James Bundles and Applications

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SUMMARY

We study cubical sets without degeneracies, which we call $\square$–sets. These sets arise naturally in a number of settings and there is a beautiful geometry associated with them; in particular a $\square$–set C has an infinite family of associated $\square$–sets $J^i(C)$, $i = 1, 2, \ldots$, which we call James complexes. There are projections p_i: $|J^i(C)| \to |C|$ which are mock bundle projections. These define classes in unstable cohomotopy which generalise the classical James-Hopf invariants of $\Omega(S^2)$ and for this reason we call these bundles James bundles. The algebra of James bundles mimics the algebra of the cohomotopy of $\Omega(S^2)$ and the reduction to cohomology defines a sequence of natural characteristic classes for a $\square$–set. The main application is to the rack space (defined in [FRS]). In this case the first James bundle is a classifying bundle for links of codimension 2 with representation in the given rack. The invariants of the rack space (including the James–Hopf invariants) provide many new invariants for racks and hence for knots and links in codimension 2. A second application, which will be explored further in a later paper is to Vassiliev theory. There is a definition of a Vassiliev invariant for any $\square$–set generalising the classical case and an obstruction theory with obstructions in the cohomology of the James complexes for extending a partially defined invariant over the whole set.

Other applications are given. There is a combinatorial interpretation for π_2 of a complex which can be used for calculations and some new interpretations of the higher homotopy groups of the 3–sphere. There is also a generalised cohomology theory associated to any $\square$–set with geometric interpretation similar to that for Mahowald orientation [M, S1].

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0 Introduction

This paper is about the geometry of □–sets, pronounced "square sets", which are cubical sets without degeneracies. These arise naturally in a number of settings and have an intricate instrinsic geometric structure associated with them. In this paper we shall describe some of this structure and give several applications. The main application is to the classification of links of codimension 2 and the consequent construction of knot and link invariants. Other applications include an obstruction theory for the construction of Vassiliev invariants, a new combinatorial description of π_2 of any space and some new geometric descriptions for the higher homotopy groups of the 3–sphere.

The content and organisation of this paper are as follows :

Section 1 : Basic definitions (page 5)

In this section we define □–sets and □–maps. We show that any □–set can be subdivided to yield a Δ–set and conversely; hence there is no restriction on the homotopy type of a □–set. Examples of □–sets are the singular □–set of a topological space, the nerve of a category, the rack space and the singular knot space. The first three examples were studied in [FRS] in a more formal setting; in particular the rack space BX is a □–set associated to any rack X, and hence to any link of codimension 2 (see [FR] for the basics and history of racks). Several properties of rack spaces were established in [FRS] including connections with classical classifying spaces for categories and crossed complexes. In this paper we are concerned with geometric applications.

For each □–set C we define an infinite family of associated □–sets $J^i(C)$, $i = 1, 2, \ldots$ with natural projections $p_i\colon |J^i(C)| \to |C|$ for each i, where $|C|$ denotes the realisation of the □–set C. These are the James complexes of C.

Section 2 : Mock bundles and transversality (page 12)

We define a mock bundle over a □–set (cf [BRS]) and observe that the projections p_i define mock bundles $\zeta^i(C)$ which embed as framed mock bundles in $|C| \times \mathbb{R}$. These are the James bundles of C. We define transversality for a map of a smooth manifold into a □–set and prove that any map can be approximated by a transverse map. Mock bundles pull-back over transverse maps to yield mock bundles whose total spaces are manifolds and in particular the first James bundle pulls back to give a self-transverse immersion of codimension 1 and the higher James bundles pull-back to give the multiple point sets of this immersion.

Section 3 : James bundles and James–Hopf invariants (page 22)

In this section we explain the reason for our choice of terminology. We show that the trivial □–set T with a unique cell in each dimension is the free monoid on S^1 and hence has the homotopy type of $\Omega(S^2)$ by the James construction [J1]; in this case the James bundles define the classical James–Hopf invariants. The general James bundles pull-back from the James bundles of T and hence define generalised James–Hopf invariants. The proof involves a more precise treatment of the embedding of $\zeta^i(C)$ in $|C| \times \mathbb{R}$.

We also give a short proof of the formula of Baues [Ba] for the cup products of the (stable) James bundles and describe the algebra which results. By considering the images of the James bundles in cohomology, we define a natural family of characteristic classes for any $\square$–set whose mod 2 reductions are pulled back from both Steifel–Whitney and Wu classes of BO.

Section 4 : Links and Diagrams (page 30)

In this section we use the transversality theorem from section 2 to prove that any $\square$–set C is the classifying space for cobordism classes of labelled link diagrams. The labelling is by the cubes of C. More precisely, the homotopy group $\pi_n(C)$ is naturally isomorphic to the group of cobordism classes of link diagrams in $\mathbb{R}^n$ labelled by C. Any such link diagram is naturally covered by an embedded codimension 2 link in $\mathbb{R}^{n+1}$ and the whole structure can be seen to coincide with the pull-back of the first James bundle $\zeta^1(C)$.

In the key example in which C is the rack space BX, there is a far simpler description : $\pi_n(BX)$ is naturally isomorphic to the group of cobordism classes of codimension 2 links embedded in $\mathbb{R}^{n+1}$ with a homomorphism of the fundamental rack to X. Thus BX is a classifying space for such links up to cobordism and moreover the first James bundle $\zeta^1(BX)$ plays the rôle of classifying bundle.

There are a similar interpretations for the bordism groups of C and BX.

In the lowest non-trivial dimension ($n = 2$) the cobordism classes can be described as equivalence classes under simple moves and this gives a combinatorial description of $\pi_2(C)$ which can be used for calculations. To illustrate this we translate the Whitehead conjecture [Wh] into a conjecture about coloured link diagrams.

Section 5 : The algebraic topology of rack spaces (page 45)

The theory of James bundles gives invariants for knots and links for the following reason. If Γ is the fundamental rack of a link L then any invariant of the rack space $B\Gamma$ is *a fortiori* an invariant of L. In particular any of the classical algebraic toplogical invariants of rack spaces are link invariants. Further invariants are obtained by considering representations of the fundamental rack in a small rack and pulling back invariants from the rack space of this smaller rack. In this section we concentrate on calculating invariants of rack spaces.

We describe all the homotopy groups of BX where X is the fundamental rack of an irreducible (non-split) link in a 3–manifold (this is a case in which the rack completely classifies the link [FR]). This description leads to the new geometric descriptions for the higher homotopy groups of the 3–sphere mentioned earlier. We also calculate π_2 of BX where X is the fundamental rack of a general link in S^3. We show that BX is always a simple space and we compute the homotopy type of BX in the cases when X is a free rack and when X is a trivial rack with n elements.

We report on further calculations given elsewhere [Fl, G ,Wi].

Remark It is important to note that there are invariants of the rack space which are not homotopy invariants, but combinatorial ones. These include the James–Hopf invariants (defined by the James bundles) and the characteristic classes mentioned above. Although not homotopy invariants, these all yield invariants of knots and links. Now, even when the rack is a complete invariant of the knot or link, it is likely that the homotopy type of the rack space is not a complete invariant. See the remarks following theorem 5.4. However the combinatorics of the rack space contain all the information needed to reconstruct the rack, so in principle combinatorial invariants should give a complete set of invariants. In the next section we describe a construction which is a combinatorial invariant and yields invariants in this spirit.

Section 6 : Homology theories from $\square$–sets (page 56)

In this section we return to the map $\Phi\colon |C| \to BO$, defined in section 2 for any $\square$–set C, under which the mod 2 reductions of the characteristic classes of C pull back from both the Steifel–Whitney and Wu classes. This defines the concept of a C–oriented manifold, namely a manifold whose stable normal bundle lifts over Φ and thus to a generalised (co)-homology theory. This theory has a geometric interpretation close to that for Mahowald orientation given in [S1]. Operations in this theory can be defined geometrically. By applying this construction in the case where $C = B\Gamma$, where Γ is the fundamental rack of a link L we obtain the notion of an L–orientation for a smooth manifold and a consequent link invariant namely the associated generalised (co)-homology theory.

Section 7 : Vassiliev theory (page 62)

We define a $\square$–set $\mathcal{I}$ whose n–cells are knots with n double points. We define the concept of a Vassiliev invariant for any $\square$–set C which, in the case of the set $\mathcal{I}$ coincides with the usual definition [V]. Given a partially defined Vassiliev invariant on a $\square$–set C there is a sequence of obstructions lying in $H^1(J^n(C))$ to extending the invariant over the whole set. This obstruction theory gives a natural setting in which to understand the results of Kontsevich and Bar-Natan [K, BN]. The theory can be extended to cover knots with triple points, but in order to do this we need to extend the concept of $\square$–set to that of a cubical CW complex, and we intend to pursue this in a further paper. Cubical CW complexes also have associated James bundles and we finish the paper with conditions on the James bundles for a cubical CW complex to be a $\square$–set.

1 Basic definitions

In this section we define $\square$–sets (and generalisations) and construct an infinite family of associated $\square$–sets, which for reasons which will be explained in section 3, we call **James** complexes. Important examples for what follows are the rack space and the knot space (see below).

A **$\square$–set** $C = \{C_n; \partial_i^\epsilon\}$ consists of a collection of sets C_n, one for each natural number $n \geq 0$, and functions $\partial_i^\epsilon : C_n \to C_{n-1}$, $1 \leq i \leq n$, $\epsilon \in \{0,1\}$, called **first order face maps**, satisfying the following *face relations*:

$$\partial_{j-1}^\eta \partial_i^\epsilon = \partial_i^\epsilon \partial_j^\eta, \quad 1 \leq i < j \leq n, \quad \epsilon, \eta \in \{0,1\}.$$

There is a notion of a **$\square$–map** $f: C \to D$ between $\square$–sets, namely a collection of maps $f_n : C_n \to D_n$ for $n = 0, 1, \ldots$, which commute with the face maps, ie $f_{n-1}\partial_i^\epsilon = \partial_i^\epsilon f_n$.

Note that a $\square$–set differs from a *cubical set*, which has a similar definition but also has degeneracy maps; for more detail on the connections between cubical and $\square$–sets, see [FRS].

There is also the notion of a **$\square$–space**. This comprises a collection $\{X_n\}$ of topological spaces and first order face maps (continuous maps) $\partial_i^\epsilon : X_n \to X_{n-1}$, $1 \leq i \leq n$, $\epsilon \in \{0,1\}$ satisfying the same face relations as above.

We shall give presently alternative definitions for $\square$–sets and maps based on cubes. First we need to establish some terminology.

The category $\square$

The **n–cube** I^n is the subset $[0,1]^n$ of $\mathbb{R}^n$.

A **p–face** of I^n is a subset defined by choosing $n-p$ coordinates and setting some of these equal to 0 and the rest to 1. In particular there are $2n$ faces of dimension $n-1$ determined by setting $x_i = \epsilon$ where $i \in \{1, 2, \ldots, n\}$ and $\epsilon \in \{0,1\}$.

A 0–face is called a **vertex** and corresponds to a point of the form $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ where $\epsilon_i = 0$ or 1 and $i \in \{1, 2, \ldots, n\}$. The 1–faces are called **edges** and the 2–faces are called **squares**.

Let $p \leq n$ and let J be a p–face of I^n. Then there is a canonical **face map** $\lambda: I^p \to I^n$, with $\lambda(I^p) = J$, given in coordinate form by preserving the order of the coordinates $(x_1, \ldots, x_p)$ and inserting $n-p$ constant coordinates which are either 0 or 1. If λ inserts only 0's (resp. only 1's) we call it a **front** (resp. **back**) face map. Notice that any face map has a unique **front–back** decomposition as $\lambda\mu$, say, where λ is a front face map and μ is a back face map. There is also a unique **back–front** decomposition. There are $2n$ face maps defined by the $(n-1)$–faces which are denoted $\delta_i^\epsilon : I^{n-1} \to I^n$, and given by:

$$\delta_i^\epsilon(x_1, x_2, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \epsilon, x_i, \ldots, x_{n-1}), \qquad \epsilon \in \{0,1\}.$$

The following relations hold:

$$\delta_i^\epsilon \delta_{j-1}^\omega = \delta_j^\omega \delta_i^\epsilon, \quad 1 \leq i < j \leq n, \quad \epsilon, \omega \in \{0,1\}. \qquad \mathbf{1.1}$$

Definition The category $\square$ is the category whose objects are the n–cubes I^n for $n = 0, 1, \ldots$ and whose morphisms are the face maps.

$\square$–Sets and their Realisations

We now give the second equivalent definitions of $\square$–sets and maps.

A **$\square$–set** is a functor $C\colon \square^{op} \to Sets$ where $\square^{op}$ is the opposite category of $\square$ and $Sets$ denotes the category of sets.

A **$\square$–map** between $\square$–sets is a natural transformation.

The equivalence of the two definitions of $\square$–sets and maps follows from the formulæ for the composition of face maps given in equation 1.1 above.

The **realisation** $|C|$ of a $\square$–set C is given by making the identifications $(\lambda^* x, t) \sim (x, \lambda t)$ in the disjoint union $\coprod_{n\geq 0} C_n \times I^n$.

We shall call 0–cells (resp. 1–cells, 2–cells) of $|C|$ vertices (resp. edges, squares) and this is consistent with the previous use for faces of I^n, since I^n determines a $\square$–set with cells corresponding to faces, whose realisation can be identified in a natural way with I^n.

Notice that $|C|$ is a CW–complex with one n–cell for each element of C_n with a canonical characteristic map from the n–cube.

However, not every CW–complex with cubical characteristic maps comes from a $\square$–set — even if the cells are glued by isometries of faces. In $|C|$, where C is a $\square$–set, cells are glued by face *maps*, in other words by canonical isometries of faces. We call a CW–complex with cubical characteristic maps, in which cells are glued by isometries of faces, a **cubical CW–complex** and we shall have more to say about these complexes in section 7.

There is a similar alternative definition of a $\square$–space namely a functor $X\colon \square^{op} \to Top$ (where Top denotes the category of topological spaces and continuous maps) and the realisation $|X|$ given by the same formula as above. Note that a $\square$–*set* is effectively the same as a $\square$–*space* in which all spaces have the discrete topology.

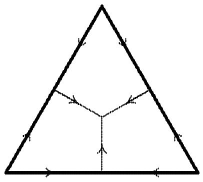
Connection with Δ–sets

The concept of a Δ–set is similar to the concept of a $\square$–set but is based on simplexes rather than cubes. A Δ–set can have an arbitrary (weak) homotopy type, see [RS2] where basic material on Δ–sets can be found. The definition of a Δ–set can be obtained from the first definition of a $\square$–set (at the start of the section) by forgetting the ϵ's and η's and allowing $i = 0$.

A $\square$–set can be subdivided to form a Δ–set by coning from centres. More precisely suppose that C is a $\square$–set. Define the Δ–subdivision, $Sd_\Delta(C)$, as follows. A k–simplex of $Sd_\Delta(C)$ is a $(k+1)$–tuple $(c, \lambda_1, \ldots, \lambda_k)$ where $c \in C_n$, some n, $\lambda_i\colon I^{n_{i-1}} \to I^{n_i}$ is a face map, $1 \leq i \leq k$, and $n_0 < n_1 < \ldots < n_k = n$. The face maps ∂_i, for $0 \leq i \leq k$, are given by

$$\partial_i(c, \lambda_1, \ldots, \lambda_k) = \begin{cases} (c, \lambda_2, \ldots, \lambda_k) & \text{for } i = 0 \\ (c, \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}\lambda_i, \lambda_{i+2}, \ldots, \lambda_k) & \text{for } 0 < i < k \\ (\lambda_k^* c, \lambda_1, \ldots, \lambda_{k-1}) & \text{for } i = k \end{cases}$$

Conversely given a Δ–set X we can define a $\square$–subdivision, $Sd_\square(X)$ of X by projecting simplexes from the origin. More precisely, let $\Delta^n = \{x \in \mathbb{R}^{n+1} | x_i \geq 0, \sum_i x_i = 1\}$ denote the standard n–simplex. Let $\rho_n \colon \Delta^n \to \partial I^{n+1}$ be the radial projection from $0 \in \mathbb{R}^{n+1}$. Then $\rho_n(\Delta^n)$ is the union of the back n–faces of I^{n+1}. A picture of the image of Δ^2 under this projection (seen from the origin) is given below, and it can be seen that Δ^2 has been subdivided into three squares. A similar projection subdivides an n–simplex into $n+1$ n–cubes and this process defines the required $\square$–subdivision.



For detailed formulæ, let $B(k,n)$ denote the set of back face maps $\lambda \colon I^k \to I^n$. If $\lambda \colon I^k \to I^n$ is a front face map let $r(\lambda) \colon \Delta^{k-1} \to \Delta^{n-1}$ denote its restriction. Define the $\square$–subdivision, $Sd_\square(X)$ as follows. Set

$$Sd_\square(X)_k = \coprod_{n>k} X_{n-1} \times B(k,n).$$

Supose given $x \in X_{n-1}$, $\lambda \in B(k,n)$ and a face map $\mu \colon I^s \to I^k$. Let $\lambda'\mu'$ be the front–back decomposition of $\lambda\mu$. Define $\mu^*(x,\lambda) = (r(\lambda')^* x, \mu')$.

1.2 Proposition

(1) For any Δ–set X the spaces $|X|$ and $|Sd_\square(X)|$ are homeomorphic,

(2) For any $\square$–set C the spaces $|C|$ and $|Sd_\Delta(C)|$ are homeomorphic.

Proof The homeomorphism $|Sd_\square(X)| \to |X|$ is induced by

$$(x, \lambda, t) \in X_{n-1} \times B(k,n) \times I^k \longmapsto (x, \rho^{-1}(\lambda(t))) \in X_{n-1} \times \Delta_{n-1}.$$

Let $\hat{c}_m = (\frac{1}{2}, \ldots, \frac{1}{2}) \in I^m$. The homeomorphism $|Sd_\Delta(C)| \to |C|$ is induced by

$$((c, \lambda_1, \ldots, \lambda_k), \mathbf{t}) \in Sd_\Delta(C)_k \times \Delta_k \longmapsto (c, t_k \hat{c}_n + \sum_{0 \leq i \leq k-1} t_i \lambda_k \ldots \lambda_{i+1} \hat{c}_{n_i}) \in X_n \times I^n$$

where $\mathbf{t} = (t_1, \ldots, t_k) \in \Delta_k$. Further details are left to the reader. $\square$

1.3 Corollary *Any space has the weak homotopy type of some $\square$–set.* $\square$

Notation We shall often omit the mod signs and use the notation C both for the $\square$–set C and its realisation $|C|$. We shall use the full notation whenever there is any possibility of confusion.

1.4 Examples of $\square$–sets

(1) Singular complex Let X be a topological spaces. The **singular complex** of X is the $\square$–set denoted $\mathrm{S}(X)$ and defined by :

$\mathrm{S}(X)_n = \{f \colon I^n \to X\}$, the set of continuous maps and

$\lambda^*(f) = f \circ \lambda$, where λ is a face map.

If $g\colon X \to Y$ is a continuous map then $\mathrm{S}(g)\colon \mathrm{S}(X) \to \mathrm{S}(Y)$ is the $\square$–map defined by $\mathrm{S}(g)(f) = f \circ g$. Thus $\mathrm{S}(.)$ is a functor from Top to the category of $\square$–sets and maps.

An important special case is the singular complex of a point. This has exactly one cell in each dimension and we shall call it the **trivial $\square$–set** and denote it T.

(2) Nerve of a category (For full details on this example, see [FRS].) Let I^n_{cat} denote the category whose objects are the vertices of I^n. Morphisms are generated by the oriented edges of I^n and subject to relations given by taking squares to be commutative diagrams. Each face map $\lambda\colon I^q \to I^n$ determines a functor $\lambda_{\mathrm{cat}}\colon I^q_{\mathrm{cat}} \to I^n_{\mathrm{cat}}$. Let $\mathcal{C}$ be a category; the **nerve** of $\mathcal{C}$, denoted $\mathcal{NC}$ is the $\square$–set defined in analogy to the singular set :

$\mathcal{NC}_n = \{f\colon I^n_{\mathrm{cat}} \to \mathcal{C}\}$, the set of functors and

$\lambda^*(f) = f \circ \lambda_{\mathrm{cat}}$, where λ is a face map.

Given a functor $g\colon \mathcal{C} \to \mathcal{D}$ between categories, we have a $\square$–map $\mathcal{N}(g)\colon \mathcal{NC} \to \mathcal{ND}$, given by $\mathcal{N}(g)(f) = f \circ g$.

(3) Rack space A **rack** is a set R with a binary operation written a^b such that $a \mapsto a^b$ is a bijection for all $b \in R$ and such that the **rack identity**

$$a^{bc} = a^{cb^c}$$

holds for all $a, b, c \in R$.

For example if $a, b \in G$ are two members of a group then conjugation, given by $a^b := b^{-1}ab$, defines a rack structure on G. If $a, b \in \mathbb{Z}$ are two integers then $a^b := a + 1$ defines a rack structure on $\mathbb{Z}$ which does not come from conjugation in a group. Another example of a rack associated to a group is the **core** of a group [Br, J] defined by $a^b := ba^{-1}b$. For more examples of racks see [FR].

If R is a rack, the **rack space** is the $\square$–set denoted BR and defined by :

$BR_n = R^n$ (the n–fold cartesian product of R with itself).

$$\partial_i^0(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$
$$\partial_i^1(x_1, \ldots, x_n) = ((x_1)^{x_i}, \ldots, (x_{i-1})^{x_i}, x_{i+1}, \cdots, x_n) \quad \text{for} \quad 1 \le i \le n.$$

More geometrically, we can think of BR as the $\square$–set with one vertex, with (oriented) edges labelled by rack elements and with squares which can be pictured as part of a link diagram with arcs labelled by a, b and a^b.

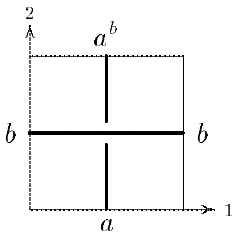


Diagram of a typical 2–cell of the rack space

The higher dimensional cubes are determined by the squares in a way which was made precise in [FRS] and which we now summarise.

A **trunk** T comprises a set T_0 of *vertices*, a set T_1 of directed *edges* (between vertices) and a set T_2 of oriented squares of edges (the *preferred squares*). Thus a trunk is essentially a cubical set truncated at dimension 2; but notice that a square is fully determined by its faces (edges). A trunk T has a **nerve** $\mathcal{N}T$ which is a $\square$–set defined in a similar way to the nerve of a category (above). The nerve has vertices, edges and squares in bijection with those of T and higher cubes determined by the 2–skeleton as follows. Let I^n_{trunk} be the trunk determined by the 2–skeleton of I^n, then $\mathcal{N}T_n = \{f\colon I^n_{\text{trunk}} \to T\}$, the set of trunk maps, with face maps given, as usual, by composition. This construction is fully compatible with the nerve of a category; indeed a category determines a trunk with preferred squares being the commuting squares; the two nerves are then identical.

Now a rack R determines the trunk $T(R)$ with one vertex, edges the set R and preferred squares of the type pictured above. The rack spaces BR and the nerve of the trunk $\mathcal{N}T(R)$ coincide. For full details here see [FRS], where the trunk formalism is used to establish connections between the rack space and other classifying spaces.

Notice that the rack space of the rack with one element has precisely one cube in each dimension. This is another description of the trivial $\square$–set.

(4) The knot space

For full details on this example, see section 7. Let $\mathcal{I}_n$ $n = 0, 1, \ldots$ denote the set of isotopy classes of smooth immersions (singular knots) K of S^1 in S^3 with the following properties.

(1) The immersion K is an embedding except at n double points none of which involve the basepoint of S^1.

(2) At each double point the two tangents are linearly independent.

Then $\mathcal{I} = \{\mathcal{I}_n\}$ can be made into a $\square$–set as follows. Let K be a singular knot defining $[K] \in \mathcal{I}_n$. Number the double points of K in order round the knot starting from the basepoint and define $\partial^1_i([K]) = [K^+_i]$ where K^+_i is obtained from K by unwrapping the i-th double point to give a positive crossing and $\partial^0_i([K])$ is similarly defined using a negative crossing. The face relations given at the start of the section are easily verified.

James Complexes

We finish this section by defining, for any $\square$–set C, an infinite family of associated $\square$–sets $J^n(C)$, $n \geq 0$ called the James complexes of C.

Projections An $(n+k,k)$–**projection** is a function $\lambda : I^{n+k} \to I^k$ of the form

$$\lambda : (x_1, x_2, \ldots, x_{n+k}) \mapsto (x_{i_1}, x_{i_2}, \ldots, x_{i_k}), \qquad \mathbf{1.5}$$

where $1 \leq i_1 < i_2 < \ldots < i_k \leq n+k$.

Let P^{n+k}_k denote the set of $(n+k,k)$–projections. Note that P^{n+k}_k is a set of size $\binom{n+k}{k}$.

Let $\lambda \in P_k^{n+k}$ and let $\mu : I^l \to I^k, \quad l \le k$ be a face map. The projection $\mu^\sharp(\lambda) \in P_l^{n+l}$ and the face map $\mu_\lambda : I^{n+l} \to I^{n+k}$ are defined uniquely by the following pull-back diagram:

$$\begin{array}{ccc} I^{n+l} & \xrightarrow{\mu_\lambda} & I^{n+k} \\ \mu^\sharp(\lambda)\downarrow & & \lambda\downarrow \\ I^l & \xrightarrow{\mu} & I^k \end{array} \qquad \mathbf{1.6}$$

Definition Let C be a $\square$–set. The n**–th associated James complex** of C, denoted $J^n(C)$ is defined by

$$J^n(C)_k = C_{n+k} \times P_k^{n+k}$$

and

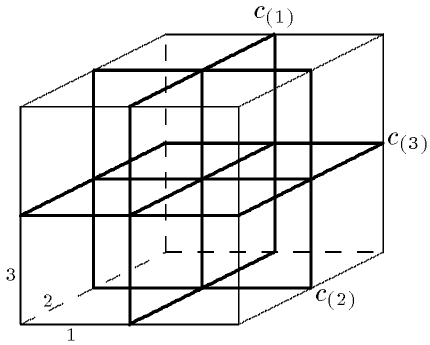
$$\mu^*(x, \lambda) = (\mu_\lambda^*(x), \mu^\sharp(\lambda))$$

where $\mu : I^l \to I^k, \quad l \le k$ is a face map.

Notation Let $\lambda \in P_k^{n+k}$ and $c \in C_{n+k}$ then we shall use the notation c_λ for the k–cube $(c, \lambda) \in J^n(C)$. When necessary, we shall use the full notation $(\lambda_1, \ldots, \lambda_n)$ for the projection λ (given by formula 1.5) where $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ and $\{\lambda_1, \ldots, \lambda_n\} = \{1, \ldots, k+n\} - \{i_1, \ldots, i_k\}$. In other words we index cubes of J^n by the n directions (in order) which are collapsed by the defining projection.

Picture for James complexes

We think of $J^n(C)$ as comprising all the codimension n central subcubes of cubes of C. For example a 3–cube c of C gives rise to the three 2–cubes of $J^1(C)$ which are illustrated in the following picture



In the diagram we have used full notation for projections. Thus for example, $c_{(2)}$ corresponds to the projection $(x_1, x_2, x_3) \mapsto (x_1, x_3)$, ($x_2$ being collasped).

This picture can be made more precise by considering the section $s_\lambda : I^k \to I^{n+k}$ of λ given by

$$s_\lambda(x_1, x_2, \ldots, x_k) = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, x_1, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, x_2, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, x_k, \tfrac{1}{2}, \ldots)$$

where the non-constant coordinates are in places $i_1, i_2, \ldots, i_k$ and λ is given by 1.5.

For example in the picture the image of s_λ where $\lambda(x_1, x_2, x_3) \mapsto (x_1, x_3)$ is the 2–cube labelled $c_{(2)}$.

Now the commuting diagram (1.6) which defines the face maps implies that the s_λ's are compatible with faces and hence they fit together to define a map

$$p_n : |J^n(C)| \to |C| \quad \text{given by} \quad p_n[c_\lambda, t] = [c, s_\lambda(t)].$$

In the next section we will see that p_n is a mock bundle projection.

2 Mock Bundles and Transversality

In this section we define mock bundles over smooth CW complexes, which include $\square$–sets, and prove a transversality theorem for $\square$–sets. This material is similar to material in [BRS; Chapters 2 and 7]. However [BRS] is set entirely in the PL category and deals only with transversality with respect to a transverse CW complex. Here we shall need to extend the work to the smooth category and prove transversality with respect to a $\square$–set (which is not quite a transverse CW complex). However many of the proofs are similar to proofs in [BRS] and therefore we omit details when appropriate. Similar material can also be found in [Fe] and [Pa].

The main technicalities concern manifolds with corners, which is where we start. We shall use **smooth** to mean C^∞.

Definition *Manifold with corners*

For background material on smooth manifolds with corners, see Cerf or Douady [C, Do]. In particular these references contain an appropriate version of the tubular neighbourhood theorem. There is a uniqueness theorem for these tubular neighbourhoods. The proof can be obtained by adapting the usual uniqueness proof.

A **smooth** n**–manifold with corners** M is a space modelled on $\mathbb{E}^n$ where $\mathbb{E} = [0, \infty)$. In other words M is equipped with a maximal atlas of charts from open subsets of $\mathbb{E}^n$ such that overlap maps are smooth. There is a natural stratification for such a manifold. Define the **index** of a point p to be the minimum q such that a neighbourhood of p in M is diffeomorphic to an open subset of $\mathbb{E}^q \times \mathbb{R}^{n-q}$. The **stratum of index** q denoted $M_{(q)}$ comprises all points of index q. Note that the dimension of $M_{(q)}$ is $n-q$ and that $M_{(0)}$ is the interior of M, $M_{(1)}$ is an open codimension 1 subset of ∂M and in general $M_{(i)}$ is an open codimension 1 subset of the boundary of the closure of $M_{(i-1)}$.

Examples I^n and $\mathbb{E}^n$ are manifolds with corners. If M and Q are manifolds with corners then so is $M \times Q$.

Definition *Maps of manifolds with corners*

Let M and Q be manifolds with corners then a **stratified map** is a smooth map $f\colon M \to Q$ such that $f(M_{(q)}) \subset Q_{(q)}$ for each q. This is the analogue for manifolds with corners of a proper map for manifolds with boundary.

An **embedding of manifolds with corners** is a smooth embedding $i\colon M \subset Q$ such that the pair is locally like the inclusion of $\mathbb{E}^p \times \mathbb{E}^t$ in $\mathbb{E}^p \times \mathbb{R}^t \times \mathbb{R}^s$ where $p+t = \dim(M)$ and $p+t+s = \dim(Q)$. Thus an embedding of manifolds with corners which is proper (ie takes boundary to boundary and interior to interior) must be a stratified embedding. However in general an embedding of manifolds with corners allows corners on M not at corners of Q, see the examples below.

A **face map of manifolds with corners** $\lambda\colon M \to Q$ is a smooth embedding which is a proper map of topological spaces (preimage of compact is compact) such that $\lambda(M_{(q)}) \subset Q_{(t+q)}$ for each q where $t = \dim(Q) - \dim(M)$. Thus a face map is a diffeomorphism of M onto a union of components of strata of Q.

Examples The map $s_\lambda : I^k \to I^{n+k}$ of λ given by

$$s_\lambda(x_1, x_2, \ldots, x_k) = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, x_1, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, x_2, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, x_k, \tfrac{1}{2}, \ldots)$$

(see the end of section 1) is a stratified embedding.

If M is a manifold with corners and Q is a smooth manifold without boundary then the projection $M \times Q \to M$ is a stratified map.

The inclusions $I^n \subset \mathbb{E}^n \subset \mathbb{R}^n$ are embeddings of manifolds with corners.

A face map (in the sense of section 1) $\lambda\colon I^k \to I^n$ is a face map of manifolds with corners.

Definition *Smooth CW complex*

We refer to [RS1; pages 13–14] for the definition and basic properties of convex linear cells in $\mathbb{R}^n$ (which we shall abbreviate to convex cells). Convex cells have well-defined faces and if e' is a face of e we write $e' < e$. Convex cells are smooth manifolds with corners. A smooth face map between convex cells (in the sense of manifolds with corners) is the same as a diffeomorphism onto a face.

A **smooth CW complex** is a collection of convex cells glued by smooth face maps. More precisely it comprises a CW complex W and for each cell $c \in W$ a preferred characteristic map $\chi_c\colon e_c \to W$ where e_c is a convex cell such that for each face $e' < e_c$ there is a cell $d \in W$ with preferred characteristic map $\chi_d\colon e_d \to W$ and a diffeomorphism $\mu\colon e_d \to e'$ such that the diagram commutes :

$$\begin{array}{ccc} e_d & \xrightarrow{\chi_d} & W \\ \downarrow \mu & & \uparrow \chi_c \\ e' & \xrightarrow{\text{inc.}} & e_c \end{array}$$

Examples A (realised) $\square$–set or Δ–set, a simplicial complex or a convex linear cell complex are all smooth CW complexes.

We say that a smooth CW complex W gives a **smooth decomposition** of a smooth manifold M (possibly with corners) if there is a homeomorphism $h\colon W \to M$ such that $h \circ \chi_c\colon e_c \to M$ is an embedding of manifolds with corners for each cell $c \in W$. Usually we identify W and M via h in this situation and say that W is a smooth decomposition of M.

2.1 Definition *Smooth mock bundle*

Let W be a smooth CW complex. A **mock bundle** ξ **over** W of codimension q (denoted ξ^q/C) comprises a **total space** E_ξ and a **projection** $p_\xi\colon E_\xi \to W$ with the following property. †

Let c be an n–cell of W with characteristic map $\chi_c\colon e_c \to W$, then there is a smooth manifold with corners B_c of dimension $n-q$ called the **block** over c and a stratified map $p_c\colon B_c \to e_c$ and a map $b_c\colon B_c \to E_\xi$ such that the following diagram is a pull-back :

$$\begin{array}{ccc} B_c & \xrightarrow{b_c} & E_\xi \\ \downarrow p_c & & \downarrow p_\xi \\ e_c & \xrightarrow{\chi_c} & W \end{array}$$

2.2 Amalgamation lemma *Suppose that ξ^q/W is a smooth mock bundle and that W is smooth decomposition of a smooth manifold with corners M^m. Then E_ξ can be given the structure of a smooth manifold with corners of dimension $m-q$ such that $p_\xi\colon E_\xi \to M$ is ε–homotopic through mock bundle projections to a stratified map.*

Proof By standard embedding theorems we may assume that each block B_c of ξ is a stratified submanifold of $e_c \times \mathbb{R}^N$ for some N and that these embeddings fit together to give an embedding of E_ξ in $M \times \mathbb{R}^N$ such that p_ξ is ε–homotopic to the restriction of projection on the first coordinate. Now fix attention on the interior of a particular block B_c° which can be thought of as a subset of E_ξ lying over c° the interior of the corresponding cell of W. The standard tubular neighbourhood theorem applied to the submanifold c° of $|W| = M$ yields a tubular neighbourhood λ of c° in $|W| = M$ formed by tubular neighbourhoods in each of the incident cells. Using the tubular neighbourhood theorem for manifolds with corners we can construct a (non-smooth) tubular neighbourhood μ of B_c° in E_ξ formed by (smooth) tubular neighbourhoods in each of the incident blocks, which extends to a tubular neighbourhood μ^+ on $c^\circ \times \mathbb{R}^N$ in $M \times \mathbb{R}^N$. By inductively applying uniqueness we can deform μ^+ near B_c° to $\lambda \times \mathbb{R}^N$ by an ε–isotopy. This carries E_ξ to a smooth submanifold of $M \times \mathbb{R}^N$ so that projection on M is a stratified map and moreover the isotopy determines a homotopy through mock bundle projections of the projection on $|W|$. □

† Note that the notation used here for dimension of a mock bundle, namely that q is *codimension*, is the negative of that used in [BRS] where ξ^q/W meant a mock bundle of *fibre dimension* q ie codimension $-q$. The notation used here is consistent with the usual convention for cohomology.

Definition *Maps of smooth CW complexes*

A **linear projection** of convex cells is a surjective map $f\colon e_1 \to e_2$, where e_1, e_2 are convex cells, which is the restriction of an affine linear map, and such that $f(e') < e_2$ for each face $e' < e_1$. Examples include simplicial maps of one simplex onto another, projections $I^{n+k} \to I^n$ and projections of the form $d \times e \to e$ where d, e are convex cells.

A **smooth projection** of convex cells is a linear projection composed with a diffeomorphism of e_1.

A **smooth map** $f\colon W \to Z$ **of smooth CW complexes** is a map such that for each cell $c \in W$ there is a cell $d \in Z$ and a smooth projection $\phi\colon e_c \to e_d$ such that the diagram commutes :

$$\begin{array}{ccc} e_c & \xrightarrow{\phi} & e_d \\ \downarrow \chi_c & & \downarrow \chi_d \\ W & \xrightarrow{f} & Z \end{array}$$

Examples include $\square$–maps, Δ–maps, simplicial maps and projections $W \times Z \to W$, where W, Z are any two smooth CW complexes.

The following lemma follows from definitions :

2.3 Pull-back lemma *Let ξ^q/Z be a smooth mock bundle and $f\colon W \to Z$ a smooth map of smooth CW complexes, then the pull-back $f^*(E_\xi) \to W$ is a mock bundle of the same codimension (denoted $f^*(\xi^q)/W$).* $\square$

Properties of mock bundles

We shall summarise properties of mock bundles. The details of all the results stated here are analogous to the similar results for PL mock bundles over cell complexes proved in [BRS].

The set of cobordism classes of mock bundles with base a smooth CW complex forms an abelian group (under disjoint union of total spaces) and there is a relative group (the total space is empty over the subcomplex). This all fits together with the pull-back construction to define a cohomology theory which can be identified with smooth cobordism (classified by the Thom spectrum MO).

If the base is a manifold then the amalgamation lemma defines a map from q–cobordism to $(n-q)$–bordism. This map is the Poincaré duality isomorphism.

Given two mock bundles $\xi^q/W, \eta^r/W$ we can define the mock bundle $(\xi \cup \eta)^{q+r}/W$ in various equivalent ways analogous to the Whitney sum of bundles. We can pull one bundle back over the total space of the other and then compose. This is equivalent to making the projection of the first bundle transverse to the projection of the second and then pulling back. We can take the external product $\xi \times \eta/W \times W$ and restrict to the diagonal. These equivalent constructions define the cup product in cobordism. If the base is a manifold then pull-back (or transversality) defines the cap product which coincides under Poincaré duality with the cup product.

Mock bundles can be generalised and extended in a number of ways. The simplest is to use orientation. If each block (and cell) is oriented in a compatible way (cf [BRS; page 82]) then the resulting theory of oriented mock bundles defines oriented cobordism (classified by MSO). More generally we can consider restrictions on the stable normal bundle of blocks and this yields the corresponding cobordism theory. A particular example of relevance here is the case when blocks are stably framed manifolds; in this case the resulting theory is stable cobordism classified by the sphere spectrum $\mathbb{S}$. By considering manifolds with singularities, the resulting theory can be further generalised and such a mock bundle theory can represent the cohomology theory corresponding to an arbitrary spectrum [BRS; chapter 7]. The corresponding homology theory is represented by the bordism theory given by using manifolds with the same allowed singularities. Coefficients and sheaves of coefficients can also be defined geometrically (see [BRS; chapters 3 and 6]).

Key example *James bundles*

At the end of the last section we defined a projection

$$p_n : |J^n(C)| \to |C|$$

Where $J^n(C)$ is the n-th James complex of the $\square$–set C.

Now if we choose a particular $(n+k)$–cell σ of C then the pull back of p_n over I^{n+k} (by the characteristic map for σ) is a k–manifold (in fact it is the $\binom{n+k}{k}$ copies of I^k corresponding to the elements of P_k^{n+k}). Therefore p_n is the projection of a mock bundle of codimension n, which we shall call the n–**th James bundle** of C denoted $\zeta^n(C)$.

2.4 Definition *Embedded mock bundle*

Let W be a smooth CW complex and Q a smooth manifold without boundary then an **embedded mock bundle** in $W \times Q$ is a mock bundle ξ/W with an embedding $E_\xi \subset W \times Q$ such that p_ξ is the restriction of the projection $W \times Q \to W$ and such that for each cell $c \in W$ the induced embedding $B_c \subset e_c \times Q$ is a stratified embedding. The proof of the amalgamation lemma (with $\mathbb{R}^N$ replaced by Q) implies that if W is a smooth decomposition of a manifold then E_ξ can be ε–isotoped to a smooth submanifold of $W \times Q$ so that projection on W is still a mock bundle projection.

We shall be particularly concerned with the case when Q is $\mathbb{R}^t$ for some t and each block is framed in $e_c \times \mathbb{R}^t$. The theory defined by mock bundles of this type is unstable cohomotopy. See in particular 3.6 and 3.7.

Key example *Embedding the James bundles in* $|C| \times \mathbb{R}$

Let C be a $\square$–set. The James bundles can be embedded in $|C| \times \mathbb{R}$. This is done by ordering the cubes of $J^n(C)$ over a particular cube of C and lifting in that order. Recall that the k–cubes of $J^n(C)$ lying over a $(k+n)$–cube are indexed by projections $\lambda = (i_1, \ldots, i_n) \in P_n^{n+k}$. These may be ordered lexicographically. The lexicographic order is compatible with face maps and can be used to define the required embedding by induction on dimension of cells of C as follows.

Suppose inductively that the embedding has been defined over cells of C of dimension $\leq k+n-1$.

Consider a $(k+n)$–cube $c \in C$ with characteristic map $\chi_c \colon I^{k+n} \to |C|$. Pulling the embedding back (where it is already defined) over χ_c gives an embedding of $\zeta^n(\partial I^{k+n})$ in $\partial I^{k+n} \times \mathbb{R}$. Now embed the centres of the k–cubes of $J^n(I^{k+n})$ at $(\frac{1}{2}, \ldots, \frac{1}{2}) \times r_\lambda$ where r_λ are real numbers for $\lambda \in P_n^{n+k}$ chosen to increase strictly corresponding to the lexicographic order on P_n^{n+k}.

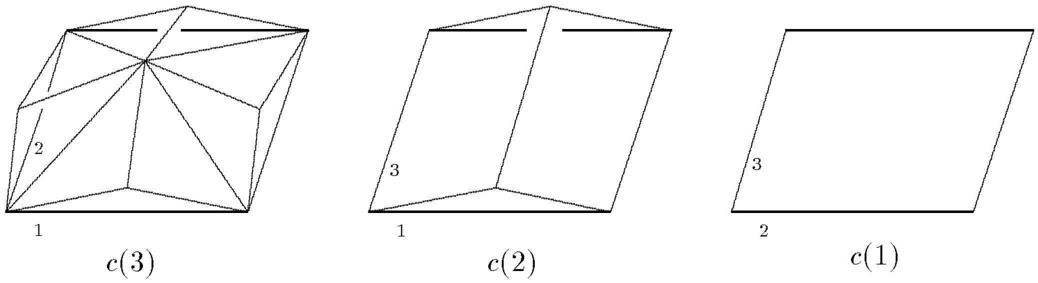
Now embed each k–cube of $J^n(I^{k+n})$ as the cone on its (already embedded) boundary. Finally smooth the resulting embedding and push it forwards to $|C| \times \mathbb{R}$ using $\chi_c \times \mathrm{id}$.

In the next section we shall give precise smooth formulæ for this embedding using a bump function.

The embedding is in fact framed. This can be seen as follows. Each k–cube $p_n(c, \lambda)$, where $\lambda = (i_1, \ldots, i_n)$, of $J^n(I^{k+n})$ is framed in I^{k+n} by the n vectors parallel to directions $i_1, \ldots, i_n$. These lift to parallel vectors in $I^{n+k} \times \mathbb{R}$ and the framing is completed by the vector parallel to the positive $\mathbb{R}$ direction (vertically up). This framing is compatible with faces and defines a framing of $\zeta^n(C)$ in $|C| \times \mathbb{R}$. The formulæ which we shall give in the next section also give formulæ for the framing.

For the special case $n = 1$ the map of $\zeta^n(C)$ to $\mathbb{R}$ can be simply described: the centre of $c_{(k)}$ is mapped to k. This determines a map, linear on simplexes of $Sd_\Delta \zeta^1(C)$, to $\mathbb{R}$. It follows that the centre of $\partial_i^\epsilon c_{(k)}$ is mapped to k if $i \geq k$ and to $k-1$ if $i < k$.

At the end of section 1 we illustrated $J^1(C)$ for a 3–cube $c \in C$. The embeddings in $|C| \times \mathbb{R}$ (before smoothing) above each of the three 2–cubes are illustrated in the following diagram.



We finish this section with a discussion of transversality with respect to a $\square$–set, which will be important for the main classification theorems of section 4. A similar treatment can be given for any smooth CW complex, but we shall not need this in this paper.

Transversality

Definition *Transverse map to a $\square$–set*

Let M be a smooth manifold (possibly with boundary) of dimension m and C a $\square$–set. Let c be an n–cell of C with characteristic map χ_c: $I^n \to |C|$ denote $\chi_c(I^n - \partial I^n)$ by c° (the interior of c) and $\chi_c(\frac{1}{2}, \ldots, \frac{1}{2})$ by $\hat{c}$ (the centre of c).

Let f: $M \to |C|$ be a map. Let M_c denote the closure of $f^{-1}(c^\circ)$ and N_c the closure of $f^{-1}(\hat{c})$.

We say f is **transverse** to c if M_c is a smooth m–manifold with corners embedded (as a manifold with corners) in M and equipped with a diffeomorphism ι_c: $N_c \times I^n \longrightarrow M_c$ such that the diagram commutes :

$$\begin{array}{ccc} N_c \times I^n & \xrightarrow{\iota_c} & M_c \\ \downarrow p_2 & & \downarrow f \\ I^n & \xrightarrow{\chi_c} & |C| \end{array}$$

where p_2 denotes projection on the second coordinate. Thus N_c is a framed submanifold (with corners) of codimension n framed by copies of I^n on each of which f is the characteristic map for c.

A map f: $M \to |C|$ is **transverse** if it is transverse to each cell $c \in C$ and the framings are compatible with face maps in the following sense. Let λ: $I^q \to I^n$ be a face map and $d = \lambda^*(c)$. Then there is a face map (of manifolds with corners) λ_c^*: $N_c \to N_d$ and the following diagram commutes :

$$\begin{array}{ccc} N_c \times I^q & \xrightarrow{\mathrm{id}\times\lambda} & N_c \times I^n \\ \downarrow \lambda_c^*\times\mathrm{id} & & \downarrow i_c \\ N_d \times I^q & \xrightarrow{i_d} & M \end{array}$$

where $i_c = \mathrm{inc.} \circ \iota_c$, using the notation established above. The last condition can be summarised by saying that M is the realisation of a $\square$–space and f is the realisation of a $\square$–map.

To see this, define a $\square$–space by $X_n = \coprod_{c\in C_n} N_c$ and $\lambda^* = \coprod \lambda_c^*$ then the diffeomorphisms ι_c define a homeomorphism ι: $|X| \to M$ and if we identify M with $|X|$ via ι then the commuting diagrams above imply that f is the realisation of a $\square$–map $|X| \to |C|$.

Remark The alert reader will have noticed that the framing compatibility condition in the definition of a transverse map is unnecessary. If f: $M \to |C|$ is transverse to each cell of C then the framings can in fact be changed to become compatible (without altering f). However, the full definition is the one that we shall need in practice.

Ellucidation To help the reader understand the (somewhat complicated) concept of a transverse map to a $\square$–set we shall describe transversality for maps of closed surfaces and 3–manifolds (possibly with boundary) into C.

A transverse map of a closed surface Σ into a $\square$–set C meets only the 2–skeleton of C. The pull-backs of the squares of C are a number of disjoint little squares in Σ (each of which can be identified with the standard square I^2 and maps by a characteristic map to a 2–cell of C). The pull-backs of the 1–cells are a number of bicollared 1–manifolds which are either bicollared closed curves or are attached to edges of the little squares (and each edge of each square is used in this way). Each bicollar line can identified with I^1 and is mapped by f to a 1–cell of C (by the characteristic map for that cell). Finally each component of $\Sigma - \{$little squares and collared 1–manifolds$\}$ is mapped to a vertex of C.

Thus we can think of the transverse map as defining a thickened diagram of self-transverse curves (with the squares at the double points). This is illustrated in the following picture. We shall explore the connection between transverse maps and diagrams in section 4.

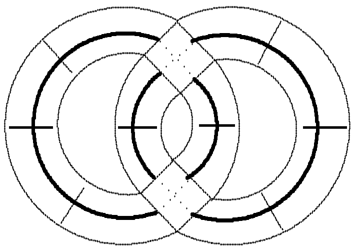


Figure 2.5

A transverse map of a closed 3–manifold into a $\square$–set C meets only the 3–skeleton of C. The pullback of the 3–cubes are a number of little cubes each of which can be identified with I^3 and which map onto 3–cubes of C by characteristic maps. The pullback of squares are framed 1–manifolds (framed by a copy of the standard square) and such that each such square is mapped onto a square of C by a characteristic map. These framed 1–manifolds are attached to the square faces of the little cubes at their boundaries. The pullback of edges are framed sheets (framed by copies of I^1 and mapped by characteristic maps to edges of C). The edges of the sheets are attached to edges of the framed 1–manifold (ie along 1–manifold $\times$ edge of square) and the two framings are required to be the same here (this is the framing compatibility condition in this case). The remainder of M is then a 3–manifold (with corners) each component of which is mapped to a vertex of C. A general view near one of the little cubes is illustrated below.

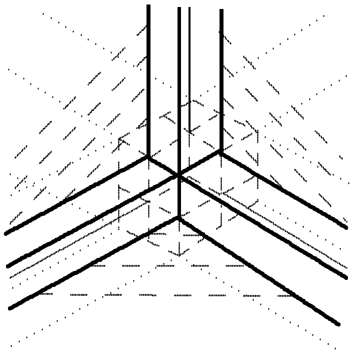


Figure 2.6

For a 3–manifold M with boundary, transversality has a similar description. In this case the little cubes are all in the interior of M, $f|\partial M$ is a transverse map of a surface in

C and the framed 1–manifolds can terminate at little squares in ∂M as well as faces of little cubes.

2.7 Theorem Transversality for $\square$–sets

Let M be a smooth manifold (possibly with boundary) and C a $\square$–set. Let $f\colon M \to |C|$ be a map. Then f is homotopic to a transverse map.

If $f|\partial M$ is already transverse, then the homotopy can be assumed to keep $f|\partial M$ fixed.

Proof We shall first prove the theorem in the case that M is a closed surface Σ, as this case contains all the ideas for the general case.

We start by using standard cellular approximation techniques to homotope f to meet only the 2–skeleton. Next we make f transverse to the centres of the squares of C. The result is that f maps a number of small squares in Σ diffeomorphically onto neighbourhoods of centres of squares in C. By radial homotopies, we can assume that each of these small squares in fact maps onto the whole of a square in C and that the rest of Σ now maps to the 1–skeleton. It is clear how to identify each little square with I^2 so that f maps each by a characteristic map. Now let Σ' denote the closure of $\Sigma - \{$small squares$\}$. Then Σ' is a surface with boundary (with corners) and $f|\partial\Sigma'$ is transverse to the centres of the edges of C. By relative transversality and further radial homotopies, we can homotope f rel $\partial\Sigma'$ so that the preimages of the centres of the edges of C are framed 1–manifolds in Σ' with framing lines mapped onto the relevant edges of C. Moreover we can identify each framing line with I^1 so that it is mapped by a characteristic map. If Σ_0 now denotes the closure of $\Sigma - \{$small squares and framed 1–manifolds$\}$ then Σ_0 is mapped to the 0–skeleton, ie each component of Σ_0 is mapped to a vertex of C. The map f is now transverse.

For the general case of an n–manifold (perhaps with boundary), we can assume inductively that $f|\partial M$ is already transverse and use cellular approximation rel ∂M to ensure that f meets only the n–skeleton. We next make f transverse to the centres of n–cells. By radial homotopies as in the 2–dimensional case we can assume that the closure of the preimage of the interiors of the n–cells denoted M_n is a collection of disjoint little n–cubes in $M - \partial M$ each of which can be identified with I^n and is mapped by a characteristic map, ie f is now transverse to the n–cells of C. Now let M' be the closure of $M - M_n$ then $f|\partial M'$ is transverse to the $(n-1)$–cells of C. By relative transversality and radial homotopies we can homotope f rel $\partial M'$ to be transverse to the $(n-1)$–cells. We then proceed by downward induction to complete the construction of a transverse map homotopic to f rel ∂M. Notice that the process produces compatible framings automatically. $\square$

Pulling back mock bundles by transverse maps

Now suppose that $f\colon M^m \to |C|$ is a transverse map and ξ^q/C is a mock bundle. Then by choosing smooth CW decompositions of each manifold N_c so that the face maps λ_c^* are inclusions of subcomplexes (notation from the definition of a transverse map, above), then the product structure on M_c for each $c \in C$ defines a smooth decomposition of M so that f is a smooth map (of smooth CW complexes). It follows from 2.2 and 2.3

that $f^*(\xi)$ is a mock bundle of codimension q over M and that $E(f^*(\xi))$ is a smooth manifold of dimension $m-q$ representing the Poincaré dual to $f^*(\xi)$. Moreover if ξ is embedded in $|C| \times Q$ then $E(f^*(\xi))$ is ε–isotopic to a smooth submanifold of $M \times Q$.

The case when ξ is a James bundle ζ^i of C will be particularly important for the rest of the paper. Let P_i denote $E(f^*(\zeta^i))$. Then P_i is a smooth manifold of dimension $m-i$ which can be assumed to be embedded smoothly in $M \times \mathbb{R}$. Moreover we can see from construction that the image of P_1 in M is a framed immersed self-transverse submanifold Q of M of codimension 1. (This is illustrated for the case $m=2$ in the figure 2.5 above.) Moreover the images of P_i are the i–tuple points of Q and this is illustrated in figure 2.6. In this figure the image of P_2 is the immersed 1–manifold defined by the double lines and P_3 is the 0–manifold of triple points.

In the next section we shall explore the James bundles from a homotopy theoretic point of view and explain their connection with classical James–Hopf invariants. Moreover we shall see a close connection with the results of [KS] which relate generalised James–Hopf invariants to multiple points of immersions. Then in section 4 the connection with links and diagrams suggested by figures 2.5 and 2.6 will be established.

3 James bundles and James–Hopf invariants

In this section we shall show that James bundles are strongly connected with classical James–Hopf invariants; indeed they define (generalised) James–Hopf invariants for any $\square$–set C and this explains our choice of terminology. We do this by looking carefully at James' original contruction [J1, J2] and in the process we give precise formulæ for the framed embedding of the James bundles in $|C| \times \mathbb{R}$.

We then examine the algebra generated by these classes and recover (stably) the results of Baues [Ba]. Finally we turn to the reduction in ordinary cohomology, which define natural characteristic classes for a $\square$–set whose mod 2 reductions are pulled back from both the Stiefel–Whitney and the Wu classes of BO.

The James construction

Definition *Free topological monoid*

Let X be a topological space based at $*$. The **free topological monoid** on X, denoted X_∞ comprises all words $x_1 \cdots x_m$, where $x_i \in X$ and $m \geq 0$ with the identifications given by regarding $*$ as a unit; ie $x_1 \cdots x_{i-1} * x_{i+1} \cdots x_m \sim x_1 \cdots x_{i-1} x_{i+1} \cdots x_m$ for $i = 1, \ldots, m$. Thus X_∞ is a quotient of the disjoint union $\coprod_{m \geq 0} X^m$ and this defines the topology. The space X_∞ is based at the empty word which we also denote by $*$.

The James map

Let X be a based topological space and suppose that there is a map $\rho_X\colon X \to I$ with the property that $\rho_X^{-1}(0) = \{*\}$. Let $\Omega(X)$ denote the loop space of X and $S(X) = X \wedge S^1$ the suspension of X. (Notice that we take the suspension coordinate second.) We shall identify S^1 with $I/\partial I$. Thus a point of $S^1 - \{*\}$ is uniquely represented as a real number t with $0 < t < 1$, and a point of $S(X) - \{*\} = (X \wedge S^1) - \{*\}$ is uniquely represented as a pair (x, t) where $x \in X - \{*\}$ and $0 < t < 1$. It is convenient to regard the suspension coordinate as vertical.

Suppose given $x_1 \cdots x_m \in X_\infty$ where each $x_i \neq *$. Let $\alpha_i = \rho_X(x_i)$, and let t_i be given by

$$t_i = \frac{\alpha_1 + \ldots + \alpha_i}{\alpha_1 + \ldots + \alpha_m}$$

for $0 \le i \le m$. Then $0 = t_0 < t_1 < \ldots < t_m = 1$.

The **James map** $k_X\colon X_\infty \to \Omega S(X)$ is given by

$$k_X(x_1 \cdots x_m)(t) = \Big(x_i, \frac{t - t_{i-1}}{t_i - t_{i-1}}\Big), \qquad \textbf{3.1}$$

for $t_{i-1} \le t \le t_i$.

In words, what k_X does is to map the word $x_1 \cdots x_m$ to a loop in $S(X)$ which comprises m vertical loops passing through $x_1, \ldots, x_m$ respectively, with the time parameters adjusted by using ρ to make the time spent on a subloop go to zero as the corresponding point x_i moves to the basepoint of X.

James proves that k_X is a homotopy equivalence if X is a countable CW complex with just one vertex. However the result extends to a weak homotopy equivalence for a considerably more general class of spaces. We shall be particularly interested in the case when X is the n–sphere and we shall abbreviate k_{S^n} to k_n.

The classical James–Hopf invariants

We make the following further identifications :

$$S^{n+1} = S(S^n) = S^n \wedge S^1 = S^1 \wedge \cdots \wedge S^1 = I/\partial I \wedge \cdots \wedge I/\partial I = I^{n+1}/\partial I^{n+1}.$$

This means that a point of $S^n - \{*\}$ is uniquely an n-tuple $x = (x_1, x_2, \ldots, x_n)$ where $0 < x_i < 1$.

We choose the map ρ_{S^n} rather carefully.

Let $\rho\colon I \to I$ be a smooth bump function with the property that $\rho^{-1}(0) = \{0, 1\}$ and all derivatives vanish at 0 and 1. Let $\rho_{S^1}\colon S^1 \to I$ be the induced map, and define $\rho_{S^n}\colon S^n \to I$ by $\rho_{S^n}(x_1, x_2, \ldots, x_n) = \rho(x_1)\rho(x_2)\ldots\rho(x_n)$.

Now let $g_n\colon S^1_\infty \to S^n_\infty$ be given by

$$g_n(x_1 \cdots x_m) = \prod_\lambda (x_{\lambda_1}, \ldots, x_{\lambda_n}), \qquad \textbf{3.2}$$

where the product is over all strictly monotone $\lambda\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ and is taken in lexicographical order.

The n**–th James–Hopf invariant** $J_n\colon [S(X), S^2] \to [S(X), S^{n+1}]$ is induced by the composition

$$\Omega S^2 \xrightarrow{\tau} S^1_\infty \xrightarrow{g_n} S^n_\infty \xrightarrow{k_n} \Omega S^{n+1},$$

where τ is a homotopy inverse to k_1.

The trivial $\square$–set T

We now turn to the connection between embedded James bundles and the James–Hopf invariants. We start by identifying the homotopy type of the trivial $\square$–set T (with exactly one cell in each dimension).

3.3 Proposition *The realisation of the $\square$–set T can be identified with S^1_∞ and hence has the homotopy type of $\Omega(S^2)$.*

Proof Denote the unique m–cube of T by c_m. Let $x \in |T|$ then $x \in c_m^\circ$ for some m and hence x has a unique expression as $(x_1, \ldots, x_m)$ where each $0 < x_i < 1$. Moreover the glue given by the face maps has the effect of omitting x_i's which become 0 or 1. It follows that the map $x = (x_1, \ldots, x_m) \mapsto x_1 \cdots x_m$ is a homeomorphism $|T| \to S^1_\infty$. $\square$

Remark Antolini [A] has generalised this result and shown that the singular $\square$–set of any topological space X has the weak homotopy type of $X \times \Omega(S^2)$.

The James bundles of T

Now consider the n-th James bundle $\zeta^n(T)$. In the last section we showed that it embeds as a framed mock bundle in $|T| \times \mathbb{R}$. If we apply the Thom–Pontrjagin construction to this framed embedding we obtain a map $|T| \times \mathbb{R} \to S^{n+1}$ and hence a map $q_n\colon S(|T|) \to S^{n+1}$. Using the last proposition we have a map

$$q_n\colon S(S^1_\infty) \longrightarrow S^{n+1}.$$

The following proposition connects the James bundles of T with the James–Hopf invariants :

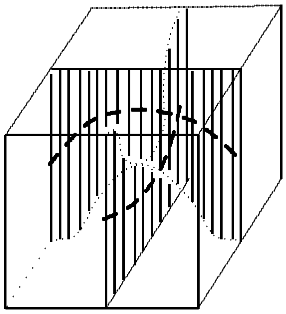
3.4 Proposition *The framed embedding of $E_{\zeta^n(T)}$ in $|T| \times \mathbb{R}$ can be chosen so that q_n is the adjoint of*

$$S^1_\infty \xrightarrow{g_n} S^n_\infty \xrightarrow{k_n} \Omega(S^{n+1})$$

.

Proof Let $h_n\colon I^{n+k} \times I \to S^{n+1}$ be the map determined by the adjoint of $k_n g_n$ restricted to the $(n+k)$–cell of T. Then h_n is transverse to $a = (\frac{1}{2}, \ldots, \frac{1}{2})$ and we shall see that the framed submanifold $h^{-1}(a)$ can be identified with the block of $E_{\zeta^n(T)}$ over I^{n+k}. Moreover the restriction of h_n to $h_n^{-1}(\mathrm{int}(I^{n+1}))$ provides a trivialisation of an open tubular neighbourhood of $E_{\zeta^n(T)}$ in $I^{n+k} \times I \subset I^{n+k} \times \mathbb{R}$. The closure of this neighbourhood is the whole of $I^{n+k} \times I$. The picture illustrates the case $n = k = 1$.

Only the vertical part of the framing is shown for clarity. In general the k–cubes M_{c_λ} are placed in lexicographical order above their images $s_\lambda(I^k)$ in I^{n+k}.



We now give a precise formula for the embedding of this block and then the statements made above can all be checked. Let $\lambda^1, \dots, \lambda^N$ be the list of λ's in the definition of g_n (3.2) with $m = n + k$, taken in order. Then from 3.1 and 3.2 we have

$$h_n(x_1, \dots, x_{n+k}, t) = \Big(x_{\lambda^i_1}, \dots, x_{\lambda^i_n}, \frac{t - t_{i-1}}{t_i - t_{i-1}}\Big),$$

$$\text{where} \quad t_i = \frac{\alpha_1 + \ldots + \alpha_i}{\alpha_1 + \ldots + \alpha_N} \quad \text{for} \quad 0 \le i \le N \quad \text{and} \quad \alpha_i = \rho_n(x_{\lambda^i_1}, \dots, x_{\lambda^i_n}).$$

Each $\lambda^i = (\lambda^i_1, \dots, \lambda^i_n)$ corresponds to a k–cube $M_{c_{\lambda^i}}$ of the n-th James complex, namely the cube $(c_{n+k}, \lambda^i) \in J^n(T)_k$, in the block of $E_{\zeta^n(T)}$ over c_{n+k}. Recall that the λ^i_j's index the directions in I^{n+k} normal to the k–cube (c_{n+k}, λ^i).

We will define an embedding

$$e_{\lambda^i}\colon I^k \times \mathrm{int}(I^n) \times \mathrm{int}(I) \to I^{n+k} \times I \subset I^{n+k} \times \mathbb{R}.$$

Let $sh_{\lambda^i}\colon I^{n+k} \to I^{n+k}$ be the isometry given by permuting coordinates as follows : sh_{λ^i} sends the $(k+s)$–th coordinate to the λ^i_s–th coordinate, for $1 \le s \le n$, and preserves order on the remaining k coordinates. Write $sh_{\lambda^i}(y) = x$, then

$$e_{\lambda^i}(y, u) = (x, ut_i + (1-u)t_{i-1}).$$

Note that the images of the e_{λ^i} are disjoint.

Now define the embedding of the k–cube $(c_{n+k}, \lambda^i) \in J^n(T)_k$ by identifying I^k with $I^k \times (\frac12, \dots, \frac12) \times \frac12 \subset I^k \times \mathrm{int}(I^n) \times \mathrm{int}(I)$ and composing e_{λ^i} with $\chi_{c_{n+k}} \times \mathrm{id}_{\mathbb{R}}$. If we do this for each of the λ^i we obtain a framed embedding of the block of $\zeta^n(T)$ over c_{n+k} and the required properties follow from comparing the formulæ above with 3.1 and 3.2. $\square$

Remark The embeddings e_{λ^i} above extend to maps (not embeddings) of $I^k \times I^n \times I \to I^{n+k} \times \mathbb{R}$ whose images fill the whole of $I^{n+k} \times I$. However the Thom–Pontrjagin construction works for such framings, since the boundary of the tubular neighbourhood is mapped to the basepoint. If a closed trivial tubular neighbourhood is required, then we can restrict to a small $(k+1)$–cube at the centre of $I^k \times I$.

We now turn to general $\square$–sets. The following proposition follows at once from definitions.

3.5 Proposition *James bundles are natural, ie given a $\square$–map $f\colon C \to D$ then $\zeta^n(C)$ is the pull-back $f^*(\zeta^n(D))$.* $\square$

The James–Hopf invariants of a $\square$–set

If C is any $\square$–set there is a canonical **constant** $\square$–map $t_C: C \to T$, so proposition 3.5 implies that $\zeta^n(C) = t_C^* \zeta^n(T)$. Further the framed embedding of $\zeta^n(T)$ in $|T| \times \mathbb{R}$ pulls back to a framed embedding of $\zeta^n(C)$ in $|C| \times \mathbb{R}$ which (by the Thom–Pontrjagin construction) determines a map $S(C) \to S^{n+1}$.

We call the homotopy class of this map $j_n(C) \in [S(C), S^{n+1}]$ the **n-th James–Hopf invariant** of the $\square$–set C.

3.6 Theorem

(1) The James–Hopf invariants of a $\square$–set are natural, ie given a $\square$–map $f: C \to D$ then $j_n(C) = S(f)^* j_n(D)$.

(2) If we identify T with S^1_∞ by proposition 3.3 and with $\Omega(S^2)$ via the homotopy equivalence k_1 then the classical James–Hopf invariant J_n (defined earlier) is induced by composition with $j_n(T)$.

(3) Let $q_C: S(C) \to S^2$ denote the adjoint of $C \xrightarrow{t_C} T = S^1_\infty \xrightarrow{k_1} \Omega(S^2)$ then $j_n(C) = J_n(q_C)$.

Proof Naturality follows at once from definitions. Moreover (2) is a restatement of proposition 3.4 after observing that $q_n \equiv j_1(T)$ (the notation q_n was used in 3.4). Finally

$$J_n(q_C) = \mathrm{adj}((k_n g_n) t_C) = S(t_C)^* \mathrm{adj}(k_n g_n) = S(t_C)^* j_n(T) = j_n(C). \qquad \square$$

Remark As we saw at the end of the last section, if $f: M \to C$ is a transverse map, then the Poincaré duals of the pull-backs of the James bundles comprise a self-transverse immersion and its multiple manifolds. The connection we have given with classical James–Hopf invariants overlaps with the results of Koschorke and Sanderson [KS] which relate generalised James–Hopf invariants to multiple points of immersions. The results overlap when considering $\Omega S(S^1)$. Here $\Omega S(S^1)$ is a special case of $|C|$. In [KS], $\Omega S(S^1)$ is a special case of $\Omega^n S^n(T(\xi))$, $n \geq 1$, where $T(\xi)$ is a Thom space.

Cup Products of James bundles

The James–Hopf invariants $j_n(C)$ of a $\square$-set C are classes in unstable cohomotopy, which stabilise to classes $\gamma_n(C) \in \mathbb{S}^n(C)$ in stable cohomotopy. There is a cup product defined for the unstable classes. Baues [Ba; page79] gives a formula for these cup products for the case $C = T$ (ie for S^1_∞). See Dreckmann [Dr; page 42] for a correction. Here we give a simple geometric proof of the formula after one suspension.

The (unstable) cup product

$$[SX, S^{m+1}] \times [SX, S^{n+1}] \to [SX, S^{n+m+1}]$$

may be described in terms of mock bundles as follows. Let ξ^m and η^n be framed mock bundles embedded in $(X - \{*\}) \times \mathbb{R}$. Assume the projections, p_ξ and p_η are transverse.

Define $E_{\xi \cup \eta} = p_\xi^* E_\eta$. Since E_η is $(n+1)$–framed in $X \times \mathbb{R}$, $E_{\xi \cup \eta}$ is $(n+1)$–framed in $E_\xi \times \mathbb{R}$. But E_ξ is $(m+1)$–framed in $X \times \mathbb{R}$, so using the last of these framing directions to embed $E_\xi \times \mathbb{R}$ in $X \times \mathbb{R}$ we have an embedding of $E_{\xi \cup \eta}$ which is $(n+m+1)$–framed in $X \times \mathbb{R}$. If the roles of ξ and η are reversed we get a homeomorphic result with a possible framing change.

However in the case of the James bundles the last framing direction is parallel to $\mathbb{R}$ in $X \times \mathbb{R}$ and we can see that the two cup products agree up to permutation of components of blocks and a sign change. They also agree (after one suspension and up to sign) with the external cup product (take the Cartesian product, which is naturally $(n+m+2)$–framed in $X^2 \times \mathbb{R}^2$ and restrict to the diagonal). This is seen by rotating the last framing vector through a right angle into the last $\mathbb{R}$–coordinate.

Now let H be an n element subset of $\{1, 2, \ldots, m+n\}$. Let $s(H)$ be the sign of the shuffle permutation of $\{1, 2, \ldots, m+n\}$ which moves H to the front and preserves order in both H and its complement. Let $\phi_{m,n} = \sum_H s(H)$ where the sum runs over all n element subsets H of $\{1, 2, \ldots, m+n\}$

3.7 Proposition *For any $\square$-set C, $Sj_m(C) \cup Sj_n(C) = \phi_{m,n} Sj_{m+n}(C)$.*

Proof We prove the result in the case $C = T$, the trivial complex. The formula then pulls back over the constant map $t_C\colon C \to T$ since the cup product is natural for cubical maps. To compute the cup product we must make the mock bundle projections $p_m\colon |J^m(T)| \to |T|$ and $p_n\colon |J^n(T)| \to |T|$ transverse. Let $b_1 = c_{(i_1,\ldots,i_m)}$ and $b_2 = c_{(l_1,\ldots,l_n)}$ be elements of $J^m(T)$ and $J^n(T)$ respectively where $c \in T_k$. In order to compute the transverse intersection of $p_m|b_1|$ with $p_n|b_2|$ in $|c|$ consider the corresponding sections $s_{(i_1,\ldots,i_m)}\colon I^{k-m} \to I^k$ and $s_{(l_1,\ldots,l_n)}\colon I^{k-n} \to I^k$. The contribution to $j_m(T) \cup j_n(T)$ over $|c|$ is given as follows. There are two possibilities. If $\{i_1, \ldots, i_m\} \cap \{l_1, \ldots, l_n\} = \emptyset$ then the sections are transverse and they meet in the image of $s_{(p_1,\ldots,p_{n+m})}$ where $\{p_1, \ldots, p_{n+m}\} = \{i_1, \ldots, i_m\} \cup \{l_1, \ldots, l_n\}$. Since, strictly speaking, p is a function $\{1, \ldots, n+m\} \to \mathbb{N}$ we can define $H = p^{-1}\{i_1, \ldots, i_m\}$. We have determined an element $c_{(p_1,\ldots,p_{n+m})} \in J^{n+m}(T)$ whose realisation is a $k-(n+m)$ cube in the total space of the mock bundle $\zeta^{n+m}(T)$. But the cup product framing differs from the natural framing by the sign of the shuffle $s(H)$. This covers the transverse case.
If say $\{i_1, \ldots, i_m\} \cap \{l_1, \ldots, l_n\} = \{q_1, \ldots, q_s\}$ with $q_1 < \ldots < q_s$ then we isotope p_n on $|c_{(j_1,\ldots,j_n)}|$ by increasing the q_1–th coordinate in I^k from $\frac{1}{2}$ to say $\frac{3}{4}$, then the intersection is empty and there is no contribution to $j^m \cup j^n$ in this case.

At this point we have proved the result provided we can cancel off contributions of opposite sign and reorder the pieces. For this we need to use the one extra dimension where there are obvious cobordisms which achieve this result. $\square$

3.8 Lemma *For $m, n \in \mathbb{N}$*

$$\phi_{m,n} = \begin{cases} \binom{[(m+n)/2]}{[n/2]} & \text{if } nm \text{ even,} \\ 0, & \text{otherwise} \end{cases}$$

In particular for all m, n $\phi_{2m,2n} = \binom{m+n}{m}$, $\phi_{m,n} = \binom{m+n}{m}$ *mod 2*, $\phi_{1,1} = 0, \phi_{1,2n} = 1$.

Proof Clearly $\phi_{0,n} = \phi_{m,0} = 1$. Set $\phi_{m,n} = 0$ if either m or n is negative. For $\phi_{m,n}$ the sum can be split up, according as the least element of H is or is not 1, giving the recurrence relation:

$$\phi_{m,n} = \phi_{m,n-1} + (-1)^n \phi_{m-1,n}.$$

The result now follows by induction. □

The computation of the algebra generated by the stabilisation of the James–Hopf invariants easily follows. We shall carry out this computation after introducing the characteristic classes of a □–set, which are the corresponding cohomology classes.

The characteristic classes of a □–set

Let C be any □–set. The n**–th characteristic class** of C, $V_n(C) \in H^n(C)$ is the class of the unit cocycle: value 1 on each n–cube.

Let $\mathbb{S}$ denote the sphere spectrum and let $\gamma_n(C) \in \mathbb{S}^n(C)$ denote the stabilisation of $j_n(C)$. The element $\gamma_n(C)$ is represented by E_{ζ^n} framed in $|C| \times \mathbb{R}^\infty$. Let $h\colon \mathbb{S}^* \to H^*$ be the Hurewicz transformation.

3.9 Proposition *For any □–set C* $h(\gamma_n(C)) = V_n(C)$.

Proof At the mock bundle level h can be described simply by counting, with sign, the components of the 0–dimensional blocks. The result follows since the 0–dimensional blocks of γ_n consist of a single point over each n-cube with standard framing. □

Denote the exterior algebra, over $\mathbb{Z}$, on one generator x by $E(x)$, then $x^2 = 0$ and elements of $E(x)$ and can be written as $p + q.x$, $p, q \in \mathbb{Z}$. Denote the divided polynomial algebra on one generator y by $D(y)$, in other words there are elements y_n, $n \geq 0$, with $y_0 = 1$ and $y_1 = y$ such that $y_n.y_m = \binom{n+m}{m} y_{n+m}$. An element of $D(y)$ is then just a linear combination of the y_n's.

3.10 Corollary *The elements $\gamma_n(T)$, $n \geq 0$, generate a subalgebra of $\mathbb{S}^*(|T|)$ of form $E(\gamma_1(T)) \otimes D(\gamma_2(T))$. In particular any element of the subalgebra can be uniquely written as a linear combination of the elements $\{1, \gamma_1\gamma_{2n}, \gamma_{2n}\}_{n\geq 0}$.*

Proof The result follows from the proposition and the lemma and the fact noted above that the Hurewicz map $h\colon \mathbb{S}^*(|T|) \to H^*(|T|)$ is non trivial on the γ_n, $n \geq 0$. In fact it follows that h restricted to the subalgebra is an isomorphism. □

The $\mathbb{Z}_2$–characteristic classes

Let v_n denote the mod 2 reduction of our characteristic class V_n. We can identify $v_n(T)$ as the Stiefel–Whitney class of a vector bundle. Background for what follows can be found in [S1, M]. Let BO be the classifying space for the infinite orthogonal group, and let B^2O be its delooping; so $\Omega B^2O \simeq BO$. Let $\omega\colon S^2 \to B^2O$ be a generator of $\pi_2 B^2O \cong \mathbb{Z}_2$. By taking $\Omega\omega$ we get a map $s\colon |T| \to BO$ with the property that s restricted to the 1-skeleton classifies the Möbius band.

3.11 Proposition *$w_n(s^*\gamma) = w_n(s^*\gamma^{-1}) = v_n(T)$, $n \geq 0$, where γ is the universal (virtual) bundle.*

Proof Consider the n–fold product $S_n = S^1 \times \ldots \times S^1$. Let $m\colon S_n \to |T|$ be given by multiplication. Then $m^*\colon H^n(|T|) \to H^n(S_n)$ is an isomorphism. Since s is an H-map $w(m^*s^*\gamma) = \prod_{i=1}^n (1 + a_i)$ where each a_i is the generator of $H^1(S^1, \mathbb{Z}_2)$, and so $w_n(s^*\gamma) \neq 0$. Since $m^*s^*\gamma$ is a product of (stable) line bundles over S^1's $m^*s^*\gamma = m^*s^*\gamma^{-1}$. □

Let $\phi_C\colon |C| \to BO$ be the composition $s \circ t$ where $t\colon |C| \to |T|$ and let $\gamma_C = \phi_C^*\gamma$. We call γ_C the **characteristic bundle** over C.

3.12 Corollary *For any □–set C, $w_n(\gamma_C) = v_n(C)$, $n \geq 0$.* □

Remark Recall that the (total) Stiefel–Whitney class , $w(\xi)$, of a vector bundle ξ is defined by $w(\xi) \cup U = Sq\,U$ where U is the Thom class. Similarly the Wu class, $\mathrm{wu}(\xi)$, is given by $\mathrm{wu}(\xi) \cup U = \chi Sq U$. From this and the Cartan formula we get the well known relation $Sq\,\mathrm{wu}(\xi) = w(\xi^{-1})$. On S_n we have $Sq = Id$ and $m^*s^*\gamma = m^*s^*\gamma^{-1}$ so both in the proposition and the corollary Whitney and Wu classes coincide.

We shall return to the study of the characteristic bundle and the associated map $\phi_C\colon |C| \to BO$ in section 6, when we use it to define the concept of a C–oriented manifold and to define an associated generalised cohomology theory.

4 Links and Diagrams

In this section we use the transversality theorem from section 2 to prove that any □–set C is a classifying space for labelled link diagrams. The labelling is by the cubes of C. More precisely, the homotopy group $\pi_n(C)$ is naturally isomorphic to the group of bordism classes of link diagrams in $\mathbb{R}^n$ labelled by C. Any such link diagram is naturally covered by an embedded codimension 2 link in $\mathbb{R}^{n+1}$ and the whole structure can be seen to coincide with the pull-back of the first James bundle $\zeta^1(C)$.

In the key example in which C is the rack space BX, there is a far simpler description : $\pi_n(BX)$ is naturally isomorphic to the group of bordism classes of codimension 2 links embedded in $\mathbb{R}^{n+1}$ with homomorphism of fundamental rack to X. Thus BX is a classifying space for such links up to bordism and moreover the first James bundle $\zeta^1(BX)$ plays the rôle of classifying bundle.

There are similar interpretations for the bordism groups of C and BX.

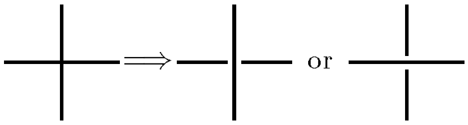
In the lowest non-trivial dimension ($n = 2$) the cobordism classes can be desribed as equivalence classes under simple moves and this gives a combinatorial description of $\pi_2(C)$ which can be used for calculations. We shall deal with this case ($n = 2$) in detail before discussing the general case.

Diagrams

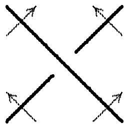
We shall need to consider diagrams and we start in the familiar setting of diagrams drawn in 2 dimensions.

Definition *Diagram on a surface*

A **diagram** D on a surface Σ is a collection of framed immersed circles in general position in Σ such that at each crossing one of components is locally regarded as the overcrossing curve and the other as the undercrossing curve.



The framing can be pictured as a transverse arrow for each arc of the diagram the direction of which is preserved through crossings.



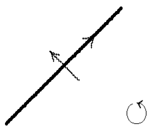
We call the component of $\Sigma - D$ the **regions** of the diagram and the components of $D - \{\text{double points}\}$ the **arcs** of D.

4.1 Lemma *A diagram in Σ determines a framed embedding in $\Sigma \times \mathbb{R}$.*

Proof Near each crossing, lift the overcrossing curve up into the positive $\mathbb{R}$ direction (thought of as vertical). Use the framing vector in Σ as the first framing vector and vertically up as the second. □

We call the framed embedding given by the lemma the **lift** of the diagram.

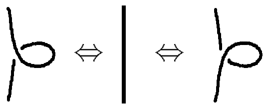
Remark If the surface is oriented then the framing determines an orientation on the arcs of the diagram (and conversely) by the left-hand rule illustrated :



When the surface and arcs are oriented then the framing given by the lemma is usually called the "blackboard framing".

Framed embeddings and diagrams

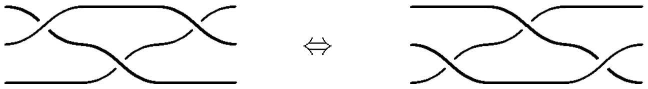
Up to isotopy any framed embedding in $\mathbb{R}^3$ is the lift of a diagram in $\mathbb{R}^2$. This is seen by choosing any diagram to represent the (unframed) link and then correcting the framing by introducing twists (Reidemeister 1–move) :



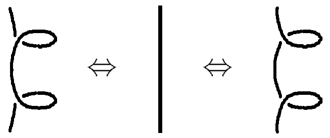
The resulting diagram is unique up to regular homotopy (or equivalently Reidemeister 2 and 3 moves) together with the double Riedemeister 1–move illustrated below. For a proof see [FR; pages 369-370]. We shall refer to these moves as R2, R3 and R11 (pronounced r-one-one) respectively :

R2

R3



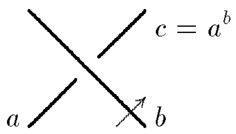
R11



There is a similar result with the same proof for any surface.

Fundamental rack of a diagram in the plane

For details here see [FR; pages 369–375]. A diagram in $\mathbb{R}^2$ determines a **fundamental rack** by labelling the arcs by generators and reading a relation at each crossing :



A framed codimension 2 embedding also determines a **fundamental rack**; moreover the rack determined by the lift of a diagram can be naturally identified with the rack determined by the diagram.

Notation If D is a diagram in $\mathbb{R}^2$, then we denote the fundamental rack of D by $\Gamma(D)$.

Labelling by a rack

We now explain how to **label** a diagram by a rack :

Definition *Diagram labelled by a rack*

Let D be a diagram. We say that D is **labelled** by a rack X if each arc of D is labelled by an element of X with compatibility at crossings as illustrated in the last diagram above, where a, b, c now denote elements of X (rather than generators of $\Gamma(D)$) and $c = a^b$ in X.

Note that there are really two versions of this diagram because the framing of the understring is immaterial.

There is a natural labelling of any diagram by its fundamental rack, which we call the **identity labelling**. More generally we have the following result which follows immediately from definitions :

4.2 Lemma *A labelling of a diagram D by a rack X is equivalent to a rack homomorphism $\Gamma(D) \to X$.* □

The lemma implies that labelling is functorial in the sense that a rack homomorphism $X \to Y$ induces a labelling by Y, it also implies that labelling is really a property of the lift, not the diagram :

4.3 Corollary *A labelling of a diagram by a rack X is equivalent to a rack homomorphism to X of the fundamental rack of the lifted framed embedding. Hence a labelling is carried canonically through moves R11, R2, R3.* □

We shall speak of a *representation in* X for a framed embedding as short for a homomorphism of the fundamental rack to X.

Labelling by a □–set

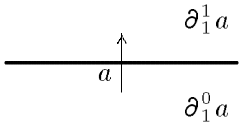
We shall need a more general notion of a labelled diagram :

Definition *Diagram on a surface labelled by □–set C*

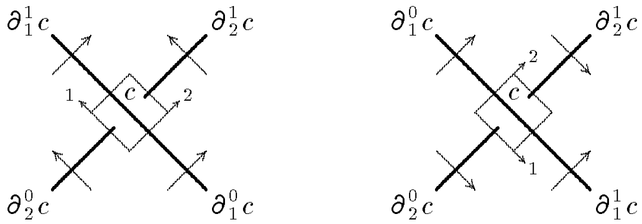
A diagram D is **labelled** by a □–set C if :

(1) The regions are labelled by vertices of C.

(2) The arcs are labelled by edges of C compatibly with adjacent regions. This means that the edge labelling an arc α is attached (in C) to the two vertices labelling the adjacent regions. To decide which vertex labels which side, identify the edge with the transverse framing arrow :



(3) The double points are labelled by squares of C compatibly with adjacent arcs. This means that the four adjacent arcs are labelled by the four 1–faces of the square. The rule for determining which face labels which arc is this : position the a copy of the standard square (ie I^2) at the double point with faces oriented correctly by the framing arrows and axis 1 parallel to the overcrossing. Now the four faces intersect the appropriate adjacent arcs, see the diagram below, where we have drawn both possible orientations for the square :



Remarks

(1) If $C = BX$, where X is a rack, then labelling in C is precisely the same as labelling in X.

(2) Every diagram has the **trivial** labelling, namely labelling by T, the trivial $\square$–set (with one cell of each dimension). This can also be regarded as labelling by the trivial rack (with one element).

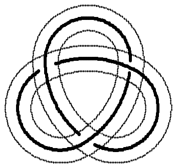
(3) Labelling is functorial : Given a diagram labelled in C and a $\square$–map $f\colon C \to D$ then f transforms the labels to a labelling in D.

Labelling and transversality

In section 2 we defined a transverse map of a surface Σ in a $\square$–set C and we observed that such a map defines a self-transverse immersed 1–manifold in Σ (see figure 2.5), which can be seen as the image of the pull-back of the first James bundle $\zeta^1(C)$. We also observed that $\zeta^1(C)$ embeds in $|C| \times \mathbb{R}$ and hence this immersed 1–manifold is covered by an embedding in $\Sigma \times \mathbb{R}$ which is framed by one vertical vector and one horizontal vector in other words it is a *diagram*. Moreover this diagram is labelled by C in a natural way. Recall that each double point is surrounded by a little square which maps to a square of C. Label the double point by that square. Each arc is framed (outside the little squares) by lines mapped to an edge of C. Label the arc by that edge. Finally each region is mapped to a vertex (outside the framed arcs and little squares). Label the region by that vertex. It can readily be checked that this labelling is compatible, indeed the definition of compatibility (above) was set up precisely in this way.

Thus a transverse map *to* C determines a diagram labelled *by* C. We now describe the converse process.

Suppose that we are given a diagram in Σ labelled by C. We construct a **neighbourhood system** for the diagram by drawing little squares around the double points and continue to construct bi-collars around the arcs.



This determines a transverse map into C by mapping the regions outside the squares and bi-collars to the labelling vertex, collapsing the bi-collars onto fibres and mapping to the labelling edge and finally mapping the little squares to the labelling squares for the double points, using the orientations for edges and squares described in the definition of labelling by a $\square$–set (above).

It is clear that this construction is unique up to minor choices which only affect the neighbourhood system up to an ambient isotopy fixing the diagram setwise. To be precise, define two transverse maps of Σ in C to be **equivalent** if they differ by an ambient isotopy fixing the pull-back diagram setwise then we have a well-defined process for turning a labelled diagram into an equivalence class of transverse maps. The two processes (transverse map to diagram, diagram to equivalence class of transverse map) are inverse. Summarising we have :

4.4 Proposition *There is a bijection between labelled diagrams in Σ labelled by a $\square$–set C and equivalence classes of transverse maps of Σ in C.*

The bijection is defined by pulling back the first James bundle $\zeta^1(C)$. $\square$

Remark Note that the lift of the diagram in $\Sigma \times \mathbb{R}$ is also obtained by pulling back the embedded first James bundle $\zeta^1(C) \subset C \times \mathbb{R}$.

In order to interpret π_2 of a cubical set, we need a based version of the proposition. Choose basepoint $* \in S^2$ and base vertex $* \in C_0$ and identify $S^2 - \{*\}$ with $\mathbb{R}^2$. We have :

4.5 Proposition *There is a bijection between labelled diagrams in $\mathbb{R}^2$ labelled by a $\square$–set C such that the non-compact region is labelled by the vertex $* \in C_0$ and equivalence classes of based transverse maps of S^2 in C.*

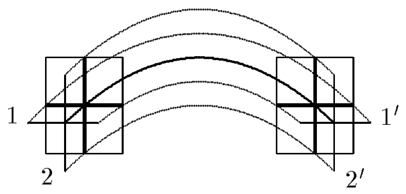
The bijection is defined by pulling back the first James bundle $\zeta^1(C)$. $\square$

Cobordism of diagrams

We now seek to interpret homotopy classes of maps into C. To do this we shall need the concepts of 3–dimensional diagrams and of cobordism of diagrams.

Definition *Diagram in a 3–manifold*

A diagram in a 3–manifold is a self-transverse immersed surface (ie transverse double curves and triple points) equipped with a compatible ordering of sheets at double curves and triple points. Compatible means that the ordering of sheets at the triple points restricts to give the ordering at adjacent double arcs. It follows that the ordering of sheets at double arcs is preserved in a neighbourhood of a triple point. So, for example, if in the following diagram sheet 2 is above sheet 1 then sheet $2'$ is above sheet $1'$.



Remarks

(1) A diagram lifts to a framed embedding in $M \times \mathbb{R}$. The proof is much the same as lemma 4.1. The order of the sheets at double arcs and triple points is used to construct the embedding. We lift a sheet, which is higher in the order, higher in the vertical direction. Compatibility ensures that the construction is coherent.

(2) As in the 2–dimensional case, we can define a fundamental rack for both the diagram and its lift and the two can be naturally identified. To read the rack from the diagram, sheets outside double arcs are labelled by generators and a relation is read from each double arc by the same rule as in the 2–dimensional case (think of a perpendicular slice); triple points are not used (cf [FR; remark (2), page 375]).

(3) In the case that M has boundary, we assume that diagrams are **proper** ie the immersed surface meets ∂M in its boundary (which thus defines a diagram in ∂M).

Definition *Cobordism of diagrams*

Diagrams D_0, D_1 in Σ are **cobordant** if there is a diagram D in $\Sigma \times I$ which meets $\Sigma \times \{0,1\}$ in D_0, D_1 respectively. We call D the **cobordism** between D_0 and D_1. It can readily be checked that diagram cobordism is an equivalence relation.

The concept of labelling by a $\square$–set for a 2–dimensional diagram extends in a natural way to 3–dimensional diagrams :

Definition *Labelling of a diagram in a 3–manifold*

Let D be a diagram in a 3–manifold M. Call the components of the double curves minus triple points **double arcs**, the components of the surface minus the double curves **sheets** and the components of M minus the surface **regions**.

A labelling of D by a $\square$–set C is a labelling of regions (resp. sheets, double arcs, triple points) by vertices (resp. edges, squares, 3–cubes) of C subject to compatibility conditions at sheets, double curves and triple points.

The compatibility conditions at sheets and double curves are the same as for a 2–dimensional diagram (imagine working in a transverse slice) whilst the compatibility condition at a triple point can be described as follows. Say the *positive* side of a sheet coincides with the head of its framing arrow. Suppose a triple point is labelled by $x \in C_3$. Suppose a nearby double curve is labelled by $y \in C_2$ and the missing sheet is number $i \in \{1,2,3\}$. Then $\partial_i^\epsilon x = y$ where $\epsilon = 1$ if the double curve is on the positive side of the missing sheet and $\epsilon = 0$ otherwise. (Notice that a small copy of the standard cube I^3 can be placed at a triple point by orienting axes according to framing of sheets and ordering axes so that axis i is perpendicular to the i^{th} sheet. If this is done then the small cube meets an adjacent double arc in the face given by the above labelling rules.)

Labelled diagrams and transverse maps

Just as in the 2–dimensional case, a transverse map of M into C determines a labelled diagram in M labelled by C. The diagram is the image of the pull-back of the first James bundle $\zeta^1(C)$. The labelling is by the cells to which the relevant item (triple point, double arc, sheet or region) is mapped.

Conversely, given a labelled diagram, we can construct a transverse map into C by first constructing a **neighbourhood system** as follows. Choose little 3–cubes around the triple points which meet neighbouring sheets in the three central 2–cubes. Each of these can be identified with I^3 in a canonical way using the ordering of sheets as remarked above. We call the portions of the double arcs outside these cubes, **reduced double arcs**. Next construct a trivial bundle with fibre a square around each reduced double arc such that each square meets neighbouring sheets in the central cross and which fits onto a face of the relevant 3–cube at boundary points (precisely how to construct these trivial bundles will be explained in lemma 4.6 below). Finally construct bi-collars around the reduced sheets (outside the square bundles) which fit onto edges of the squares. (See figure 2.6 for a view of part of this construction.)

Now map the 3–cubes to the labelling 3–cube of C, collapse the square bundles onto a single square and map to the labelling square, and likewise collapse the bi-collars and map to labelling edges and finally map the reduced regions to labelling vertices. The result is the required transverse map to C.

The two processes are again inverse. The only element of choice in the contruction was the choice of the neighbourhood system. In the next lemma we show how to choose this system using collaring arguments. Using uniqueness of collars we can then see that the system is unique up to ambient isotopy fixing the diagram setwise.

4.6 Lemma Constructing neighbourhood systems by collars

Let D be a diagram in a 3–manifold M. Then D has a neighbourhood system and any two such systems differ by an ambient isotopy of M fixing D setwise.

Proof Start by choosing a collar for each triple point in each double arc (ie an interval). Now concentrate on a particular triple point p and label the extended sheets near p 1,2,3 (in the order given by the diagram) and use the label 12 for example for the extended double arc $1 \cap 2$ etc. The collar on p in 12 is a double interval J say. Choose a bi-collar on J in 1 extending the chosen collar in 13. This defines a square S say in 1. Do the same for 12 in 2 and 13 in 3. We now have three mutually perpendicular squares. Complete the little cube at p by choosing a bi-collar on S in M extending the chosen (partially defined) collars on 12 in 2 and 13 in 3.

To define the square bundle on a reduced double arc α say, choose bi-collars in both intersecting sheets (extending collars given by the little cubes on ∂J) and then extend one of these to a bi-collar on the total space of the other collar (extending the collars given by the squares over ∂J). Finally construct the bi-collars over the reduced sheets extending the already constructed collars (given by the square bundles) over the boundary.

It is clear that a neighbourhood system defines all the above collars and the uniqueness part of the lemma now follows from uniqueness of collars. $\square$

Summarising the above discussion we have the following result in which transverse maps are again defined to be **equivalent** if they differ by an ambient isotopy of M fixing the diagram setwise :

4.7 Proposition *There is a bijection between equivalence classes of labelled diagrams in a 3–manifold M labelled by a $\square$–set C and transverse maps of M in C.*

If M has boundary then the bijection coincides with the bijection for ∂M given by proposition 4.4.

The bijection is defined by pulling back the first James bundle $\zeta^1(C)$. $\square$

Labelling by a rack

Labelling by a rack X is defined to be labelling by the rack space BX. Notice that this is the same as labelling sheets by elements of X with the same compatibility requirement at double curves as for double points of 2–dimensional diagrams (see the earlier diagram) — think of a transverse slice. Triple points play no part in this because if $C = BX$ then there is no need to label triple points since a 3–cube is determined by its faces (see the discussion in 1.4 example (3)). Also there is no need to label double curves explicitly since a square in BX is determined by its edges. The compatibility condition at double curves ensures that the required square exists.

As in the 2–dimensional case labelling by a rack X is equivalent to a homomorphism of the fundamental rack to X either of the diagram or of the covering embedding.

There is also the concept of labelling by a *trunk* T in other words labelling by the nerve $\mathcal{N}T$ (see [FRS]). In this case regions are labelled by vertices and sheets by edges (between the vertices labelling adjacent regions) such that at double curves the adjacent edges form a preferred square. As for racks, triple points play no part in the labelling.

Definition $\mathcal{D}(2, C)$ – *The group of cobordism classes of labelled diagrams*

Let C be a $\square$–set with a basepoint $* \in C_0$. Diagrams D_0, D_1 labelled by C are **labelled cobordant** if there is a cobordism between them labelled by C. Define the set $\mathcal{D}(2, C)$ to be the set of labelled cobordism classes of diagrams in $\mathbb{R}^2$ (labelled by C) such that the non-compact region is labelled by the vertex $*$.

Now the set of diagrams in $\mathbb{R}^2$ has an addition given by disjoint union. To be precise, given diagrams D_1, D_2 choose copies in disjoint half planes, then define $D_1 + D_2$ to be $D_1 \amalg D_2$. This addition is well-defined up to cobordism, is compatible with cobordism and makes $\mathcal{D}(2, C)$ into an abelian group.

In the case when $C = BX$, a rack space, we abbreviate $\mathcal{D}(2, BX)$ to $\mathcal{D}(2, X)$.

We are now ready to state and prove our first main classification theorem.

4.8 Theorem Classification of labelled diagrams

Let C be a $\square$–set with basepoint $* \in C_0$. There is a natural isomorphism between $\pi_2(C)$ and $\mathcal{D}(2, C)$. The isomorphism can be described as given by pulling back the first James bundle $\xi^1(C)$ which thus plays the rôle of classifying bundle for labelled diagrams.

Proof By proposition 4.5 a labelled diagram in $\mathbb{R}^2$ such that the non-compact region is labelled by the vertex $* \in C_0$ determines a based (transverse) map of S^2 in C. Similarly by proposition 4.7 a cobordism determines a homotopy. Thus we have a function $\Phi\colon \mathcal{D}(2, C) \to \pi_2(C)$. By transversality (theorem 2.7) Φ is a bijection. The rest of the theorem follows from definitions. $\square$

The theorem cuts both ways. It classifies diagrams up to cobordism and also provides an interpretation of $\pi_2(C)$ which can be used for calculations. For this purpose we need to break a cobordism into a sequence of combinatorial moves which we now describe.

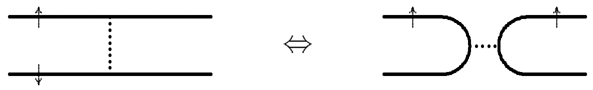
Cobordism by moves

Suppose that we are given a cobordism (a diagram D in $\mathbb{R}^2 \times I$) between diagrams D_0, D_1. Think of $\mathbb{R}^2 \times I$ as a sequence of copies of $\mathbb{R}^2$. This breaks D into a sequence of slices. By general position this is an isotopy apart from a finite number of critical slices which are maxima, minima and saddles of the sheets, maxima and minima of the double curves and triple points. Corresponding to these are the diagram moves listed below :

Diagram moves

BD Births and deaths of little circles : $D \Leftrightarrow D \cup O$.

Br Bridge between arcs with compatible framing :

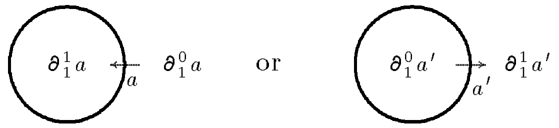


R2 move

R3 move

If the cobordism is labelled then the labelling before and after a move satisfies the following compatibility conditions :

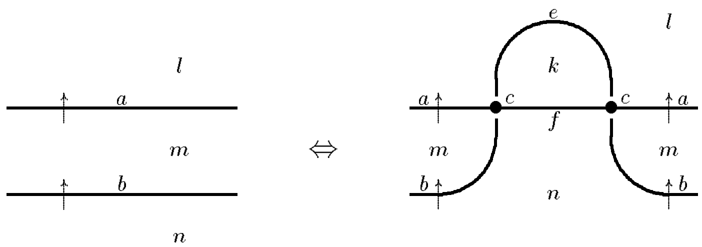
BD Births must correspond to an edge of C :



There is no condition for deaths.

Br Bridges must be between arcs with the same label.

An **R2** move must involve two double points labelled by the same square (with opposite orientations) :



In the figure the following face equalities hold : $n = \partial_1^0 b = \partial_1^0 f$, $m = \partial_1^1 b = \partial_1^0 a$, $l = \partial_1^1 a$, $a = \partial_1^1 c$, $f = \partial_1^0 c$, $e = \partial_2^1 c$, $b = \partial_2^0 c$, $k = \partial_1^1 f = \partial_1^0 e$.

An **R3** move must correspond to a 3–cube of C.

4.9 Proposition *Diagrams labelled in C are cobordant iff they differ by compatible moves BD, Br, R2, R3 (above).* □

Note If the labelling is by a rack then R2 and R3 moves are always compatible (this is essentially what the rack laws are designed for [FR; section 4]), and births can have arbitrary labels.

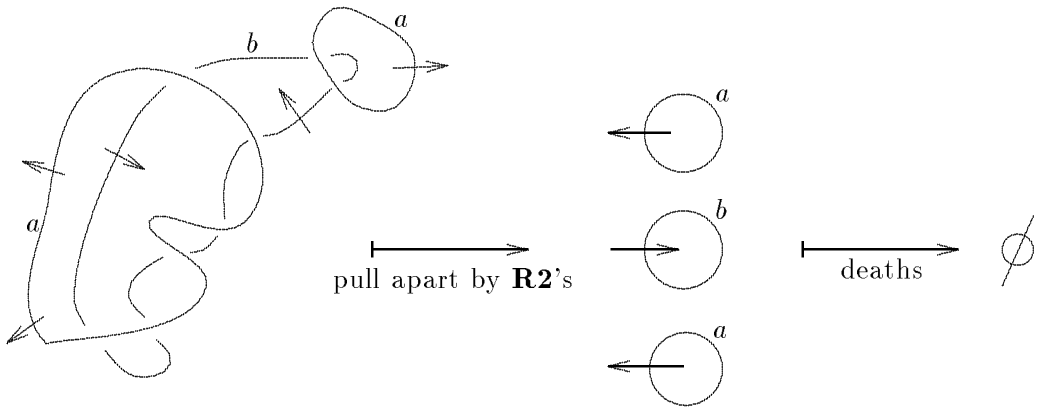
Digression on π_2

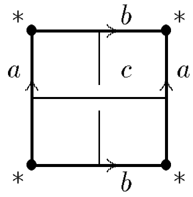
Remark Theorem 4.8 and description of cobordisms in terms of moves allows us to make calculations of π_2. See for example theorem 5.15 for a report of calculations made by this method. Note at once that the writhe of a diagram is invariant under moves and hence we always have a map to $\mathbb{Z}$. The **writhe** is defined (as usual) to be the number of double points counted with sign — a right-hand crossing counting $+1$ and a left-hand one -1.

This map to $\mathbb{Z}$ is not always onto as the illustrative example below shows. The writhe has the following interpretation : Let $t_C : C \to T$ be the constant map as usual. Then writhe is the same as $\pi_2(C) \xrightarrow{(t_C)_*} \pi_2(T) = \pi_2(\Omega(S^2)) = \pi_3(S^2) = \mathbb{Z}$.

Example *Calculation of π_2 (torus)*

As an illustration of the use of diagrams to calculate π_2 of a $\square$–set we calculate π_2 (2–torus).



where $T^2 =$ 

The Whitehead conjecture

To illustrate the applicability of this method in general, we now give a translation of the Whitehead conjecture [Wh] into a conjecture about coloured diagrams.

Recall that the Whitehead conjecture states that if K is a subcomplex of the 2–dimensional complex L then $\pi_2(L) = 0$ implies $\pi_2(K) = 0$. By proposition 1.2 it is sufficient to establish the conjecture for 2–dimensional $\square$–sets and subsets.

We consider plane diagrams up to moves Br, BD and R2. No R3's are allowed because the dimension is at most 2.

A **colouring** of a diagram is a colouring of arcs, regions and crossings. The outside (infinite) region is always coloured white say. A **colour scheme** is a list of allowable colours (three lists one each for regions, arcs and crossings) and rules about neighbouring colours. The rules prescribe the two neighbouring colours for a given colour on an edge and all the neighbouring colours for a given colour on a crossing.

Moves are **allowable** if Br and BD moves respect the colour scheme and R2 moves involve two crossings with the same colour (but opposite orientation).

A diagram is **reducible** if it can be changed to the empty (all white) diagram by allowable moves.

Using proposition 4.9, the Whitehead conjecture now has the following equivalent statement :

4.10 **Conjecture** *Suppose that for a given colour scheme all diagrams are reducible, then any diagram can be reduced without using any colours not already used in the diagram.*

Classification of links

Definitions We say that framed links L_0, L_1 in $\mathbb{R}^3$ are **cobordant** if there is a framed surface Σ properly embedded in $\mathbb{R}^3 \times I$ which meets $\mathbb{R}^3 \times \{0,1\}$ in L_0, L_1 respectively.

Let X be a rack, then there is an analogous notion of cobordism of links with representation in X, namely a cobordism with a representation in X (ie a homomorphism of the fundamental rack in X) extending the given representations on the ends.

It can readily be checked that cobordism is an equivalence relation and we denote the set of cobordism classes of framed links in $\mathbb{R}^3$ with representation in X by $\mathcal{F}(3, X)$.

There is an addition on $\mathcal{F}(3, X)$ given by disjoint union. To be precise, given links L_1, L_2 choose copies in disjoint half spaces, then define $L_1 + L_2$ to be $L_1 \amalg L_2$. We need to explain how to represent $L_1 \amalg L_2$ in X. The simplest way to do this is to use diagrams. If L_i is given by the diagram D_i then $L_1 \amalg L_2$ is given by $D_1 \amalg D_2$ and representations correspond to labellings of D_1, D_2 which define a labelling of $D_1 \amalg D_2$ in the obvious way. We can also define the required representation using the fact that the fundamental rack of $L_1 \amalg L_2$ is the free product of the fundamental racks of L_1, L_2 (see [FR; page 357]). The homomorphisms of the two factors determine a homomorphism on the entire rack.

The addition on $\mathcal{F}(3, X)$ is well-defined up to cobordism, is compatible with cobordism and makes $\mathcal{F}(3, X)$ into an abelian group.

We are now ready for our second main classification therorem :

4.11 **Theorem** Classification of links

Let X be a rack. There is a natural isomorphism between $\pi_2(BX)$ and $\mathcal{F}(3, X)$. The embedded first James bundle plays the rôle of classifying bundle.

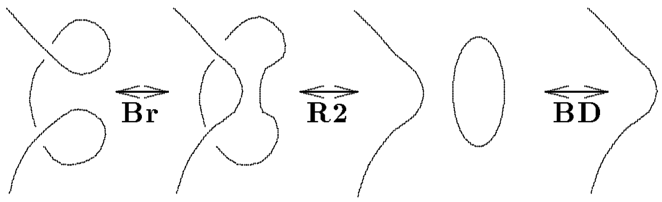
Proof The classification theorem for labelled diagrams says that $\pi_2(BX)$ is the same as cobordism classes of labelled diagrams in $\mathbb{R}^2$ with labels in X. Moreover a diagram labelled in X lifts to a framed link with representation in X and a cobordism of diagrams to a cobordism of links. Thus we have a homomorhpism :

$$\Psi \colon \pi_2(BX) \longrightarrow \mathcal{F}(3, X)$$

Ψ is surjective since any framed link is the lift of a diagram (unique up to moves R11, R2, R3). To see that Ψ is injective we have to show that labelled diagrams whose lifts are cobordant are themselves cobordant (as diagrams) or equivalently (using proposition 4.9) that they differ by moves BD, Br, R2 and R3. We shall show this by analysing the cobordism. Since the representation (or labelling) in X plays no real rôle in the proof it will henceforth be suppressed.

Now the cobordism is a framed 3–manifold in $\mathbb{R}^3 \times I$ and by slicing by parallel $\mathbb{R}^3$'s we obtain (using general position) a sequence of framed links with the following critical stages (where a slice contains a critical point of the projection to I) : Bridge moves, births and deaths of small circles. Moreover by rotating small neighbourhoods of these critical points (if necessary) we can assume that near the critical point nearby slices are lifts of diagrams; ie that the R1 moves needed to correct the framing are away from the critical points.

Thus we can choose diagrams near these critical stages whose lifts are isotopic to the nearby slices and which differ by diagram moves BD or Br. Now away from critical stages each link in the sequence corresponds to a diagram unique up to moves R2, R3, R11 and combining the two sets of moves, we see that the cobordism corresponds to diagram moves BD, Br, R2, R3, R11. But the following sequence of pictures shows how to achieve an R11 as a combination of a bridge move, an R2 and a death. The rest of the theorem is clear. $\square$



4.12 Remark Both the main theorems extend to a surface M to classify diagrams in M labelled by a $\square$–set and links in $M \times \mathbb{R}$ with representation in a rack, both being taken up to cobordism. We shall describe the extension of theorem 4.10 and leave the reader to extend theorem 4.8. Framed links L_0, L_1 in $M \times \mathbb{R}$ are cobordant if there is a framed surface Σ properly embedded in $M \times \mathbb{R} \times I$ such that $\Sigma \cap M \times \mathbb{R} \times \{i\} = L_i, i = 1, 2$.

Theorem *There is a bijection between cobordism classes of links in $M \times \mathbb{R}$ with representation in a rack X and the set of homotopy classes $[M, |BX|]$.*

(The proof is analogous to 4.10.)

We can also classify links up to simultaneous cobordism of both link and surface. This set is in bijection with $\mathfrak{N}_2(BX)$ (the unoriented bordism group). The reader can readily state and prove other similar classification theorems.

Higher dimensional results

The work done above for diagrams in $\mathbb{R}^2$ and links in $\mathbb{R}^3$ can all be extended to $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. Large parts of this extension are simple modifications of the low dimensional versions and will be treated very briefly. Where details are omitted they are essentially the same as the low dimensional details.

We start by defining diagrams in arbitary dimensions. First we need the concept of a self transverse immersion. Let $\boxplus^p$ be the p–cube I^p together with the p hyperplanes $x_i = \frac{1}{2}, i = 1, \ldots, p$ and let T^p denote the union of the p hyperplanes. Then $\boxplus^p \times I^{n-p}$ consists of an n–cube with p central hyperplanes meeting in an $n-p$ dimensional subspace called the **core**. An immersed smooth manifold Q of dimension $n-1$ in a manifold M of dimension n is called **self transverse** if each point x of Q has a neighbourhood in M like $\boxplus^p \times I^{n-p}$ in which the point x corresponds to an interior point of the core and in which $T^p \times I^{n-p}$ corresponds to the image of Q. The integer p is called the **index** of x and the set of points of index p is called the p–**stratum**. Points of index one are locally embedded and the closures of components of index one points will be called the **sheets** of the immersed manifold Q. Sheets can be locally continued through points of higher index. At a point of index p there are locally p such extended sheets which meet in a manifold of codimension p.

Remark Any immersed manifold can be regularly homotoped so that it is self transverse (see Lashof and Smale [LS]).

Definition *Higher dimensional diagrams* An (unlabelled) **diagram** is a framed immersed self transverse manifold such that the sheets are locally totally ordered and this ordering is preserved in a neighbourhood of points of higher index.

A diagram in M lifts to a framed embedding in $M \times \mathbb{R}$ with the same fundamental rack.

A diagram is **labelled by a $\square$–set** C if for each p each component of the p–stratum is labelled by a p–cube of C with compatibility conditions. These are similar to the conditions for a diagram in a 3–manifold. If a component of the p–stratum is labelled by $c \in C_p$ then the neighbouring components of the $(p-1)$–stratum are labelled by $\partial_i^\epsilon(c)$ with the rule for determing i and ϵ being analogous to the 3–dimensional case given earlier. A labelling by BX where X is a rack is equivalent to a homomorphism of the fundamental rack in X and the labelling is determined by just the labels on the sheets with the usual compatibility condition at 2–strata (think of a perpendicular 2–dimensional slice).

(Labelling by a trunk is also analogous to the 3–dimensional case.)

A **transverse map** of an n–manifold into a $\square$–set C determines a diagram in M labelled by C by pulling back the first James bundle and there is a bijection between equivalence classes of transverse maps and labelled diagrams.

Let C be a $\square$–set with a basepoint $* \in C_0$. The abelain group $\mathcal{D}(n, C)$ of cobordism classes of diagrams in $\mathbb{R}^n$ labelled by C (such that the non-compact region is labelled by $*$) is defined in analogy with the 2–dimensional case and we again abreviate $\mathcal{D}(n, BX)$ to $\mathcal{D}(n, X)$ if X is a rack.

The abelian group $\mathcal{F}(n+1,X)$ of framed cobordism classes of framed codimension 2 links in $\mathbb{R}^{n+1}$ with representation in X is defined in analogy with the 3–dimensional case.

4.13 Theorem Classification of higher dimensional labelled diagrams

Let C be a $\square$–set with a basepoint $* \in C_0$. There is a natural isomorphism between $\pi_n(C)$ and $\mathcal{D}(n,C)$. The isomorphism can be described as given by pulling back the first James bundle $\xi^1(C)$ which thus plays the rôle of classifying bundle for labelled diagrams. □

For the classification of high-dimensional links we shall not analyse the diagrams and cobordisms by using classical moves (as in the dimension 3 case above); instead we use a new result. Let M be a framed embedding in $Q \times \mathbb{R}$. We call M **horizontal** if the last framing vector is always vertically up. Note that a horizontal embedding covers an immersion in Q.

4.14 Compression Theorem *Let M be a framed embedding in $Q \times \mathbb{R}$, then M is isotopic (by a small isotopy) to a horizontal embedding; moreover if M already horizontal in the neighbourhood of some compact set, then the isotopy can be assumed fixed on that compact set.*

Proof See [RS3]. □

4.15 Theorem Classification of higher dimensional links

Let X be a rack. There is a natural isomorphism between $\pi_n(BX)$ and $\mathcal{F}(n+1,X)$. The embedded first James bundle plays the rôle of classifying bundle.

Proof The compression theorem together with self-tranversality [LS] implies that framed cobordism (with representation in X) is equivalent to diagram cobordism (with label in X) and the result follows from the previous theorem. □

5 The algebraic topology of rack spaces

In this section we turn to invariants of rack spaces. These are important because any invariant of rack spaces automatically becomes a knot or link invariant by calculating it for the rack space of the fundamental rack of the knot or link. Moreover, further invariants can be derived by considering representations of the fundamental rack in other racks, for example, finite racks.

All the invariants considered in this section are homotopy type invariants. There is good reason to believe that the homotopy type of the rack space is not a complete link invariant for links in S^3 even when the rack itself is (see the remarks on theorem 5.4 below). In this context it is worth reiterating that the combinatorial structure of the rack space is equivalent to the rack itself and therefore it is valuable to construct combinatorial invariants of a $\square$–set which are not homotopy type invariants. The main new invariants introduced in this paper (the James–Hopf invariants), and also the associated generalised cohomology theories considered in the next section, are combinatorial invariants of this type.

Here we start by identifying the fundamental group of rack spaces and proving that they are simple. We then turn to calculations of homotopy groups. We calculate all the homotopy groups of the rack space of an irreducible link in a 3–manifold and the second homotopy group of the rack space of any link in S^3. We determine the homotopy type of the rack space of an irreducible link in an irreducible 3–manifold with infinite fundamental group. We also determine the homotopy type of the rack space of a free rack and of the trivial rack with n elements. (Note that Wiest [Wi] has also determined the homotopy type of the rack space of an irreducible link in a general 3–manifold.)

We conclude with some results on homology groups and a review of results of Flower [Fl] and Greene [G].

Fundamental group

The fundamental groupoid of a $\square$-set is discussed in [FRS]. We repeat the computation of the fundamental group of the rack space.

Recall from [FR] that the **associated group** $As(X)$ to a rack X is the group generated by the elements of X subject to the relations $a^b = b^{-1}ab$.

5.1 Proposition *The fundamental group $\pi_1(BX)$ of the rack space of a rack X is isomorphic to the associated group $As(X)$ of X.*

Proof Recall that the rack space BX has a single vertex and edges in bijection with the elements of X which therefore generate $\pi_1(BX)$. Moreover the relations given by the squares of BX are $a^b = b^{-1}ab$ for $a, b \in X$. The result now follows from the definition of $As(X)$. $\square$

Simplicity of the rack space

We next prove that BX is a simple space for any rack X.

5.2 Proposition *Let X be any rack then the action of the fundamental group $\pi_1(BX)$ on $\pi_n(BX)$ is trivial.*

Proof Let $\beta \in \pi_n(BX)$, and let $\alpha \in \pi_1(BX)$ be a generator corresponding to $x \in X$ as above. We may represent β by a labelled diagram D in $\mathbb{R}^n$. Then $\alpha\cdot\beta$ is represented by the diagram which comprises a framed standard sphere in $\mathbb{R}^n$ labelled by x and containing D in its interior. But the following diagram cobordism shows that the two diagrams are equivalent. Pull the sphere under D to one side (without changing any labels on D) and then eliminate it. $\square$

Notation Let X be a rack. Recall from section 2 that $\mathcal{D}(n,X)$, $\mathcal{F}(n+1,X)$ denote the group of cobordism classes of diagrams in $\mathbb{R}^n$ labelled by X and the group of framed cobordism classes of framed codimension 2 links in $\mathbb{R}^{n+1}$ with representation in X, respectively.

We shall use the notation $[D,\lambda]$ for an element of $\mathcal{D}(n,X)$, where D denotes a diagram and λ a labelling and we shall use the notation $[L,F,\rho]$ for an element of $\mathcal{F}(n+1,X)$ where L is a codimension 2 link (an $(n-1)$–manifold) in $\mathbb{R}^{n+1}$, F is a framing of L and ρ a representation in X, ie a homomorphism of the fundamental rack of (L,F) to X.

If $x \in X$ is an element of X, then we denote by λ^x the labelling obtained from λ by operating on all labels by x. That this is also a labelling follows from the rack law. Similarly we denote by ρ^x the representation obtained by composing ρ with the automorphism $a \mapsto a^x$ of X.

5.3 Corollary *Let X be a rack. Let $[D,\lambda] \in \mathcal{D}(n,X)$ and let $[L,F,\rho] \in \mathcal{F}(n+1,X)$ and $x \in X$. Then $[D,\lambda]=[D,\lambda^x]$, and $[L,F,\rho]=[L,F,\rho^x]$.*

Proof To see that $[D,\lambda]=[D,\lambda^x]$ simply pull the sphere over instead of under in the proof of 5.2. The other result now follows from the isomorphism of the two groups (4.15). $\square$

Remark Notice that in the case that X is the fundamental rack of a link L in S^n then the action of x is given by $a^x = a \cdot \partial x$ where ∂ is the augmentation to $\pi_1(L) := \pi_1(S^3-L)$, hence the corollary also implies invariance of $\mathcal{D}(n,X)$ and $\mathcal{F}(n+1,X)$ under the action of $\pi_1(L)$ in this case.

Homotopy groups of the rack space of an irreducible link

Let L be a framed link in a 3–manifold M^3. We say that L is **irreducible** if each embedded 2–sphere in M^3-L bounds a 3–ball (ie $M-L$ is an irreducible 3–manifold). We say that L is **trivial** if $M=S^3$ and L is equivalent to the unknot in S^3 with zero framing.

Let Λ be a framed submanifold of $\mathbb{R}^{n+1}$ with framing $F\colon \Lambda \times D^2 \to \mathbb{R}^{n+1}$. Write $N(\Lambda)=\mathrm{im}(F)$ for the tubular neighbourhood of Λ and $\Lambda^c = \overline{\mathbb{R}^{n+1}-N(\Lambda)}$ for the closure of the complement. Also write $\Lambda^+ = F(\Lambda \times \{(1,0)\})$ for the **parallel manifold** to Λ.

5.4 Theorem *Let M^3 be a 3–manifold and let L be a framed non-trivial irreducible link in M^3. Let $\Gamma(L)$ denote the fundamental rack of L. Then for $n>1$,*

$$\pi_n(B\Gamma(L)) \cong \pi_{n+1}(M^3).$$

Proof Recall from theorem 4.15 that $\pi_n(B\Gamma(L)) \cong \mathcal{F}(n+1,\Gamma(L))$.

Define the homomorphism

$$\phi\colon \pi_{n+1}(M^3) \to \mathcal{F}(n+1, \Gamma(L))$$

as follows. Let $\alpha \in \pi_{n+1}(M^3)$ be represented by a map $f\colon \mathbb{R}^{n+1} \to M^3$ which is constant outside a compact set. Homotope f to be transverse to L. Then $\Lambda = f^{-1}L$ is a framed submanifold with framing F such that f is compatible with the framings. Let $\rho\colon \Gamma(\Lambda) \to \Gamma(L)$ be the induced homomorphism of racks. We define $\phi(\alpha) = [\Lambda, F, \rho]$. If f and g are both transverse representatives of α then a homotopy between f and g can also be made transverse producing a bordism and we see that ϕ is well defined.

ϕ is surjective

For each component Λ_i^+ of Λ^+ let L_i^+ be the corresponding component of L which is in the image of Λ_i^+. Notice that the L_i^+'s may not be distinct. For each i choose an embedded path p_i from Λ_i^+ to the base point representing an element of $\Gamma(\Lambda)$ so that the chosen paths only meet at the base point. Similarly choose a path q_i for L_i^+, where $\rho[p_i] = [q_i]$. Corresponding to our choices we have longitudinal and meridinal subgroups $\Lambda_i(l), \Lambda_i(m)$ of $\pi_1(\Lambda^c)$ and subgroups $L_i(l), L_i(m)$ of $\pi_1(L^c)$. Now $As(\Gamma(\Lambda)) \cong \pi_1(\Lambda^c)$ and $As(\Gamma(L)) \cong \pi_2(M, L^c)$. Then $\rho\colon \Gamma(\Lambda) \to \Gamma(L)$ induces a homomorphism $\rho_1\colon \pi_1(\Lambda^c) \to \pi_1(L^c)$ by composing with the boundary map in the homotopy exact sequence of the pair (M, L^c). See [FR; page 360].* The homomorphism has restrictions $\rho_i(l)\colon \Lambda_i(l) \to L_i(l)$ and $\rho_i(m)\colon \Lambda_i(m) \to L_i(m)$.

We claim that the hypotheses on L imply that the group $L_i(l)$ is infinite cyclic. To see this, suppose not. Then some multiple of the longitude is null homotopic in L^c. By the loop theorem there is a closed essential curve in the neighbourhood of the longitude L_i^+ which bounds a disc in L^c. But this neighbourhood is an annulus and the only possibility is that L_i^+ itself bounds a disc. By irreducibility, we then see that $L = L_i$ and is trivial, contradicting our hypotheses.

The space L_i^+ together with its embedded 'tail' q_i is an Eilenberg-MacLane space so there is a unique map, up to homotopy, $f_0\colon \Lambda_i^+ \cup p_i(I) \to L_i^+ \cup q_i(I)$ inducing the homomorphism $\rho_i(l)$. Further we can assume, for later convenience, $f_0\Lambda_i^+ \subset L_i^+$ and $f_0 p_i = q_i$. Since ρ_1 is induced by sending meridians to meridians there is a unique extension $f_1\colon N(\Lambda) \to N(L)$ which preserves framing. We can finally extend f over Λ^c since L^c is also an Eilenberg-MacLane space.

ϕ is injective

First we observe that the map f constructed above is unique up to based homotopy. To see this suppose alternative choices p_i' and q_i' are made in place of p_i and q_i respectively and assume end points agree. Consider $f_0'\colon \Lambda_i^+ \cup p_i'(I) \to L_i^+ \cup q_i'(L)$. We can assume the set of p_i's together with the p_i''s do not meet on interiors. Now suppose $f_0 \circ (\overline{p_i} \cdot r \cdot p_i) \simeq \overline{q_i} \cdot l^{n_i} \cdot q_i$ where r is some loop in Λ_i^+ and l is 'once round' L_i^+. Notice $p_i' \simeq p_i \cdot (\overline{p_i} \cdot p_i')$.

* It has been pointed out by Wiest that the proof of the result used here [FR; proposition 3.2] contains a misleading statement. To be precise the statement made at the top of page 361 in [FR] is open to misinterpretation. The paths can also be varied by an isotopy which moves one little disc around another – essentially a pure braid automorphism. This can be realised by two interchanges and is implicit in the subsequent text.

Then from the fact that a homomorphism of racks preserves the action of the associated groups we have, after an easy calculation, $\rho_1[\overline{p'}_i \cdot r \cdot p'_i] = [\overline{q'}_i \cdot l^{n_i} \cdot q'_i]$. It follows that f_0 and f'_0 can be taken to agree on Λ_i^+ and f_1 and f'_1 agree on $N(\Lambda)$.

Now in completing the constructing of f consider the construction over the 1-skeleton. If we have $fp_i = q_i$ then essentially the same argument shows we can assume $fp'_i = q'_i$. We are now ready to prove ϕ is injective. Suppose (W, F, ρ) is a bordism between (W_0, F_0, ρ_0) and (W_1, F_1, ρ_1) and the latter are associated with maps g_0 and g_1 respectively. Now repeat the proof that ϕ is onto but this time using $S^{n+1} \times I$ and W and ρ in place of S^{n+1} and Λ and ρ. The resulting bordism then must give g_0 and g_1 on the two ends by the above observations. □

Remarks on and consequences of theorem 5.4

It is worth pointing out that irreducibility of the *link* L in M does not imply irreducibility of the *3–manifold* M. Indeed any connected 3–manifold contains an irreducible link (see [FR; page 380 remark (2)]). The higher homotopy groups of a general (non-irreducible) 3–manifold can be very complicated. Each separating 2–sphere potentially determines a copy of $\pi_*(S^2)$ and these are likely to interact in complicated ways (see in particular lemma 5.9 below for some information on π_2). To our knowledge, the subject of the higher homotopy groups of 3–manifolds is largely untouched.

The theorem therefore implies that the higher homotopy groups of rack spaces of irreducible links can in general be complicated. In the case that the 3–manifold *is* irreducible its higher homotopy groups are either all zero (the case when the fundamental group of M is infinite) or coincide with the homotopy groups of the 3–sphere. Turning first to infinite fundamental group case, we deduce :

5.5 **Corollary** *Let L be a framed non-trivial irreducible link in an irreducible 3–manifold M with infinite fundamental group and let $\Gamma(L)$ denote the fundamental rack of L. Then $B\Gamma(L)$ is a $K(\pi, 1)$ where π is the kernel of $\pi_1(M - L) \to \pi_1(M)$.*

Proof By the theorem and the remarks above, $B\Gamma(L)$ is a $K(\pi, 1)$ where π is the associated group of $\Gamma(L)$ by proposition 5.1. But since $\pi_2(M) = 0$, [FR; proposition 3.2] implies that the associated group is the kernel of $\pi_1(M - L) \to \pi_1(M)$. □

In the case when M is irreducible and has finite fundamental group (eg when $M = S^3$), then (as remarked earlier) the theorem implies that the higher homotopy groups of $B\Gamma(L)$ coincide with the higher homotopy groups of S^3 (with an index shift of 1). Thus the theorem gives a plentiful supply of new geometric interpretations for these groups. We now describe one such interpretation which does not need the concept of fundamental rack for its statement.

We need to define the writhe of a framed link in a higher dimensional sphere. Let M^{n-1} be a framed submanifold of S^{n+1}. Let M^+ denote the parallel manifold to M as usual and let $\phi\colon M^c \to S^1$ be the map defined by any Seifert bounding manifold for the codimension 2 submanifold M. Then the composition $M \to M^+ \xrightarrow{\phi|M^+} S^1$ defines an element of $H^1(M)$ called the **writhe**. There is a similar notion for the writhe of a cobordism and a cobordism which preserves writhe.

5.6 Corollary *Fix an integer $w > 0$ then $\pi_{n+1}(S^3)$ is isomorphic to the set of equivalence classes of framed $(n-1)$–manifolds embedded in S^{n+1} with writhe divisible by w, under framed cobordism also with writhe divisible by w.*

Proof Let U_w denote the unknot in S^3 with framing w. Then U_w is irreducible, non-trivial and the theorem applies. But writhe divisible by w is the same as having a representation in the cyclic rack of order w which is $\Gamma(U_w)$. □

Since we now know that $\pi_{n+1}(S^3)$ is isomorphic to the set of equivalence classes of framed $(n-1)$–manifolds embedded in S^{n+1} with writhe divisible by w we can consider the forgetful map which ignores the writhe condition on framings.

5.7 Corollary *The forgetful map is multiplication by w.*

Proof This follows from observing that the map from $\pi_3(S^3)$ to $\pi_3(S^2)$ which is given by applying the Thom–Ponrjagin construction to U_w is w times the Hopf map. But the forgetful map is effectively composition with this map. □

By contrast in the case $w = 0$, the cobordism groups are all zero. This is essentially what theorem 5.13 below says.

Some interesting questions about the homotopy type of rack spaces now arise. Consider the reef knot R (square knot in American English) and granny knot G in S^3 (both taken with framing zero for definiteness). They have the same fundamental group but different fundamental racks (the rack is a classifying invariant in this case). By the theorem all the homotopy groups of $B\Gamma(R)$ and $B\Gamma(G)$ coincide.

Question Do $B\Gamma(R)$ and $B\Gamma(G)$ have the same homotopy type? More generally are there examples of different irreducible links in S^3 with homotopy equivalent rack spaces?

Wiest [Wi] has given an example of two different irreducible links, in an irreducible 3–manifold with infinite fundamental group, which have the same associated group and hence (by corollary 5.5) have homotopy equivalent fundamental rack spaces. Moreover these links have different plain fundamental racks. In this case, the plain rack is not a complete invariant (the augmented rack is needed). Nevertheless this suggests that the homotopy type of the rack space does not classify irreducible links in the case when the rack does and motivates the study of combinatorial invariants of rack spaces.

Wiest [Wi] has given an independent proof of the theorem without using the compression theorem 4.14. His proof shows in addition that the augmented rack space $B_G X$ (where G is $\pi_1(M - L)$) has the same homotopy type as $\Omega(M^3)$. This implies that BX has the homotopy type of of $\Omega(M^3)$ factored by an action of G.

The second homotopy group for links in S^3

Now let L be a framed link in S^3. We say that L is **non-split** or **irreducible** if no embedded 2-sphere in $S^3 - L$ divides the components of L into two non-empty subsets. (This is consistent with the usage of irreducible for links in M^3 given earlier.)

In general a link can be written as a union $L = L_1 \cup \ldots \cup L_k$ where each L_i is a maximal irreducible sublink. We call the sublinks L_i the **blocks** of L. A block is said to be **trivial** if it is equivalent to the unknot with zero framing.

5.8 Theorem *Let L be a framed link in S^3. Then $\pi_2(B(\Gamma(L))) \cong \mathbb{Z}^p$ where p is the number of non-trivial blocks of L. Furthermore a basis of $\pi_2(B(\Gamma(L))$ is given by diagrams representing these blocks.*

Paraphrased, the theorem says that any link in S^3 with representation in $\Gamma(L)$ is cobordant respecting the representation to a unique *standard* link comprising a number of separate copies of the blocks of L.

Proof Case 1: L is irreducible and non-trivial

This is a special case of theorem 5.4.

Case 2: L is irreducible and trivial

In this case the homomorphism $\phi\colon \pi_3(S^3) \to \mathcal{F}(n+1, \Gamma(L))$ defined in case 1 is surjective by exactly the same argument. But $\phi[\mathrm{id}]$ is represented by the trivial link with identity representation which is null cobordant.

This completes case 2 and we now turn to the general case $L = L_1 \cup \ldots \cup L_k$ where each L_i is maximal irreducible. We need the following lemmas.

5.9 Lemma *Let M be an irreducible 3-manifold. Let M_0 be M with the interior of k balls $B_1, \ldots, B_k$ removed. Then $\pi_2(M_0)$ is generated as a $\mathbb{Z}[\pi_1(M_0)]$ module by the spheres ∂B_i.*

Proof Let $\widetilde{M}$ and $\widetilde{M_0}$ be the universal covers of M and M_0 respectively. Then $\widetilde{M}$ can be obtained from $\widetilde{M_0}$ by filling in holes with copies gB_i of B_i one for each element g in $\pi_1(M_0)$, $i = 1 \ldots n$. Since M is irreducible $\pi_2(M) \cong H_2(\widetilde{M}) \cong 0$. Then using a Mayer–Vietoris sequence we see that as an abelian group $\pi_2(M_0) \cong H_2(\widetilde{M_0})$ has one generator for each pair (g, B_i) where $g \in \pi_1(M_0)$. □

5.10 Lemma *Let M be the connected sum $M = M_1 \sharp \ldots \sharp M_k$ of k irreducible 3-manifolds each with non trivial fundamental group. Then as a $\mathbb{Z}[\pi_1(M)]$ module $\pi_2(M)$ is generated by the separating spheres $S_1, \ldots, S_{k-1}$.*

Proof Let an element of $\pi_2(M)$ be represented by a map $f\colon S^2 \to M$ of the 2-sphere into M which we may assume is transverse to the separating spheres. Consider an innermost disc D in S^2 which has boundary in the intersection of $f(S^2)$ and S_i say. Let D' be a (singular) disc in S_i which bounds ∂D. Then by the previous lemma the homotopy class of the sphere $D \cup D'$ is in the subgroup generated as a $\pi_1(M)$ module by the separating spheres $S_1, \ldots, S_{k-1}$. By subtracting this element, we may perform a homotopy to remove this intersection curve. We can now argue by induction on the number of intersections. □

Returning now to the proof of the main theorem we look at π_3 and attempt to construct as before a map $f: \mathbb{R}^3 \to S^3$. This time the complement of L may not be an Eilenberg-MacLane space. The construction of the mapping on the tubular neighbourhood of Λ runs as before and there is no obstruction to extending to the 2-skeleton of the complement but obstructions to mapping in the 3-cells may be non zero. By the above lemma it will be sufficient to consider the case of a single 3-cell, which may be taken to be far away from Λ and where the map on the bounding 2-sphere is as follows. The map is constant on the equator and the upper hemi-disc is wrapped around a separating sphere with degree ± 1. The separating sphere contains just one component L_i of L. Maps on great arcs running from the south pole to the equator give the same path in S^3. We can now map the lower hemi 3-ball to the image of this path so that the map is constant on the equatorial 2-disc. At this point, by considering the upper hemisphere together with the equatorial 2-disc, we see that the problem is reduced to the case where the bounding 2-sphere is mapped to the separating 2-sphere. But now if we add to Λ a copy of L_i in the 3-ball the map extends in the obvious way. Notice that the representation on the copy of $\Gamma(L_i)$ which this determines is conjugated by a fixed element of $\pi_1(L^c)$, Call such a link *conjugated.* This means that the map f can be defined for the extended link. Indeed we may further extend the link by adding copies of L so that f has degree zero. If we now make the resulting null homotopy transverse to L we construct a bordism between $[\Lambda, F, \rho]$ and and a number of disjoint copies of conjugated links L_i. By proposition 5.2 and the result in case 1 we can assume that these added links are labelled by identities and not conjugated. Thus the L_i with identity labelling form a generating set for π_2. Note that any trivial sublinks can be eliminated by a bordism as in case 2.

Now suppose some non-trivial linear combination of the $[L_i, F_i, id_i]$ is bordant to zero. We need the following lemma.

5.11 **Lemma** *Let X be the free product of racks X_i and let Y_j be the union of the orbits in X determined by X_j. Then there is a retraction of Y_j onto X_j.*

Proof This follows from the definition of the free product of racks [FR; page 357]. The retraction is defined by setting the action of other orbits equal to the trivial action. □

The rack homomorphism commutes with the action of the associated groups and components correspond to orbits of the action. Thus the null bordism cannot mix components. Now observe that $\Gamma(L)$ is the free product of the $\Gamma(L_i)$ and therefore there is a retraction of the orbit determined by L_i onto $\Gamma(L_i)$ by the lemma. Applying this retraction to the appropriate pieces of the null bordism we see that $[L_i, F_i, id_i] = 0$. This contradicts case 1.

It follows that the $[L_i, F_i, id_i]$ form a basis for π_2 and theorem 5.9 is proved. □

Homotopy type of the space of a trivial rack

Let $X(n) = \{1, \ldots, n\}$ be the trivial rack with n elements; so that $x^y = x$ for all x and y.

5.12 **Theorem** *The classifying space $BX(n)$ has the homotopy type of $\Omega(S^2 \vee \cdots \vee S^2)$, the loop space on the wedge of n copies of S^2.*

Proof First observe the simple form of the faces in $BX(n)$;

$$\partial_i^\epsilon(x_1,\ldots,x_n)=(x_1,\ldots,\hat{x}_i,\ldots,x_n)\quad\text{for}\quad 1\le i\le n,\quad \epsilon\in\{0,1\}$$

Now $\Omega(S^2\vee\ldots\vee S^2)\simeq\Omega S(S^1\vee\ldots\vee S^1)$, but by the James construction $\Omega S(S^1\vee\ldots\vee S^1)$ has the homotopy type of $(S^1\vee\ldots\vee S^1)_\infty$ the free monoid on $(S^1\vee\ldots\vee S^1)\backslash\{*\}$. Identify S^1 with $I/\partial I$ and let (k,t) denote the point t in the k-th copy of S^1 in $S^1\vee\ldots\vee S^1$. Then there is a homeomorphism $BX(n)\to(S^1\vee\ldots\vee S^1)_\infty$ given by

$$[(i_1,\ldots,i_n),(t_1,\ldots,t_n)]\to(i_1,t_1)\cdot\ldots\cdot(i_n,t_n).$$

It follows that $\pi_k(B(X(n)))$ can be viewed geometrically as bordism classes of framed codimension two manifolds in S^{k+1} with components labelled in $\{1,\ldots,n\}$. □

Remark For $k=2$, by the Hilton-Milnor theorem [H], $\pi_2(BX(n))\cong\mathbb{Z}^{n+\binom{n}{2}}$ where the second set of generators are Whitehead products. Geometrically, $\pi_2(BX(n))$ is interpreted as cobordism classes of links with components labelled by n distinct labels, or equivalently as a link divided into n disjoint sublinks, and the first n integer invariants are total writhes of the sublinks and the remaining $\binom{n}{2}$ are the mutual linking numbers. For more details, see [S2].

Homotopy type of the space of a free rack

5.13 Theorem *Let FR_n denote the free rack on n elements. Then BFR_n has the homotopy type of $S^1\vee\cdots\vee S^1$, the wedge of n copies of S^1.*

Proof Recall from theorem 4.15 that $\pi_n(BFR_n)\cong\mathcal{F}(n+1,FR_n)$.

We observe that FR_n is the fundamental rack of D_n, which is the link comprising n framed points in the 2–disc D^2. The proof of theorem 5.4 shows that if (Λ,FR_n,ρ) represents an element of $\pi_n(BFR_n)$ (for $n\ge2$) then (Λ,F,ρ) pulls back from a transverse map of $\mathbb{R}^{n+1}$ to D_n. But since D^2 is contractible this map is null homotopic and applying relative transversality we obtain a null cobordism of (Λ,FR_n,ρ). By 5.1 $\pi_1(BFR_n)\cong *_n\mathbb{Z}$ (the free product of n copies of $\mathbb{Z}$). The result follows. □

Remark Recall from [FR; page 376] that there is the concept of an **extended free rack**, which has free operator generators in addition to the usual free rack generators. This can be identified with the fundamental rack of a number of framed points in an orientable surface. A similar proof (using the fact that the higher homotopy groups of a surface vanish) then shows that if F is an extended free rack then BF has the homotopy type of a wedge of circles. Moreover both proofs extend with a little care to arbitrary free (or extended free) racks (in other words there is no need for the generating sets to be finite). Thus BF has the homotopy type of a 1–complex where F is any free (or extended free) rack.

A remark on homology groups

Let X be any rack. Recall that there is the concept of the **extended** rack space $B_X X$ [FRS; example 3.1.1] which is a covering space of BX [FRS; theorem 3.7]. Now there is a chain equivalence between $C^*(BX,*)$ and $C^{*-1}(B_X X)$ where $* \in BX_0$ is the unique vertex. There are corresponding isomorphisms of homology and cohomology groups (with a shift of one dimension). This is defined by mapping $(x, x_1, \dots, x_n) \in B_X X^{(n-1)}$ to $(x, x_1, \dots, x_n) \in BX^{(n)}$ using the description given in [FRS; examples 3.4.1 and 2]. Note that $\partial_{i-1}^{\epsilon}$ in $B_X X$ coincides with ∂_i^{ϵ} in BX for $i = 2, 3, \dots, n$ whilst $\partial_1^1 = \partial_1^0$ in BX (both are given by $(x_1, x_2, \dots, x_n) \mapsto (x_2, \dots, x_n)$) and that these two cancel out as a pair in the boundary formula.

Thus we have :

5.14 **Theorem** *The rack space BX of any rack admits a covering space with the same homology groups but in dimensions all shifted one lower.* □

The chain equivalence can be realised using a map $|B_X X| \times S^1 \to |BX|$ defined as follows. Embed the $(n-1)$–cube $(x, x_1, \dots, x_n)$ of $B_X X$ as the central $(n-1)$–cube perpendicular to the first direction in the n–cube $(x, x_1, \dots, x_n)$ of BX. Then use the remaining coordinate to extend to $B_X X \times [-1, 1]$. Since $\partial_1^1 = \partial_1^0$ in BX this map factors via $B_X X \times [-1, 1]/-1 \sim 1$ that is $B_X X \times S^1$. Then the above chain equivalence is given by crossing with the fundamental class of S^1 and using this map.

It is an interesting question to characterise spaces which have the property of admitting a covering space with the same homology groups shifted one dimension.

Permutation and Dihedral racks

Let $\rho: P \to P$ be a fixed permutation of the set P. Then the **permutation rack** P_ρ is P with $i^j = \rho(i)$ for all i, j.

Combining results of Flower [Fl] and Greene [G], which were proved using the cobordism by moves technique of section 4, we have:

5.15 **Theorem** *For a permutation rack P_ρ*

(1) *$\pi_2(BP_\rho)$ is freely generated by one twisted unknot for each finite orbit, with the number of twists equal to the length of the orbit, together with a pair of linked unknots (each unknot having a single twist) for each unordered pair of unequal orbits.*

(2) *$H_2(BP\rho)$ is as in (1) except that there is a generator for each ordered pair of unequal orbits.*

□

Let D_n denote the **dihedral rack** on n elements: $D_n = \{0, 1, \dots, n-1\}$, and $i^j = 2j - i \bmod n$ for all i, j.

5.16 Theorem [G]

$$H_2(BD_n) = \begin{cases} \mathbb{Z} & \textit{for } n \textit{ odd} \\ H_2(BD_n) = \mathbb{Z}^4 & \textit{for } n = 2 \, mod \, 4 \\ H_2(BD_n) \geq \mathbb{Z}^4 & \textit{otherwise} \end{cases}$$

□

Remark A calculation using maple shows $H_2(BD_4) = \mathbb{Z}^4 + {\mathbb{Z}_2}^2$, so the inequality of the theorem can be strict. For further details see [Fl,G].

6 Homology Theories from □-sets

In this section we describe how a □–set may be used to construct homology and cohomology theories. More precisely we shall define a spectrum MC for any □–set C which then defines a generalised homology theory MC_*. This theory has a good description as a bordism theory in terms of manifolds and labelled diagrams. The description is particularly nice when C is the classifying space of a rack. These theories have natural operations, which as we shall see are analogous to $\mathcal{X}Sq$ operations in ordinary cohomology.

In the case $C = T$, the trivial □–set, the spectrum MT is a ring spectrum and is one of a family studied by Mahowald [M].

The spectrum MC

We recall from section 3 that there is a map $s\colon |T| \to BO$ (where T is the trivial □–set with just one cell in each dimension and BO is the classifying space for stable vector bundles) and this defines a map $\phi_C = s \circ t_C\colon |C| \to BO$ for any □–set C, where $t_c\colon C \to T$ is the canonical □–map. The map ϕ_C has the property that it pulls both the universal Stiefel–Whitney and Wu classes back to the mod 2 reductions of the characterstic classes of C, see 3.11.

Let γ be the universal (virtual) bundle over BO and γ_C (the characteristic bundle) its pull-back over C by ϕ_C. The spectrum MC is defined to be the associated Thom spectrum to γ_C. It defines a generalised homology theory, which we call $C-$**bordism**, and denote MC_* and a corresponding theory of $C-$**cobordism** denoted MC^*.

Connection with standard spectra

We next see how the new spectrum fits with the sphere spectrum (which classifies stable cohomotopy) and the Eilenberg-MacLane spectrum which classifies ordinary (mod 2) cohomology.

The universal Thom class determines a map of spectra $MO \to H\mathbb{Z}_2$, where $H\mathbb{Z}_2$ is the mod 2 Eilenberg-MacLane spectrum. Hence we get a natural map $\theta_C\colon MC \to H\mathbb{Z}_2$. Let $\mathbb{S}$ denote the sphere spectrum, and let $\mathbb{S}\mathbb{Z}_2$ denote the sphere spectrum with mod 2 coefficients.

6.1 Proposition *For any □–set C:*

(1) The Hurewicz map factors as $\mathbb{S} \to MC \to H\mathbb{Z}_2$,

(2) If there is an element $x \in C_1$ with $\partial_1^0 x = \partial_1^1 x$ then the mod 2 Hurewicz map factors as $\mathbb{S}\mathbb{Z}_2 \to MC \to H\mathbb{Z}_2$.

Note The condition in part (2) is always satisfied for rack spaces.

Proof The sphere spectrum $\mathbb{S}$ may be regarded as the Thom spectrum pulled back from BO by the inclusion of a point. The first result follows from naturality of θ by choosing a vertex in $|C|$.

For the second part let D be the □–set which is the circle with the obvious □–set structure, ie D has a single vertex, a single edge and no higher cells. From the property that ϕ_D pulls back the Stiefel–Whitney class to the mod 2 characteristic class of D, it can be seen that $\phi_D\colon |D| \to BO$ classifies the (stable) Möbius band over the circle $|D|$. The Thom complex of the Möbius band is the projective plane and MD is its suspension spectrum, ie $\mathbb{S}\mathbb{Z}_2$. The transformation $\theta_D\colon MD \to H\mathbb{Z}_2$ can now be identified with the inclusion of the (stable) 1–skeleton ie the mod 2 Hurewicz map. The result now follows from naturality of θ by considering the inclusion of D in C given by mapping the edge of D to x. □

Geometric description of the cycles of MC

The generalised homology theory defined by MC is the bordism theory defined by a certain class of manifolds. We shall describe this class.

We need to define what it means for a manifold to have its normal bundle **twisted** by a diagram. For simplicity we describe the case of a diagram with no multiple points first. So let $Q \subset M$ be a codimension 1 submanifold equipped with a bicollar neighbourhood $N(Q)$. Denote by ρ/I the **Möbius bundle** which is the 1–dimensional vector bundle constructed by gluing the trivial bundles over $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ by multiplication by -1 in the fibre over $\frac{1}{2}$. Thus although ρ/I is the trivial bundle it is equipped with opposite trivialisations on the two halves of I. Denote by $\hat{\rho}/I$ the stable version. We call $\hat{\rho}/I$ the (stable) **Möbius twist**.

Now construct the stable bundle $\hat{\rho}/M$ by taking the trivial bundle over $Q^c = \overline{M - N(Q)}$ and gluing in $\hat{\rho}/I$ on each collar line. Alternatively we can take the trivial line bundle on $M - Q$ and glue at Q by multiplication by -1 and then stabilise. We say that the stable normal bundle ν/M of M is **twisted** by Q if it is identified with $\hat{\rho}/M$.

The concept of twisting by a diagram is similar except that at k–tuple points there are k mutually perpendicular Möbius twists taking place corresponding to the directions perpendicular to the k sheets.

For the detailed definition we shall contruct a stable Möbius twist bundle over I^n for each n. To do this we need a function $v\colon J^1(I^n) \to S^{n-1}$ with an orthogonality property. Denote $v|I^n_{(i)}$ by v_i which we think of as a variable unit vector in $\mathbb{R}^n$. We require that, for each point $p \in I^n$ covered by points $p_{i_1}, \dots p_{i_t}$ in $I^n_{(i_1)}, \dots I^n_{(i_t)}$, the corresponding vectors $v_{i_1}, \dots v_{i_t}$ should form an orthonormal t–frame. We also require that v commutes with face maps, ie the diagram commutes

$$\begin{array}{ccc} J^1(I^{n-1}) & \xrightarrow{v} & S^{n-2} \\ \downarrow {\scriptstyle J^1(\delta_k^\epsilon)} & & \downarrow {\scriptstyle i} \\ J^1(I^n) & \xrightarrow{v} & S^{n-1} \end{array}$$

where $i\colon S^{n-2} \to S^{n-1}$ is the usual inclusion.

We define v by induction on n and suppose that $v\colon J^1(I^{n-1}) \to S^{n-2}$ is already defined. Then the commuting diagram defines v on ∂I^n. We extend to the interior of I^n by using the cubical subdivision of I^n determined by the image of $J^1(I^n)$ in I^n, which cuts I^n into 2^n subcubes. Note that $n-i$ sheets of the image of J^1 meet at each i–cube in $\mathrm{int} I^n$ in this subdivision (see the diagram near the end of section 1). We define v by induction over skeleta. At the unique 0–cell in $\mathrm{int} I^n$ we choose $v_1, \dots, v_n$ to be any orthonormal n–frame. In general we have an orthonormal $(n-i)$–frame defined over the boundary of a typical i–cell and we extend over the i–cell using the fact that $\pi_{i-1}(V_{n,n-i}) = 0$.

This completes the definition of v. Notice that the choices made in the definition are homotopic if S^{n-1} is replaced by S^n and hence the stable twisted bundles constructed below do not depend on these choices.

We now construct the n–dimensional vector bundle ρ/I^n by taking the trivial bundle off the image of J^1 and gluing by mapping $v_i \mapsto -v_i$ as the i-th sheet (the image of $I^n_{(i)}$) is crossed and leaving the orthogonal complement of v_i fixed. Near a point $p \in I^n$ covered by points $p_{i_1}, \dots p_{i_t}$ in $I^n_{(i_1)}, \dots I^n_{(i_t)}$ the various trivial bundles are being glued by reversing the signs of the appropriate subset of $v_{i_1}, \dots v_{i_t}$. By orthogonality, these t simultaneous gluings are all independent. We denote the stable version of ρ/I^n by $\hat{\rho}^n$. This is the n–**fold Möbius twist** over I^n.

By construction, Möbius twists have the property that for each face map $\lambda\colon I^t \to I^p, \lambda^*(\hat{\rho}^p) = \hat{\rho}^t$. This implies that if X is a □–space then we can construct a bundle $\hat{\rho}/|X|$ by gluing together the bundles $X_n \times \hat{\rho}^n$ mimicking the definition of $|X|$. Note that $\hat{\rho}/|X|$ is trivialised over X_0 and at X_n there is an n–fold Möbius twist taking place.

Remark We can describe $\hat{\rho}$ as an explicit subbundle of the infinite trivial bundle in a similar way to the analogous construction in [S1]. To be precise we can embed $\hat{\rho}^n$ as a subbundle of $I^n \times \mathbb{R}^n \times \mathbb{R}^n$ as follows. Choose a small neighbourhood system for the image of $J^1(I^n)$ in I^n, regarded as usual as made of (trivial) cube bundles of varying dimensions with fibres which we shall call "small" cubes (to distingush them from the bigger cubes which comprise I^n and $J^1(I^n)$). Without loss assume that the function v is constant on the preimage of each small cube. Think of the neighbourhood system as covered by double collars on the $I^n_{(i)}$ where each collar line lies in a small cube parallel to an edge. Now over each such line (corresponding to $p_i \in I^n_{(i)}$ say) turn the vector v_i over in the plane determined by v_i and $\overline{v}_i$ (which is the copy of v_i in the other copy of $\mathbb{R}^n$) in other words make an explicit Möbius bundle in this plane over the line. Where vectors in two or more such bundles lie over the same point of I^n they are contained in perpendicular planes, so there is no interference. The construction of $\hat{\rho}/|X|$, where X is a □–space, now yields $\hat{\rho}$ as an explicit subbundle of the infinite trivial bundle.

Now suppose that D is a diagram in M and $N(D)$ is a neighbourhod system (as described in lemma 4.6). Recall that $N(D)$ gives a □–space structure to M and hence we have a stable bundle $\hat{\rho}/M$. We say that the stable normal bundle ν/M of M is **twisted by** D if it is identified with $\hat{\rho}/M$.

We can now define a C–**manifold**, where C is a □–set. This is a manifold M containing a diagram D which is labelled by C and such that the stable normal bundle of M is twisted by D. There is an obvious concept of C–**cobordism** between C–manifolds and then we can define the C–bordism groups of a space X by mapping C–manifolds and cobordisms into X in the usual way. There is a dual concept of C–cobordism given by considering mock bundles with fibres C–manifolds.

6.2 Theorem *The theory of C–bordism given by C–manifolds coincides with the theory MC_* defined by MC. Similary the two cobordism theories coincide.*

Before proving the theorem we shall recall another theory considered by Mahowald [M] for which a geometric description is given in [S1].

The Mahowald spectrum

The spectrum MA (another of the spectra defined by Mahowald [M]) is pulled back from MO by $\Omega^2\nu$ where $\nu\colon S^3 \to B^3O$ is a generator. Mahowald [M] and Priddy [Pr] have proved that the composition $MA \to MO \to H\mathbb{Z}_2$ is an equivalence.

6.3 Proposition *$MC \to MO$ factors as $MC \to MA \to MO$.*

We shall prove the proposition and the theorem together. We shall need to recall the geometric description of $MA_*(-)$ given in [S1]. We shall translate the language of [S1] into the language of this paper. Roughly speaking, “good position” as in [S1] corresponds to the neighbourhood systems used here.

Consider a framed self-transverse immersion of codimension 1 $Q \to M$. A neighbourhood system of Q in M can be constructed exactly as in 4.6. This does not give a □–space structure to M, because this is equivalent (as we saw in sections 2 and 4) to the immersion being a diagram (ie the sheets being ordered at multiple points). However it does define a somewhat more general structure : It decomposes M into I^n bundles for varying n, glued together along the analogue of faces. We can now define twisting by Q in a very similar way to the case of a diagram. The details can be obtained from [S1].

An A–**manifold** is a manifold M with a framed self-transverse immersion of codimension 1 $f\colon Q \to M$ covered by an embedding $\tilde{f}\colon Q \to M \times \mathbb{R}^2$ and such that ν_M is twisted by Q. The following result is proved in [S1].

6.4 Theorem *The bordism theory determined by A–manifolds is the theory MA_*. Similarly the cohomology theories coincide.* $\square$

Proofs of theorem 6.2 and proposition 6.3

An element of the homology group $MC_m(Y)$ is determined by a closed m–manifold M^m together with a map $f\colon M \to Y$ and an isomorphism $F\colon \nu_M \to \phi_C^*\gamma$ where ν_M is the stable normal bundle of M and γ is the universal bundle over BO. Now F restricts to a map $M \to |C|$ which may be assumed to be transverse by 2.7 and determines a diagram labelled by C by 4.13.

It remains to prove that the isomorphism $F\colon \nu_M \to \phi_C^*\gamma$ is equivalent to ν_M being twisted by the diagram. To see this we shall use the spectrum MA and the geometric description given above.

There is a well known configuration space model $C_2(S^1)$ for $\Omega^2 S^3$.

$$C_2(S^1) = (\coprod C_{2,k} \times_{\Sigma_k} I^k)/\sim$$

where $C_{2,k}$ denotes ordered subsets of $\mathbb{R}^2$ containing k points and

$$[x_1, \ldots, x_k, t_1, \ldots, t_k] \sim [x_1, \ldots, \hat{x}_i, \ldots, x_k, t_1, \ldots, \hat{t}_i, \ldots, t_k] \ \text{ if } \ t_i \in \{0, 1\}.$$

The space of unordered sets $C_{2,k}/\Sigma_k$ can be identified with the classifying space of the braid group Bbr_k. There is a filtration of $C_2(S^1)$:

$$* \subset F_1C_2(S^1) \subset \ldots \subset F_kC_2(S^1) \ldots C_2(S^1) \simeq \Omega^2 S^3.$$

where F_k is obtained from F_{k-1} by attaching the bundle of cubes $C_{2,k} \times_{\Sigma_k} I^k \to Bbr_k$ along its sphere bundle $C_{2,k} \times_{\Sigma_k} \partial(I^k) \to Bbr_k$.

There is a similar model for ΩS^2. Simply replace $C_{2,k}$ by $C_{1,k}$. There are maps $\phi_k\colon I^k \to C_{1,k} \times_{\Sigma_k} I^k$ given by $(t_1, \ldots, t_k) \mapsto [(0,0), \ldots, (k,0), t_1, \ldots, t_k]$. These maps combine to give a homotopy equivalence $|T| \to C_1(S^1)$. The composition

$$|T| \to C_1(S^1) \subset C_2(S^1) \simeq \Omega^2 S^3 \xrightarrow{\Omega^2\nu} BO$$

is our standard map $s\colon |T| \to BO$, and so we have

$$MC \to MA \to MO \to H\mathbb{Z}_2$$

and it follows that the map $M \to BO$ factors via MA.

The required result now follows from theorem 6.4 above after noting that the descriptions of twisting given above and in [S1] are effectively the same. $\square$

Remark The maps $\mathbb{S} \to MC$ and $MC \to MA \simeq H\mathbb{Z}_2$ can now be described geometrically. To get from $\mathbb{S}_*(-)$ to $MC_*(-)$ simply take $N = \emptyset$ and the label on M is the chosen vertex in C. To pass from $MC_*(-)$ to $MA_*(-)$ forget labels in C and allow framed normal bundles to be classified by the cube bundles over Bbr_k.

Restrictions on normal bundles

The manifolds which can occur in the definition of $MC_n(X)$ are restricted by the condition on the normal bundles. In particular we have the following result.

6.5 Proposition *If M is a C–manifold then* $\mathrm{wu}(M) = w(M)$*, and* $w_i(M) = 0$ *for* $2i > n$.

Proof Recall that $\mathrm{wu}(M) = \mathrm{wu}(\nu(M))$ and $w(M) = w(\tau(M))$. The result now follows from the remark after 3.12 and the fact that Wu classes of manifolds vanish above half dimension. □

Rack and link homology theories

In the case when $C = BX$ where X is a rack, the geometric description of MC_* can be simplified. Instead of labelling the diagram by C we only need a representation of the fundamental rack in X and for this we can consider the covering framed embedding (see the discussion in section 4). If X is the fundamental rack of a link L, the we can regard the resulting theory as L–bordism, and this provides many more new invariants of links by calculating the values of this theory on test spaces.

There is also a "truncated theory" given by using only the 2–skeleton of the rack space and this also makes good sense for any trunk as well. The resulting spectrum can be identified with $\mathbb{S} \wedge X$ where X is the Thom complex of a plane bundle over the 2-skeleton.

Operations

The geometric description of C–manifolds given above allows us to define operations on these theories, which cover inverse Steenrod square operations in $\mathbb{Z}_2$–cohomology.

6.6 Theorem *Let C be a □–set. There is an operation $MC_n \to MQ_{n-k}$ where $Q = J^k(C)$ and a similar operation $MC^n \to MQ^{n+k}$. These operations correspond to a map of spectra $MC \to \Sigma^k MJ^k(C)$.*

In case $C = T$ the trivial □–set we can compose with $J^k(T) \to T$ to get a map $MT \to \Sigma^k MT$ which fits into the following commutative diagram

$$\begin{array}{ccc} MT & \longrightarrow & H\mathbb{Z}_2 \\ \downarrow & & \downarrow \mathcal{X}Sq^k \\ \Sigma^k MT & \longrightarrow & \Sigma^k H\mathbb{Z}_2 \end{array}$$

where horizontal maps are the map $MT \to MA \cong H\mathbb{Z}_2$ considered above and its k–fold suspension.

Proof If we pass from a manifold with a diagram labelled by C to the k-tuple point manifold defined by the diagram, what we get is a manifold labelled by $J^k(C)$. Moreover the twisting condition is inherited by the induced diagram in this manifold and this defines the operations.

Now the map $MT \to MA \cong H\mathbb{Z}_2$ is induced by the universal Thom class of MT and the commutativity of the diagram then follows from the results of [S1], see in particular Remark 3.6. $\square$

Remark The operation $MT \to \Sigma^k MT$ defined by theorem 6.6 coincides with one of the operations defined by Mahowald on p552 of [M].

7 Vassiliev theory

In this section we shall show how to define a Vasiliev function for a $\square$–set. This is a family of cochains satisfying the Vassiliev identity (below). Vassiliev functions for $\square$–sets generalise Vassiliev invariants for classical knots as studied by Vassiliev, Birman and Lin, Bar-Natan, Kontsevich [V, BL, BN, K] and others, because, as we shall show, there is a $\square$–set $\mathcal{I}$ for which a Vassiliev function is precisely the same as a Vassiliev invariant of classical knots.

The context of $\square$–sets is a natural setting for studying Vassiliev invariants. Indeed we can construct a simple obstruction theory for extending a partially defined obstruction over the whole set. There is, however a more natural setting and that is a special sort of cubical CW complex, which we call an *arrow set*. We shall mention these briefly at the end of the section and establish conditions for an arrow set to be a $\square$–set. Arrow sets allow a proper formulation of the four term relation and we intend to write a further paper containing a complete treatment of this material.

Chains and cochains on $\square$–Sets

Let G be an abelian group and X a $\square$–set. Define $C_n(X) \equiv C_n(X;G) = \bigoplus_{X_n} G$, the group of n–chains on X with coefficients in G, and let $C^n(X) = \mathrm{Hom}(C_n(X;\mathbb{Z}), G)$ be the group of n–cochains with coefficients in G. Define $\partial\colon C_n(X) \to C_{n-1}(X)$ by the formula

$$\partial(x) = \sum_{i=1}^{i=n} (-1)^i (\partial_i^0(x) - \partial_i^1(x)).$$

Let $\delta\colon C^n(X) \to C^{n+1}(X)$ be defined by composition with ∂. In this way a chain complex, homology groups and cohomology groups can be defined for any $\square$–set X. We now look at a particularly important family of cochains.

Definition *Vassiliev functions*

Let X be a $\square$–set. A **Vassiliev function** $V = \{V_n\}$ on X with values in an abelian group G is a sequence of functions $V_n : X_n \to G$ for each $n = 0, 1, \ldots$ satisfying the **Vassiliev identity**

$$V_n(x) = V_{n-1}(\partial_i^1 x) - V_{n-1}(\partial_i^0 x) \quad \text{for} \quad i = 1, 2, \ldots, n, \quad x \in X_n. \qquad \mathbf{7.1}$$

This identity holds for all $n \geq 1$ unless the function is only **partially defined** ie defined for *some* values of n. Then the identity holds whenever it makes sense.

This definition is analogous to the definition in the literature on Vassiliev functions and we shall make clear the connection later in the section.

Clearly any Vassiliev function defines cochains in $C^n(X)$. These have the following property.

7.2 Lemma *Let V be a Vassiliev function then*

$$\begin{aligned} \delta V_n &= 0 \quad \textit{if } n \textit{ is odd,} \\ &= V_{n+1} \quad \textit{if } n \textit{ is even.} \end{aligned}$$

Proof

$$\begin{aligned} \delta V_n(x) &= \sum_{i=1}^{n+1} (-1)^i (V_n \partial_i^0(x) - V_n \partial_i^1(x)) \\ &= -\sum_{i=1}^{n+1} (-1)^i V_{n+1}(x) \end{aligned}$$

$\square$

Differentiation

We now consider the conditions under which a partially defined Vassiliev function can be extended to other values of n. If a Vassiliev function is defined for two adjacent values of n or for $n = 0$, then it can easily be extended to larger values of n. This process is called **differentiation** by Bar-Natan.

7.3 Proposition *A partial Vassiliev function defined for two adjacent values of n or for $n = 0$ can be extended uniquely to a Vassiliev function for all larger values of n.*

Proof Given a Vassiliev function defined for $n-1$ then the Vassiliev identity (7.1) defines it for n. However it will not in general be well-defined, since the value for an n–cube will depend on the choice of i in 7.1. But if it also defined for $n-2$ then it a simple consequence of the face relations that it is well-defined, for let $x \in X_n$ then if $i < j$

$$\begin{aligned} V(\partial_j^1 x) - V(\partial_j^0 x) =& V(\partial_i^1 \partial_j^1 x) - V(\partial_i^0 \partial_j^1 x) - V(\partial_i^1 \partial_j^0 x) + V(\partial_i^0 \partial_j^0 x) \\ =& V(\partial_{j-1}^1 \partial_i^1 x) - V(\partial_{j-1}^1 \partial_i^0 x) - V(\partial_{j-1}^0 \partial_i^1 x) + V(\partial_{j-1}^0 \partial_i^0 x) \\ =& V(\partial_i^1 x) - V(\partial_i^0 x). \end{aligned}$$

If it only defined for $n=0$ then the Vassiliev identity extends it to $n=1$ (and here there is no choice so it is well-defined). Then the extension to larger values of n follows as before. In this case there is a convenient formula for the resulting function :

$$V_n(x) = \sum_{\epsilon} (-1)^{|\epsilon|} V_0(\partial_1^{\epsilon_1} \cdots \partial_n^{\epsilon_n} x)$$

where ϵ runs over all choices of the ϵ_i, and $|\epsilon| = \sum_i \epsilon_i$. $\square$

Integration

The opposite process to differentiation is **integration** whch is considerably more difficult. A **Vassiliev function of level** n on X is a partially defined Vassiliev function defined on X_m for $m \geq n$.

Integration Question: *Let V be a Vassiliev function on a $\square$–set X of level n. Under what conditions can V be extended to a complete Vassiliev function?*

An important special case is the following. A **special Vassiliev function** is a Vassiliev function of level n, for some n, which is identically zero on X_m for $m > n$.

Under what conditions does a special Vassiliev function extend to a complete Vassiliev function?

Special Vassiliev functions are important for studying **finite type** Vassiliev functions. These are Vassiliev functions identically zero on X_m for $m > n$, for some n.

Obstruction theory for integrating Vassiliev functions

The theorem below gives an obstruction theory for integrating Vassiliev functions by starting with a function of level n for some n and then working down. It can be applied to construct all Vassiliev functions of finite type by starting with the zero level n Vassiliev function and working downwards. The obstructions lie in the cohomology of the James complexes of X.

7.4 **Theorem** *Let X be a $\square$–set. A given Vassiliev function on X of level $n+1$ extends to a Vassiliev function of level n if and only if an obstruction in $H^1(J^n(X); G)$ vanishes. Moreover the choice of extension is equivalent to a choice of element of $H^0(J^n(X); G)$.*

Proof A path, Φ, in $J^n(X)$ can be thought of as a sequence of $n+1$–cubes and their faces,

$$\Phi : \quad c_n^0, c_{n+1}^0, c_n^1, c_{n+1}^1, \ldots, c_n^k,$$

where c_n^i, c_n^{i+1} are opposite n–faces of the $n+1$ cube c_{n+1}^i. We attempt to define V_n along Φ using equation 7.1. We assign a value $V_{n,0}$ to c_n^0 and inductively set $V_{n,i+1} = V_{n+1}(c_{n+1}^{i+1}) + V_{n,i}$ and define $\mathrm{int}_\Phi V_{n+1} = V_{n,k} - V_{n,0}$. In order for $V_n(c_k) := V_{n,k}$ to be well defined it is necessary that int vanishes on cycles. It can be readily checked that int is zero on boundaries and so defines an element of $\mathrm{Hom}(H_1(J_n), G) \cong H^1(J_n; G)$. The choice only depends on the initial value in a path component and so on $H^0(J_n)$. $\square$

The classical theory

Choose a basepoint $* \in S^1$ and for each $n = 0, 1, \ldots$ let $\mathcal{K}^n$ denote the space of smooth immersions of S^1 in S^3 with the following properties.

If $K \in \mathcal{K}^n$ then

(1) The immersion K is an embedding except at n double points none of which involve the basepoint of S^1.

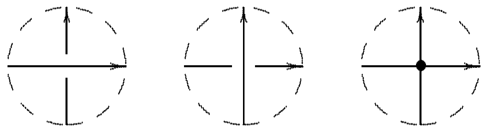
(2) At each double point the two tangents are linearly independent.

Note In the literature condition (2) is often termed transverse, but this is sloppy use of language since such immersions are embeddings!

We shall call an element $K \in \mathcal{K}^n$ a **singular knot of type** n.

Let $\mathcal{K} = \cup_0^\infty \mathcal{K}^n$. The spaces $\mathcal{K}^n$ have a natural topology and we can think of $\mathcal{K}$ as being an infinite dimensional manifold with $\mathcal{K}^n$ as a codimension n stratum.

Any smooth path in $\mathcal{K}^n$ is an isotopy between its endpoints. Any path in $\mathcal{K}^n \cup \mathcal{K}^{n+1}$ creates a double pont when it crosses $\mathcal{K}^{n+1}$. Conversely at any double point there are 'unfoldings' which eliminate the double point and create either a positive crossing or a negative crossing. This process is well defined up to isotopy. With this in mind define a **Vassiliev triple** $\{K_+, K_-, K_\bullet\}$ to be a triad of immersed knotted curves which are locally related as in the three diagrams below. The three curves are equal outside the dotted circles.



K_+ K_- $K_\bullet$

a Vassiliev triple

A **Vassiliev invariant** is a sequence of functions $V_n(K)$ defined on $K \in \mathcal{K}^n$ with values in an abelian group G which is invariant under isotopy and satisfies the following basic **Vassiliev identity** on Vassiliev triples,

$$V_n(K_+) - V_n(K_-) = V_{n+1}(K_\bullet) \qquad \textbf{7.5}$$

By using local models for the unfoldings, $\mathcal{K}$ can be made into a $\square$–space. However for our purposes here it is easier to consider $\mathcal{I}_n$, the set of isotopy classes of $\mathcal{K}_n$. Because double points avoid the basepoint of S^1, $\mathcal{I} = \{\mathcal{I}_n\}$ can be made into a $\square$–set as follows. Let K be a singular knot defining $[K] \in \mathcal{I}_n$. Number the double points of K in order round the knot starting from the basepoint. Define $\partial_i^1([K]) = [K_i^+]$ where K_i^+ is obtained from K by unfolding the i-th double point to give a positive crossing and $\partial_i^0([K])$ is similarly defined using a negative crossing. Because unfolding is well-defined up to isotopy, these face maps are well-defined. The face relations given in section 1 are easily verified.

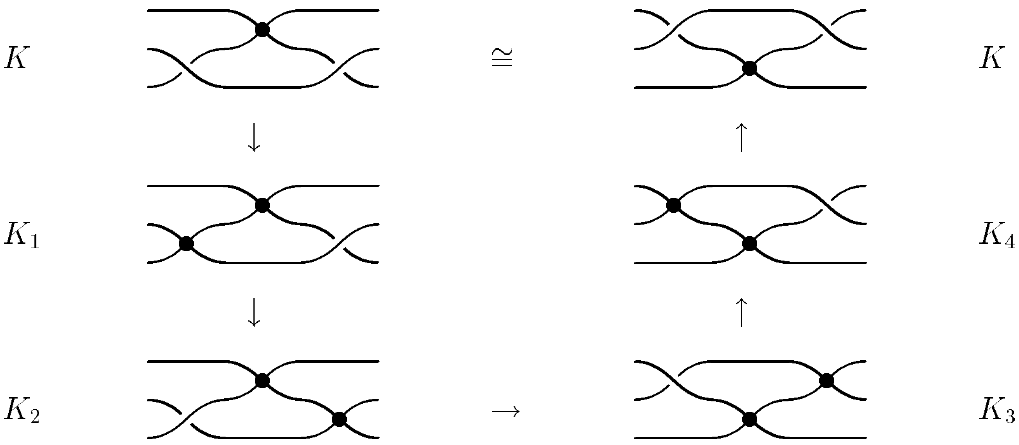
The theory of Vassiliev invariants on singular knots coincides with the theory of Vassiliev functions on this $\square$–set.

So what are the obstructions to integrating Vassiliev functions in $\mathcal{I}$? By the discussion above we need to look at 1–cycles in the James bundle. It turns out that for this classical theory there are two important types which need to be considered.

7.6 Framing independence *Let K be represented by a plane diagram $\mathcal{D}$. Let $\mathcal{D}_+$ be the result of a Reidemeister move of type 1 which introduces a new positive crossing point. Let $\mathcal{D}_-$ be a similar diagram with a new negative crossing point and let $\mathcal{D}_\bullet$ be the immersion with a new double point intermediate between the two other diagrams. See the figure below. Let $K_+, K_-, K_\bullet$ be the corresponding Vassiliev triple of knot types. Because K_+ is isotopic to K_- it follows from the basic condition that any Vassiliev function must be zero on $K_\bullet$.*

This condition can be avoided by considering invariants of framed knots. Then the framing of K_+ and K_- differ by 2 and so are different framed knots.

The other condition is a more serious obstruction to the desired extension. Consider an immersed curve K with a fixed double point. Now let a third arc circle the double point so that its final position resembles a Reidemeister 3 move on the initial position, see the figure below. This movement will create 4 singular knots with one more double point. Let these be K_1, K_2, K_3, K_4 respectively.



A quick calculation using the basic formula shows the following.

7.7 The four term condition. *Using the notation above any Vassiliev function must satisfy*

$$V(K_1) - V(K_2) - V(K_3) + V(K_4) = 0,$$

in order to extend downwards.

In the case of special level n Vassiliev functions with values in the reals, these are the only obstructions to extending the function. A special level n Vassiliev function satisfying framing independence and the four term condition is called a **weight system**.

7.8 Theorem Kontsevich, Bar-Natan [BN,K]

A weight system with values in $\mathbb{R}$ extends to a Vassiliev function defined on the whole of $\mathcal{I}$.

Cubical CW complexes and arrow sets

The singular knots indicated in the figure above are all adjacent to a knot with a triple point in which all three strands pass through one point. The figure then defines a square with the knot containing the triple point at the centre and K_1 to K_4 on the edges. There are three such squares adjacent to the triple point any two of which meet in a common axis. The whole construction can be modelled using a 3–cube. This suggests extending $\mathcal{I}$ to include singular knots with a triple point. However this cannot be done using the machinery of $\square$–sets because of problems with ordering the double points near a triple point. Thus we shall need to enlarge the concept of $\square$–set somewhat to deal with triple points.

Consider smooth CW complexes in which the preferred characteristic map of each cell is from a cube I^n for some n and in which faces are glued by arbitrary isometries of I^p onto a face of I^q for some p, q. (Ie the maps μ in the definition, above 2.1, are isometries.) Call such a CW complex a **cubical** CW complex.

The 1–cells (edges) of a cubical CW complex divide into parallelism classes. That is two edges are **parallel** if there is a sequence 'edge, square, opposite edge, square, ... opposite edge,' joining one to the other. Define an **arrowing** of K to be an orientation of each edge consistent with its parallelism class. A cubical set equipped with an arrowing is an **arrow set**.

Cubical sets and arrow sets have James complexes and James bundles defined in a similar way to those for $\square$–sets. However they enjoy rather different properties – in particular they are not always framed.

7.9 Proposition *A cubical CW complex K has an arrowing if and only if its first James bundle $\zeta^1(K)$ is framed. If K is arrowed then it can be given the structure of a $\square$–set if and only the second and third James bundles $\zeta^2(K), \zeta^3(K)$ can be framed coherently.*

Proof If K is arrowed then the edges normal to a cell of $J^1(K)$ define a framing of $\zeta^1(K)$ and conversely.

If K is a $\square$–set then its edges can be totally ordered in any cell in such a way that the ordering on any face is consistent and conversely. Thus the ordering of any pair of co-original edges which span a 2–cell can be carried around $J^2(K)$ showing that the bundle $\zeta^2(K)$ is framed. Similarly $\zeta^3(K)$ is framed.

Conversly if both bundles are framed then each pair of co-original edges which span a 2–cell are ordered and the coherent ordering of triples implies a total order. $\square$

Vassiliev functions on arrow sets

If K is arrowed then every $n-1$ face of an n cube is either positive (tip of an arrow) or negative (base of an arrow) and parallel $n-1$ faces come in (positive, negative) pairs, ∂_i^+ and ∂_i^- for some index i. For any pair of indices (i, j) there is a distinct pair (k, l) such that $\partial_i^\epsilon \partial_j^\eta = \partial_k^\eta \partial_l^\epsilon$. In fact the homology theory and the obstruction theory for Vassiliev functions carry through unaltered. Indeed arrowed complexes are the natural vehicle for the study of Vassiliev functions.

One advantage of using arrow sets is that there is no longer any need to use basepoints in defining singular knots. A second advantage is the ability to define complexes corresponding to knots with a triple point and we shall briefly sketch how this is done. Define a framed singular knot of type $(1, n-3)$ to be a smooth framed immersion of S^1 in S^3 with one triple point and $n-3$ double points (all with independent tangents). Let $\mathcal{T}_n$ be the set of isotopy classes of framed singular knots of types n and $(1, n-3)$ then $\mathcal{T} = \{\mathcal{T}_n\}$ can be made into an arrow set, face maps being defined by unfolding a double point or unwrapping the triple point to give two adjacent double points (which can be done in three ways). The subcomplex $\mathcal{D}$ is defined by considering singular knots with no triple points. A Vassiliev function on $\mathcal{T}$ coincides with a classical Vassiliev invariant of (framed unbased) knots.

However, notice that the four term relation is now a consequence of the Vassiliev identity applied to squares containing a triple point. Thus we can now define a weight system in a very elegant way.

Definition A weight system is a level n Vassiliev function on $\mathcal{T}$ which is special when restricted to $\mathcal{D}$.

Moreover the result of Bar-Natan and Kontsevich quoted earlier can be restated as follows :

7.10 Theorem *Any level n Vassiliev function on $\mathcal{T}$ of finite type with values in $\mathbb{R}$ can be extended to a full Vasiliev function.*

Proof Consider the top non-zero level in $\mathcal{D}$. This is a weight system (of level m say) and by theorem 7.8 extends to a full Vassiliev function. Subtracting this from the given partially defined function yields a level n function whose top non-zero level in $\mathcal{D}$ is $m-1$. Proceed by downward induction until we obtain a genuine weight system. □

As we remarked earlier, this material will be covered more fully in a subsequent paper.

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