small – there is no arc crossing $AB$ so as to force it be an *under* arc. Also, by braid planar isotopy we shift $A$ slightly *higher*, so as to come to the position where we can undo an $L_o$–move (see figure 16). We undo it, so the final braid – up to a small braid planar isotopy – can be seen as the braid that we would have obtained from the original diagram with the free up–arc labelled with 'u' instead of 'o'.

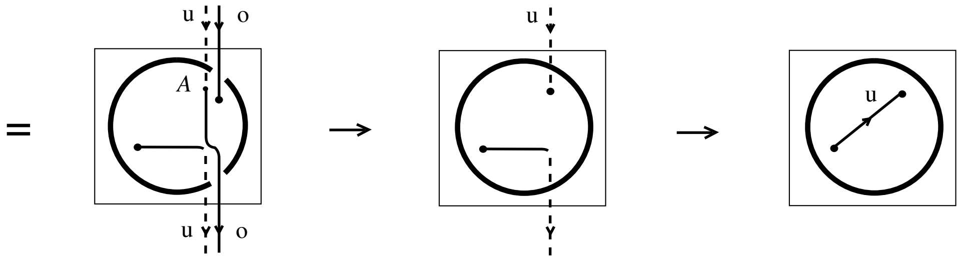



figure 16

Notice that, if the original up–arc were an 'u' we would perform an $L_o$–move at $P'$. To complete the proof of the lemma we assume that the sliding triangle of our up–arc lies over or under other arcs of the original diagram. In this case we subdivide it (using Lemma 4.1) into arcs small enough to ensure that all the sliding triangles are clear; we give all new arcs the labelling of the original one. Then we change the labelling of each up–arc using the above and, using Lemma 4.1 again, we eliminate all the new subdividing points (figure 17).

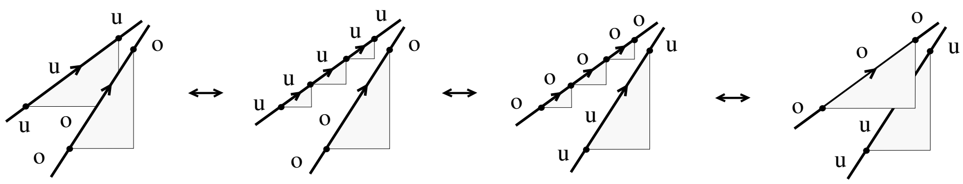



figure 17

□

**Corollary 4.3** *If we have a chain of overlapping sliding triangles of free up–arcs so that we have a free choice of labelling for the whole chain then, by Lemmas 4.1 and 4.2, this choice does not affect – up to L–equivalence – the final braid.*

**Corollary 4.4** *If by adding a subdividing point on an up–arc we have a choice for relabelling the resulting new up–arcs so that the triangle condition is still satisfied then, by Lemmas 4.1 and 4.2, the resulting braids are L–equivalent.*

**Corollary 4.5** *Given any two subdivisions, $S_1$ and $S_2$, of a diagram which will satisfy the triangle condition with appropriate labellings, the resulting braids are L–equivalent.*

**Proof.** By Remark 3.2 (4) this can be easily seen if we consider the subdivision $S_1 \bigcup S_2$ and apply the lemmas above. □

Corollary 4.5 proves independence of subdivision and labelling for diagrams, and thus we are done with the static part of the proof. Using Corollary 4.5 we have also proved independence of $\Delta$–moves related to condition 3 of Definition 3.3 (recall proof of Lemma 3.5).

It remains to consider the effect of the rest of the $\Delta$–moves. We shall check first the planar $\Delta$–moves. These will include the examination of $\Delta$–moves related to conditions 1 and 2 of Definition 3.3. Now, a $\Delta$–move can be regarded as a continuous family of diagrams (see figure 18). From the above we shall assume that the triangle condition is not violated. Also, by symmetry, we only have to check the $\Delta$–moves that take place in the first quarter of the plane, above the edge $AB$, and w.l.o.g. $AB$ is an up–arc. Therefore, the $\Delta$–moves we want to check can split into moves that take place above $AB$

within the vertical zone defined by $A$ and $B$ and moves that take place outside this zone. All resulting braids will be compared with the braid obtained by subdividing $AB$ at an appropriate point $P$. I.e. we have:

*(i) Inside subdivision* Consider the continuous family of triangles with edge $AB$ and all the new vertices lying on the vertical line of an interior point $P$ of $AB$ (figure 18). A new vertex cannot aligne horizontally with $B$. So, if $P'B$ is a new edge before the horizontal position from below, we can see $L$–equivalence of the corresponding braids by introducing an extra vertex in $AB$ at $P$ (figure 19) and applying Lemma 4.1.

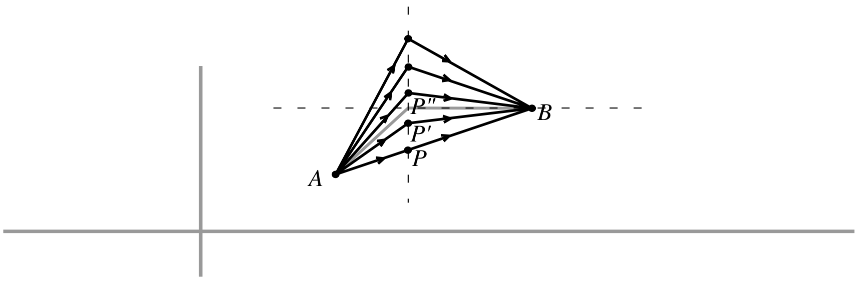



figure 18

If $P''B$ has passed above the horizontal position and therefore it becomes a down–arc (cf. Definition 3.3, condition 1), we introduce an $L$–move at the point $B$ and we show that again we obtain the same braid as if we had originally subdivided $AB$ at $P$ (see figure 19). The type of the $L$–move is over/under according as $AB$ is an over/under up–arc.

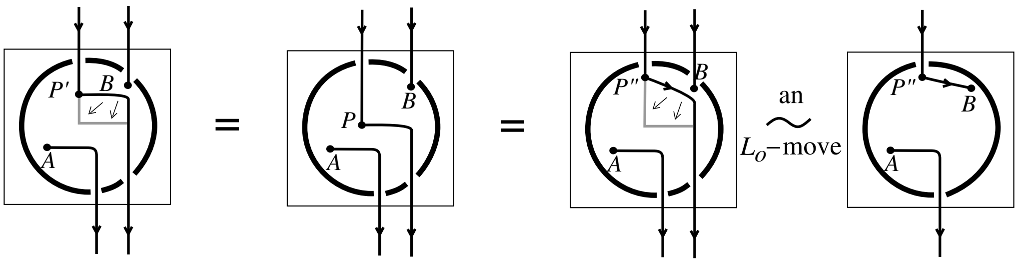



figure 19

Note that, if $AB$ were originally a down–arc, then the new edges would pass from down–arcs to opposite arcs through the horizontal position. In this case, according to the way we do the braiding and to the triangle condition, we may need to subdivide further the new opposite arcs and apply lemmas 4.1 and 4.2.

Before continuing with the proof we introduce some extra notation. We shall denote by $(APB)$ the braid obtained by completing the braiding on the arcs $AP$ and $PB$. Suppose now we subdivide $AB$ at a different point, $Q$ say, so that between the vertical lines of $P$ and $Q$ there is no other braid strand; then $(APB) = (AQB)$. If in between there is a braid strand (cf. Definition 3.3, condition 2), we argue that $(APB)$ and $(AQB)$ both differ by an $L$–move from $(APQB)$.

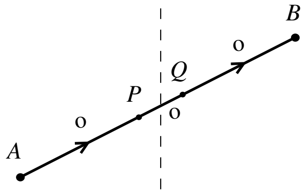



figure 20

*(ii) Outside 'subdivision'* Consider now the continuous family of triangles with edge $AB$ and all the new vertices lying on the vertical line of a point $P$ that lies outside the vertical zone defined by $A$ and $B$ (see figure 21). By symmetry we only need to examine the $\Delta$–moves taking place on the side of $B$, say. Assume first that $P$ is close enough to $B$ so that no braid strand passes in between the vertical lines of $P$ and $B$. Then obviously $(APB) \sim (AB)$ (delete the $L$–move at $P$). Reasoning as above, if $P'$ is a new vertex such that $P'B$ is below the horizontal position, then $(AP'B)$ differs from $(APB)$ by an $L$–move introduced at $B$. Also, if $P''$ is a new vertex such that $P''A$ is below the horizontal, then $(AP''B)$ differs from $(AP'B)$ by an $L$–move introduced on $AP''$ at the point hitting the vertical line of $B$.

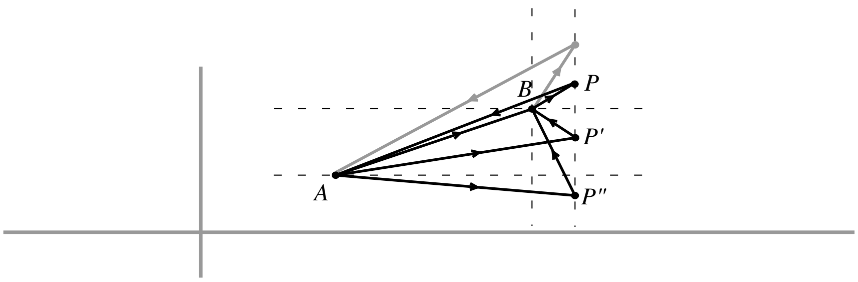



figure 21

Assume now that the $\Delta$–move introduces a new vertex $Q$ so that between the vertical lines of $P$ and $Q$ there is a braid strand (cf. Definition 3.3, condition 2). In this case we see the $L$–equivalence by introducing and deleting two $L_o$–moves (figure 22).

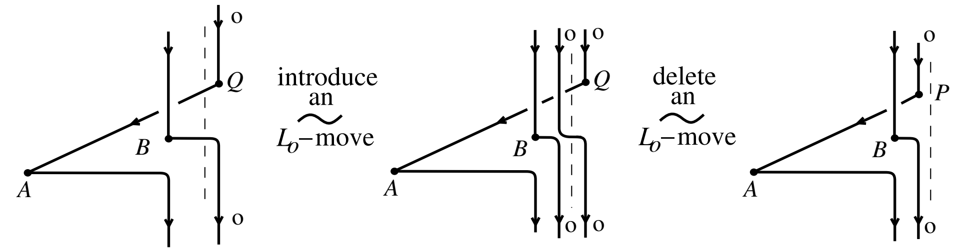



figure 22

We shall now check the Reidemeister moves (recall figure 3).

To check the first two moves we follow similar reasoning as above, but we shall demonstrate it here for completeness. For the first one, we illustrate below that the braid obtained after the performance of the move is equivalent to the one obtained before the move, up to braid isotopy and one $L_o$–move:

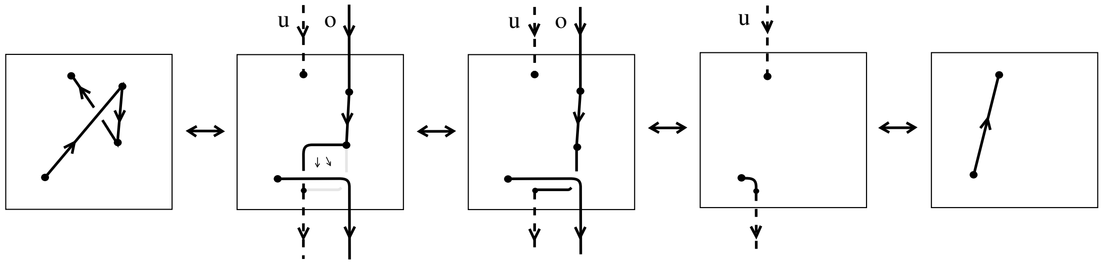



figure 23

For the second move we complete the braiding of the left-hand side diagram and we notice that, using braid isotopy, we can undo an $L_u$–move. We then obtain the braid that we would obtain from the right-hand side diagram with label ‘u’ for the up–arc.

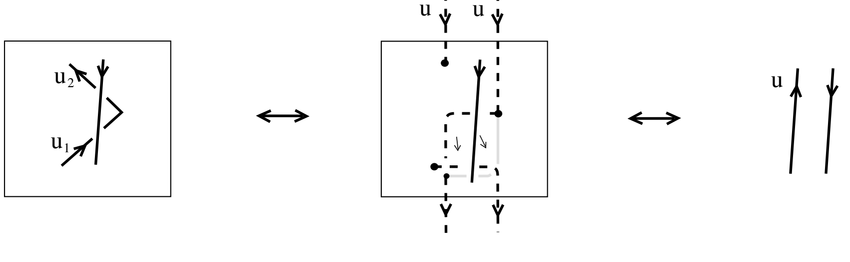



figure 24

We shall now check the third type of Reidemeister moves or 'triple point moves'. If all three arcs are down–arcs then the move is a braid isotopy. If either of the outer arcs is an up–arc then the move is invisible in the braid. It remains to check the case where both outer arcs are down–arcs and the middle arc is an up–arc. But then the following trick changes the situation to the one where all three arcs are down–arcs.

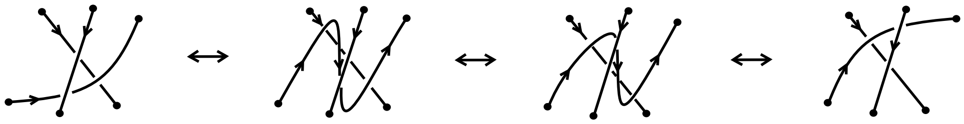


figure 25

We have now proved that the function $\mathcal{B}$ from isotopy types of link diagrams to $L$–equivalence classes of braids is well-defined, and the proof of Theorem 2.3 is now completed. $\square$

## 4.1 A comment on conjugation

The classical Markov braid theorem states that: *Isotopy classes of (oriented) link diagrams are in 1-1 correspondence with certain equivalence classes in the set of all braids, the equivalence being given by the following two algebraically formulated moves between braids in* $\bigcup_{n=1}^{\infty} B_n$*:*

*(i) Conjugation: If* $\alpha, \beta \in B_n$ *then* $\alpha \sim \beta^{-1}\alpha\beta$.

*(ii) Markov moves or M–moves: If* $\alpha \in B_n$ *then* $\alpha \sim \alpha\sigma_n^{+1} \in B_{n+1}$ *and* $\alpha \sim \alpha\sigma_n^{-1} \in B_{n+1}$.

It is clear from Remark 2.2 that an $M$–move is a special case of an $L$–move. Also, it follows from the proof of Theorem 2.3 that conjugation can be realized by a (finite) sequence of $L$–moves. Indeed, let $\alpha$ and $\beta^{-1}\alpha\beta$ be two conjugate braids. Then their closures $\mathcal{C}(\alpha)$ and $\mathcal{C}(\beta^{-1}\alpha\beta)$ are isotopic diagrams and so they differ by a sequence of $n$, say, $\Delta$–moves; so, for each stage of isotopy we obtain the following sequence of link diagrams

$$\mathcal{C}(\beta^{-1}\alpha\beta) \sim L_1 \cdots \sim L_{n-1} \sim \mathcal{C}(\alpha)$$

which we turn into the braids $B_0, B_1, \cdots, B_{n-1}, B_n$, say, using our algorithm. From the above, every consecutive pair of these braids differs by a finite sequence of $L$–moves. Now, the way we defined the closure of a braid (recall figure 6) guarantees that $B_0$ is $\beta^{-1}\alpha\beta$ and $B_n$ is $\alpha$. Thus, conjugation is only a redundant 'auxilliary' move used to bring the $M$–moves inside the braid box, and therefore Theorem 2.3 indeed sharpens the classical result.

As a concrete example, we illustrate below the main instances of the isotopy sequence from the closure of a braid that is conjugate to $\alpha$ by an elementary crossing, up to $\mathcal{C}(\alpha)$.

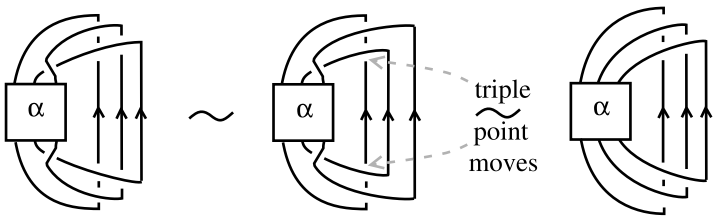



figure 26

**Remarks 4.6** (1) The proof of Theorem 2.3 may be clearly used as an alternative proof of the classical theorem. Indeed, whenever in the proof appears an $L$–move we use conjugation to bring it to the position of an $M$–move. The same reasoning holds for all Markov-type theorems that follow in the sequel.

(2) It should be stressed that Theorem 2.3 sharpens the classical result only if we work with open braids. If we work with closed braids we do not need conjugation in the braid equivalence. We should also stress that vertical alignment of top vertices is the most important case to check in the proof of Theorem 2.3 because it corresponds precisely to conjugation by a generator $\sigma_i$ in $B_n$.

## 4.2 Relative version of Markov's theorem

As a consequence of the proof of Theorem 2.3 we can prove a relative version of the result. By a *braided portion* of a link diagram we mean a finite number of arcs in the link which are all oriented downwards (so that the braid resulting from our braiding process will contain this braided portion).

**Theorem 4.7** *Let $L_1$, $L_2$ be oriented link diagrams which both contain a common braided portion $B$. Suppose that there is an isotopy of $L_1$ to $L_2$ which finishes with a homeomorphism fixed on $B$. Suppose further that $B_1$ and $B_2$ are braids obtained from our braiding process applied to $L_1$ and $L_2$ respectively. Then $B_1$ and $B_2$ are $L$–equivalent by moves that do not affect the common braided portion $B$.*

**Proof.** Assume first that the isotopy from $L_1$ to $L_2$ keeps $B$ fixed; the result then follows immediately from the proof of Theorem 2.3 because the braided portion does not participate in the proof. We shall use some standard PL topology to reduce the general case to this special case.

Let $N$ denote a relative regular neighbourhood of $B$ relative to the ends of the arcs (i.e. $N$ comprises a number $t$ say of 3–balls each of which contains an arc of $B$ as an unknotted subarc). Think momentarily of these balls as small balls centred at points $P_1, \ldots, P_t$. The isotopy restricted to $P_1, \ldots, P_t$ determines a loop in the configuration space of $t$ points in $\mathbf{R}^3$. But this configuration space is well-known to be simply-connected and therefore we may assume that the isotopy is fixed on $P_1, \ldots, P_t$. By the regular neighbourhood theorem we may now assume that the isotopy fixes $N$ setwise.

Now restrict attention to one of the balls. The isotopy on this ball determines a loop in the space of the PL homeomorphisms of the 3–ball. But $\pi_1$ of this space is generated by a rotation through $2\pi$ about some axis. We may suppose that this axis is the corresponding unknotted arc of $B$ and hence that the isotopy fixes this subarc pointwise. Thus we may assume that the whole of $B$ is fixed pointwise. The result now follows from the special case. $\square$

# 5 Extension of results to other 3–manifolds

In this section we use the methods of Theorem 2.3 to prove a second relative version (relative to a fixed closed subbraid) and deduce an analogue of Markov's theorem for isotopy of oriented links in knot/link complements in $S^3$. This is then used for extending the results to closed connected orientable 3–manifolds.

## 5.1 Markov's theorem in knot complements

Let $S^3 \backslash K$ be the complement of the oriented knot $K$ in $S^3$. By 'knot complement' we refer to complements of both knots and links. Using Alexander's theorem and the definition of ambient isotopy, $S^3 \backslash K$ is homeomorphic to $M = S^3 \backslash \widehat{B}$, where $\widehat{B}$ is isotopic to $K$ and it is the closure of some braid $B$. Let now $L$ be an oriented link in $M$. If we fix $\widehat{B}$ pointwise on its projection plane we may represent $L$ unambiguously by the *mixed link* $\widehat{B} \bigcup L$ in $S^3$, that is, a link in $S^3$ consisting of the *fixed part* $\widehat{B}$ and the *standard part* $L$ that links with $\widehat{B}$ (for an example see figure 27a).

**Definition 5.1** A *mixed link diagram* is a diagram $\widehat{B} \bigcup \widetilde{L}$ of $\widehat{B} \bigcup L$ projected on the plane of $\widehat{B}$ which is equipped with the top-to-bottom direction of $B$.

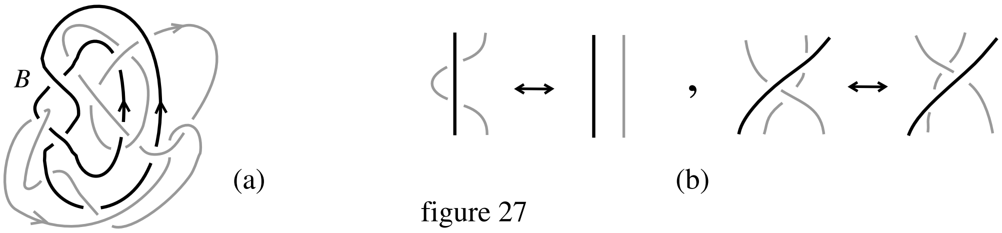



figure 27

Let now $L_1$, $L_2$ be two oriented links in $M$. It follows from standard results of PL Topology that $L_1$ and $L_2$ are isotopic in $M$ if and only if the mixed links $\widehat{B} \bigcup L_1$ and $\widehat{B} \bigcup L_2$ are isotopic in $S^3$ by an ambient isotopy which keeps $\widehat{B}$ pointwise fixed. See for example [17]; chapter 4. In terms of mixed diagrams this isotopy will involve the fixed part of the mixed links only in the moves illustrated in figure 27b. These shall be called 'extended Reidemeister moves'. Therefore Reidemeister's theorem generalizes as follows in terms of mixed link diagrams.

**Theorem 5.2** *Two (oriented) links in $M$ are isotopic if and only if any two corresponding mixed link diagrams (in $S^3$) differ by a finite sequence of the extented Reidemeister moves together with the planar $\Delta$–moves and the Reidemeister moves for the standard parts of the mixed links.*

We now wish to find an analogue of Markov's theorem for links in $M$. Since $\widehat{B}$ must remain fixed we need a braiding process for mixed link diagrams that maps $\widehat{B}$ to $B$ (and *not* an $L$–equivalent braid or a conjugate of $B$). Such braids shall be called *mixed braids* (see figure 28c for an example). Also the subbraid of a mixed braid that complements $B$ shall be called *the permutation braid.*

**Theorem 5.3 (Alexander's theorem for $S^3 \backslash \widehat{B}$)** *Any (oriented) link in $S^3 \backslash \widehat{B}$ can be represented in $S^3$ by some mixed braid $B_1 \bigcup B$, the closure of which is isotopic to a mixed link diagram $\widetilde{L_1} \bigcup \widehat{B}$ representing the link.*

**Proof.** The fixed part $\widehat{B}$ may be viewed in $S^3$ as the braid $B$ union an arc, $k$ say, at infinity. This arc is the identification of the two horizontal arcs containing the endpoints of $B$ and thus it realizes the closure of $B$. (In figure 28a the braid $B$ is drawn curved but this is just for the purpose of picturing it.) Let $N(k)$ be a small regular neighbourhood of $k$ and let $L$ be a link in $M$. By general position $L$ misses $N(k)$ and therefore it can be isotoped into the complement of $N(k)$ in $S^3$.

By expanding $N(k)$ we can view its complement as the cylinder $T = D^2 \times I$ that contains $B$ (figure 28b). We then apply our braiding in $T$. This will leave $B$ untouched but it will braid $L$ in $T$. Finally we cut $\widehat{B}$ open along $k$ in order to obtain a mixed braid (figure 28c).

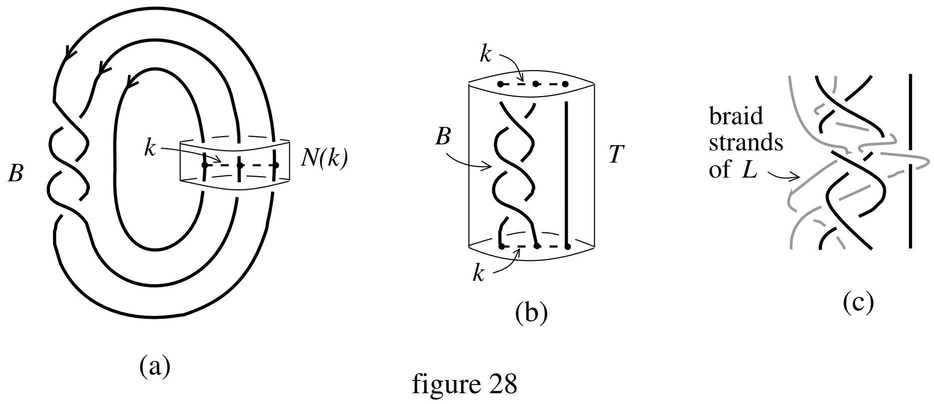



figure 28

□

**Note 5.4** An alternative more 'rigid' proof modifying the braiding of section 3 is given in [8],[9][3].

We are now ready to prove the second relative version of Markov's theorem and a version for knot complements.

**Theorem 5.5** *Two oriented links in $S^3$ which contain a common subbraid $B$ are isotopic rel $B$ if and only if any two corresponding mixed braids $B_1 \bigcup B$ and $B_2 \bigcup B$ are $L$–equivalent by moves that do not touch the common subbraid $B$.*
*Further any oriented links in $S^3 \backslash \widehat{B}$ are isotopic in $S^3 \backslash \widehat{B}$ if and only if any two corresponding mixed braids $B_1 \bigcup B$ and $B_2 \bigcup B$ in $S^3$ are $L$–equivalent by moves that do not touch the common subbraid $B$.*

[3]Theorem 5.3 can be also proved using for example Alexander's original braiding ([1], [3]) or alternatively Morton's threading ([14]), both resulting closed braids, so that the braid axis/thread goes through a point on the plane around which all strings of $\widehat{B}$ have counter-clockwise orientation; then we cut the braid open along a line that starts from the central point and cuts through the closing side of $\widehat{B}$.

**Proof.** We prove the relative version. The version for knot complements then follows immediately from the discussion above.

Consider an isotopy of a link $L$ in the complement of $\widehat{B}$. By a general position argument we may assume that this isotopy only crosses the arc $k$ a finite number of times and that these crossings are clear vertical cuts (figure 29a). The neighbourhood $N(k)$ misses also the isotopy apart from those clear crossings. As in the proof above, we expand $N(k)$ and thus the isotopy now takes place inside $T$ apart from the crossings with $k$. Applying now the proof of Theorem 2.3, the isotopies within $T$ can be converted into a sequence of $L$–moves that do not affect $B$. Therefore it only remains to check $L$–equivalence at a crossing of $L$ with $k$. But after expanding $N(k)$ such a crossing looks like exchanging a braid strand, $l$ say, that runs along the front of the mixed braid for one with the same endpoints that runs along the back of the braid (figure 29b).

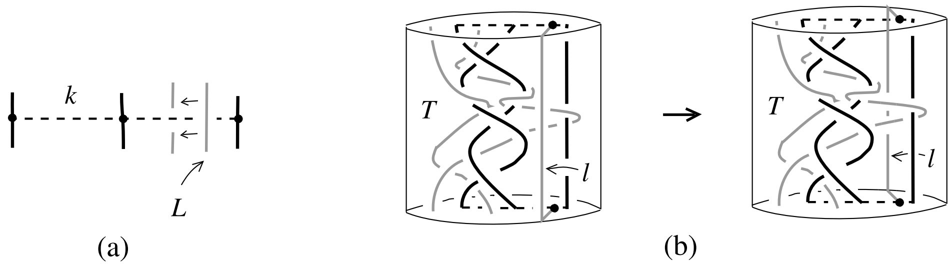



figure 29

If $l$ is an up–arc it can be made into a free up–arc and so it dissappears after the braiding. If $l$ is a down–arc we introduce an $L_u$–move at its top part so that between the endpoints of the new strand and the endpoints of $l$ there is no other braid strand. We then undo an $L_o$–move as depicted below, and the proof is concluded. Note that it is necessary to use the version of an $L$–move with a crossing (recall figure 5).

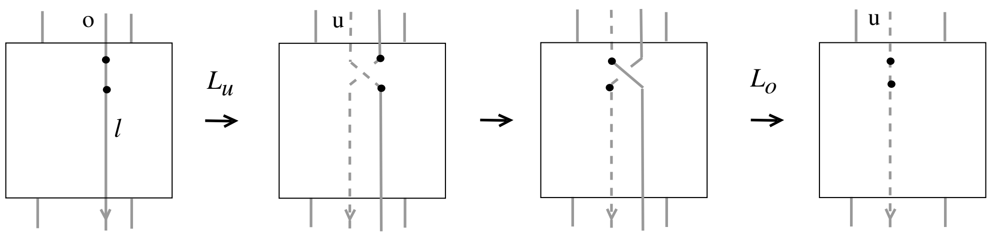



figure 30

□

## 5.2 Markov's theorem in oriented 3–manifolds

Let $M$ be a closed connected orientable (c.c.o.) 3–manifold. It is well–known [12], [22], [16] that $M$ can be obtained by surgery on a framed link in $S^3$ with integral framings and w.l.o.g. this link is the closure $\widehat{B}$ of a braid $B$. We shall refer to $\widehat{B}$ as the *surgery link* and we shall write $M = \chi(S^3, \widehat{B})$. Moreover by the proof given in [12] it can be assumed that all the components of the surgery link are unknotted and furthermore that they form the closure of a pure braid. Thus we may assume that $B$ is a pure braid. So $M$ may be represented in $S^3$ by the framed $\widehat{B}$ and if we fix $\widehat{B}$ pointwise we have that links/braids in $M$ can be unambiguously represented by mixed links/braids in $S^3$ – exactly as discussed in 5.1. Thus Theorem 5.3 holds also in this case. Only here, the braid $B$ is framed and we refer to it as the *surgery braid.*

Now a link $L$ in $M$ may be seen as a link in $S^3 \backslash \widehat{B}$ with the extra freedom to slide across the 2–discs bounded in $M$ by the specified longitudes of the components of $\widehat{B}$.

**Definition 5.6** Let $b$ be the oriented boundary of a ribbon and let $L_1 \bigcup \widehat{B}$ and $L_2 \bigcup \widehat{B}$ be two oriented mixed links, so that $L_2 \bigcup \widehat{B}$ is the band connected sum (over $b$) of a component, $c$, of $L_1$ and the specified (from the framing) longitude of a surgery component of $\widehat{B}$. This is a non–isotopy move in $S^3$ that reflects isotopy between $L_1$ and $L_2$ in $M$ and we shall call it *band move.*

A band move can be thus split in two steps: firstly, one of the small edges of $b$ is glued to a part of $c$ so that the orientation of the band agrees with the orientation of $c$. The other small edge of $b$, which we shall call *little band* (in ambiguity with the notion of a band), approaches a surgery component of $\widehat{B}$ in an arbitrary way. Secondly, the little band is replaced by a string running in parallel with the specified longitude of the surgery component in such a way that the orientation of the string agrees with the orientation of $b$ and the resulting link is $L_2 \bigcup \widehat{B}$.

Since we consider oriented links in $M$ there are two types, $\alpha$ and $\beta$ say, of band moves according as in the second step the orientation of the string replacing the the little band agrees (type $\alpha$) or disagrees (type $\beta$) with the orientation of the surgery component – and implicitely of its specified longitude. (See figure 31, where $p$ is an integral framing of the surgery component.)

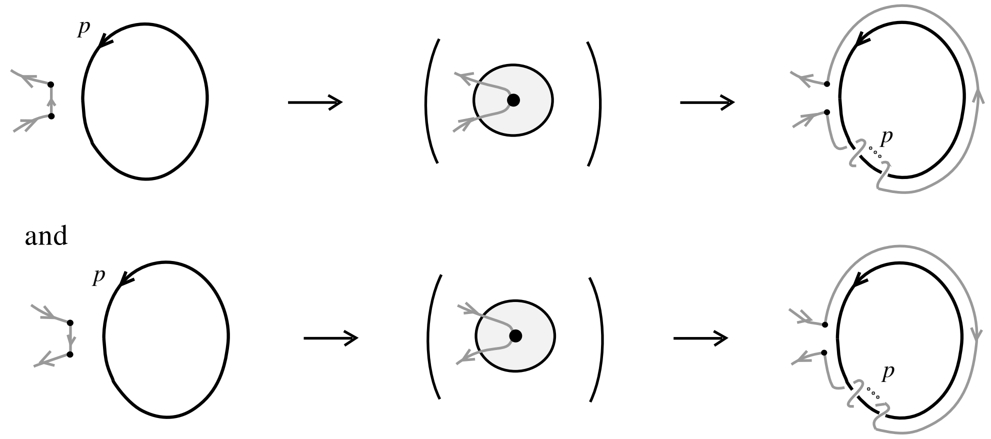



figure 31

The two types are related in the following sense.

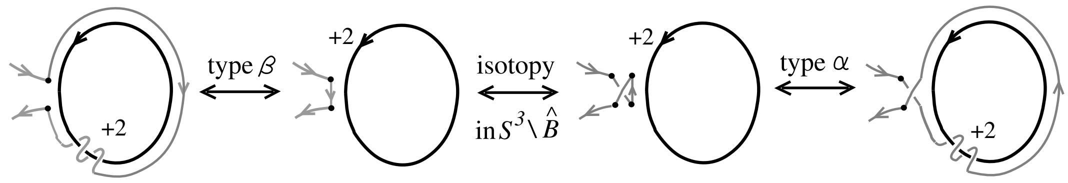



figure 32

**Remarks 5.7** (1) Let $L_1 \bigcup \widehat{B}$ and $L_2 \bigcup \widehat{B}$ represent the isotopic links $L_1$ and $L_2$ in $M$. If in the isotopy sequence there is no band move involved on the mixed link level, then $L_1$ and $L_2$ are also isotopic in $S^3\setminus\widehat{B}$. In particular, the first step of a band move reflects isotopy in $S^3\setminus\widehat{B}$. So, from now on whenever we say 'band move' we will always be referring to the realization of the second step of a band move.

(2) A band move, that is, the second step of the move described in Definition 5.6, takes place in an arbitrarily thin tubular neighbourhood of the component of the surgery link that contains no other part of the mixed link; and since $\widehat{B}$ is pointwise fixed it follows that this tubular neighbourhood projects the same for any diagram of $L_2 \bigcup \widehat{B}$. So by 'band move' we may unambiguously refer to both the move in the 3–space and its projection.

The discussion can be summarized by saying that link isotopy in $M$ can be regarded as isotopy in $S^3\backslash\widehat{B}$ together with a finite (by transversality) number of band moves. Therefore Theorem 5.2 extends to:

**Theorem 5.8 (Reidemeister's theorem for $\chi(S^3,\widehat{B})$)** *Two (oriented) links in $M$ are isotopic if and only if any two corresponding mixed link diagrams differ by a finite sequence of the band moves, the extented Reidemeister moves and also by planar $\Delta$–moves and the Reidemeister moves for the standard parts of the mixed links.*

Note now that neither of the two types of band moves can appear as a move between braids; so in order to state our extension of Markov's theorem we modify the band move of type $\alpha$ appropriately by twisting the little band before performing the move. So we have the following.

**Definition 5.9** A *braid band move* or, abbreviated, a *b.b.–move* is a move between mixed braids that reflects isotopy in $M = \chi(S^3,\widehat{B})$ and is described by the following picture (where the middle stage is only indicative).

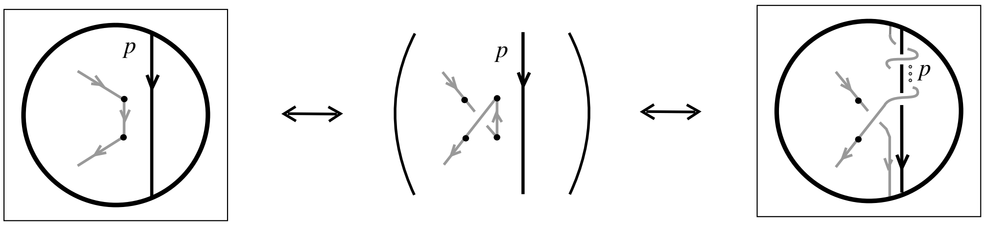



figure 33

Note that a braid band move can be positive or negative depending on the type of crossing we choose for performing it.

**Theorem 5.10 (Markov's theorem for $M = \chi(S^3,\widehat{B})$)** *Let $L_1$, $L_2$ be two oriented links in $M$ and let $B_1\bigcup B$, $B_2\bigcup B$ be two corresponding mixed braids in $S^3$. Then $L_1$ is isotopic to $L_2$ if and only if $B_1\bigcup B$ is equivalent to $B_2\bigcup B$ by the braid band moves and under $L$–equivalence that does not affect $B$.*

**Proof.** Let $\widetilde{L_1}\bigcup\widehat{B}$ and $\widetilde{L_2}\bigcup\widehat{B}$ be two mixed link diagrams of the mixed links representing $L_1$ and $L_2$. By Theorems 5.8 and 5.5 we only have to check that if $\widetilde{L_1}\bigcup\widehat{B}$ and $\widetilde{L_2}\bigcup\widehat{B}$ differ by a band move then $B_1\bigcup B$ and $B_2\bigcup B$ differ by b.b.–moves and $L$–equivalence that does not affect $B$. In $\widetilde{L_1} \bigcup \widehat{B}$ the little band would be like

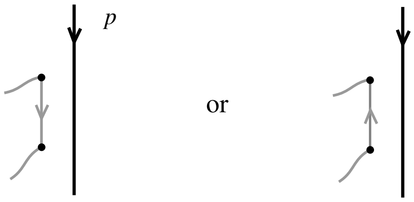



figure 34

depending on the orientation. If the little band is an opposite arc, w.l.o.g. we may assume that it satisfies the triangle condition. The algorithm we use ensures that we may assume that $\widetilde{L_1} \bigcup \widehat{B}$ and $\widetilde{L_2} \bigcup \widehat{B}$ are braided everywhere except for the little band in $\widetilde{L_1} \bigcup \widehat{B}$ (if it is an opposite arc) and its replacement after the performance of the band move. This happens because the band move takes place arbitrarily close to the surgery strand; so we can produce such a zone locally in the braid (figure 35), and consequently a b.b.–move cannot create problems with the triangle condition.

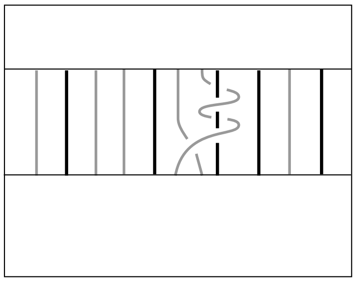


figure 35

Moreover the new strand from the band move (as far as other crossings are concerned) behaves in the same way as the surgery string itself. So, whenever we meet other opposite arcs we label them in the same way that we would do if the new strand were missing.

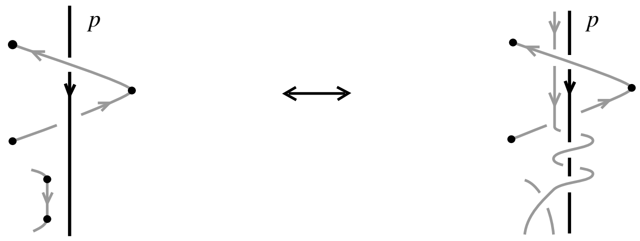



figure 36

Hence the different cases of applying a band move to $\widetilde{L_1} \bigcup \widehat{B}$ amount to the following (with proofs).

*(i)*

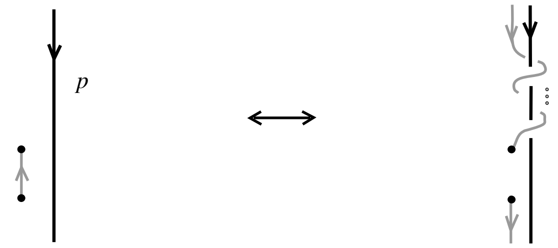



figure 37

**Proof.** We start with the left part of move *(i)* and we twist the little band (isotopy in $S^3 \backslash \widehat{B}$) using a *negative* crossing. Then we perform a b.b.–move with a *positive* crossing and we end up with the right part of move *(i)* (see figure 38).

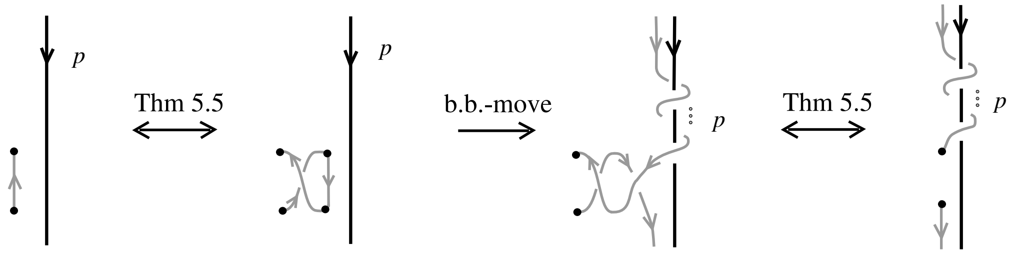



figure 38

*(ii)*

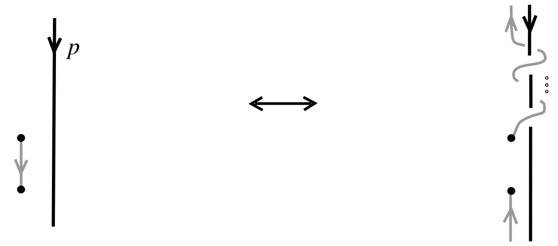



figure 39

**Proof.** We start with the right part of move *(ii)*. In the front of the otherwise braided link we do a twist of the new string using a *negative* crossing (see figure 40). Then we consider the little twisted arc as a little band and we perform another b.b.–move around the same surgery component. This second band move takes place closer to the surgery component than the first one.

Now, the shaded region in the picture below is formed by two similar sets of opposite twists of the same string around the surgery string. So it bounds a disc (together with the little band that is missing), the circumference of which is *not* linked with the surgery component; but this is isotopic in $S^3\backslash\widehat{B}$ to the left part of move *(ii)*. I.e. move *(ii)* is a finite sequence of $L$–moves and b.b.–moves.

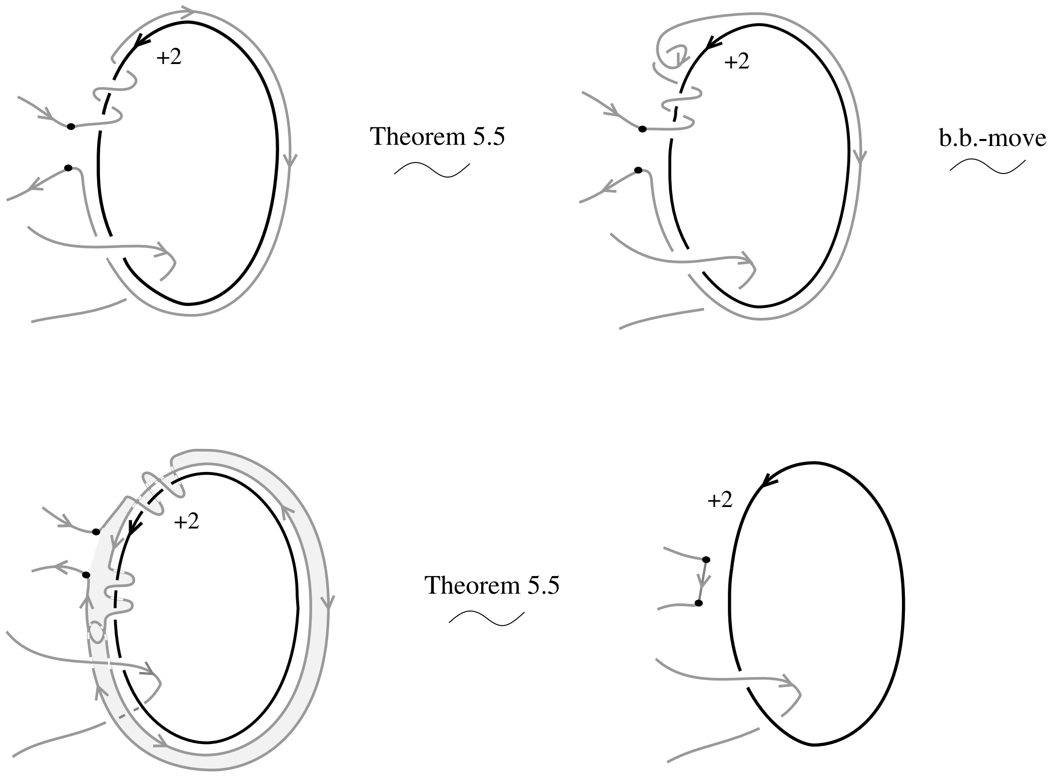



figure 40

Note that in the pictures above we have included another string of the mixed braid that links with the surgery component. Clearly this does not affect the proof. The proof of Theorem 5.10 is now concluded. □

**Remarks 5.11** (1) The proof holds even if the surgery braid is not a pure braid. In this case a b.b.–move is modified so that the replacement of the little band links only with one of the strings of the same surgery component and runs in parallel to all remaining strings of the surgery component.

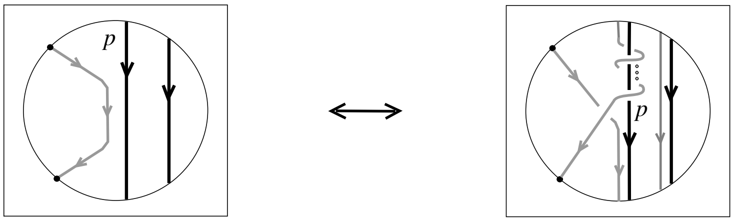



figure 41

(2) An analogue of Reidemeister's theorem for c.c.o. 3–manifolds is given in [19] and an analogue of Markov's theorem for c.c.o. 3–manifolds is given in [18], both using the intrinsic structure of the manifold.

(3) Theorems 5.5 and 5.7 above are described only geometrically. As shown in [8], [10] if our manifold is the complement of the unlink or a connected sum of lens spaces then in the set of mixed braids we have well-defined group structures, while in the other 3–manifolds we have coset structures, which enable us to give the above theorems an algebraic description. In particular, if our space is a solid torus (i.e. the complement of the unknot) or a lens space $L(p, 1)$ the related braid groups are the Artin groups of type $\mathcal{B}$ (for details and Jones-type knot invariants we refer the reader to [8], [9], [6]).

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S.L.: 16 Mill Lane, DPMMS, University of Cambridge, Cambridge CB2 1SB, U.K., E-mail sofia@pmms.cam.ac.uk

C.P.R.: Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K., E-mail cpr@maths.warwick.ac.uk