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A new approach to immersion theory

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1. Introduction

The classification of smooth immersions by Smale and Hirsch [3, 6] in the late 1950's was one of the early spectacular successes of differential topology. They classified immersions of one smooth manifold M in another Q in terms of *tangential maps* of M in Q. A tangential map of M in Q is a map with a given extension to a map of the tangent bundle TM in TQ which is a *bundle monomorphism*, ie which embeds each fibre of TM linearly in a fibre of TQ.

An immersion of M in Q gives a tangential map by differentiating and the classification theorem states that differentiation induces a 1–1 correspondence between regular homotopy classes of immersions of M in Q and homotopy classes of tangential maps of M in Q. To see the power of this result observe that, given a map of M in Q, the problem of extending it to a tangential map is purely homotopy theoretic: one just has to construct a cross-section of an appropriate associated Stiefel manifold bundle. A famous corollary is that there is only one regular homotopy class of maps of the 2–sphere S^2 in $\mathbb{R}^3$ since the obstruction to deforming one tangential map into another lies in $\pi_2(SO(3)) = 0$. In other words a sphere "can be turned inside out". (This result is due to Smale, who classified immersions of spheres in Euclidean spaces; Hirsch extended Smale's result to the classification for general M and Q.)

In the proof a new machine was invented—the *immersion theory machine*. The key to the machine is to prove far more than is needed. Consider the space $\mathrm{Imm}(M,Q)$ of all immersion of M in Q and the space $\mathrm{Mon}(TM,TQ)$ of bundle monomorphisms of TM in TQ, then differentiation gives a map $d\colon \mathrm{Imm}(M,Q) \to \mathrm{Mon}(TM,TQ)$. The machine proves that d is a (weak) homotopy equivalence and the classification theorem follows by just considering path-components. The proof is by induction on a handle decomposition of M. For a 0–handle the result is easily seen by shrinking to a neighbourhood of a point. Now let M be obtained from M_0 by attaching a handle. It is an easy exercise to prove that restriction $\mathrm{Mon}(TM,TQ) \to \mathrm{Mon}(TM_0,TQ)$ is a fibration. Suppose that we can prove the same result for $\mathrm{Imm}(M,Q) \to \mathrm{Imm}(M_0,Q)$, then comparing the two fibrations and using the five lemma, we can deduce the result for $\mathrm{Imm}(M,Q) \to \mathrm{Mon}(TM,TQ)$. Thus the machine reduces the result to proving that restriction $\mathrm{Imm}(M,Q) \to \mathrm{Imm}(M_0,Q)$ is a fibration where M is obtained from M_0 by adding a handle. For this result you need that the index of the handle is smaller than the dimension of Q, which means that the theory only applies if m has codimension at least one in Q or if each component has non-empty boundary.

To prove the fibration result is a simple geometric trick. One exploits a dimension perpendicular to the core of the handle to pull out a "fold" which then allows the necessary freedom of movement.

The immersion theory machine is an extremely useful and versatile tool. It has been used to prove similar classification theorems for a wide range of structures—submersions, smoothings, triangulations—and extended to both PL and topological categories. However the proof of the classification of immersions that it provides has a very serious drawback. Because of the complicated nature of the proof, if one is given a tangential map of M in Q, there is no way to visualise the final immersion of M in Q which results from the theory. Indeed it was several years after Smale proved his theorem before a satisfactory picture for the process of turning the sphere inside out was found by Shapiro. This was developed by Morin, see [2] and also [4]. (Eventually a description of a way of turning the sphere inside out which is essentially a visualisation of the Smale proof was given by Thurston using "corrugations". This description is the basis for the film [7].)

Recently we have found a new proof of immersion theory which does not have the drawback that the classical proof has. The immersion of M in Q is constructed in a completely explicit way and by following through the proof in any given example one can explicitly picture the resulting immersion. The purpose of this paper is to give an exposition of this new proof. Our proof uses a new result, the "Compression Theorem", which is stated and proved in section 2. This is extended in section 3 to the "Multi-compression Theorem" which is the version which is used in section 4 for the new proof of immersion theory. We finish in section 5 with an explicit example. Following through the proof, we explain how to convert the familiar non-immersion of the projective plane in $\mathbb{R}^3$, as a cross-cap, into Boy's surface.

Throughout this paper we work in the smooth (C^∞) category. We shall consider only proper embeddings or immersions (ie ones taking boundary to boundary) of codimension at least 1 with compact domain. The tangent bundle of a manifold W is denoted TW and the tangent space at $x \in W$ is denoted $T_x(W)$. For full details of all results sketched here, for addenda and extensions to the non-compact and codimension 0 cases, see [5].

2. The Compression Theorem

Compression Theorem. *Let M^m be a compact manifold embedded in $Q^q \times \mathbb{R}$ and equipped with a normal vector field. Assume $q - m \geq 1$ then the vector field can be straightened (ie made parallel to the positive $\mathbb{R}$ direction) by an isotopy of M and normal field in $Q \times \mathbb{R}$.*

We think of $\mathbb{R}$ as vertical and the positive $\mathbb{R}$ direction as upwards. The theorem moves M to a position where it is *compressible*, ie where it projects by vertical projection to an immersion in Q.

The Compression Theorem solves an old problem [1, problem 6]. In addition to giving the new proof of immersion theory sketched in this paper, it also gives a new proof for loops–suspension theory and a new approach to the old problem of classifying embeddings of one manifold in another, for full details see [5].

Proof of the Compression Theorem. Assume that Q is equipped with a Riemannian metric and use the product metric on $Q \times \mathbb{R}$. Call a normal field *perpendicular* if it is everywhere orthogonal to M.

A perpendicular vector field α is said to be *grounded* if it never points vertically down. More generally α is said to be ε*–grounded* if it always makes an angle of at least ε with the downward vertical, where $\varepsilon > 0$.

We need the following lemma.

Lemma 2.1. *Under the hypotheses of the Compression Theorem the normal field may be assumed to be perpendicular and grounded.*

Proof. The lemma follows from general position. Call the vector field α and without loss assume that α has unit length everywhere. Note that the fact that α is normal (ie independent of the tangent plane at each point of M) does not imply that it is perpendicular; however we can isotope α without further moving M to make it perpendicular. α now defines a section of $T(Q \times \mathbb{R}) \,|\, M$ and vertically down defines another section. The condition that $q - m > 0$ implies that these two sections are not expected to meet in general position. $\square$

To prove the theorem first apply the lemma which results in the normal vector field α being perpendicular and grounded. By compactness of M, α is in fact ε–grounded for some $\varepsilon > 0$.

We now perform an operation on α given by rotating it towards the upward vertical: Choose a real number μ with $0 < \mu < \varepsilon$. Consider a point $p \in M$ at which $\alpha(p)$ does not point vertically up and consider the plane $P(p)$ in $T_p(Q \times \mathbb{R})$ defined by the vector $\alpha(p)$ and the vertical. Define the vector $\beta(p)$ to be the vector in the plane $P(p)$ obtained by rotating $\alpha(p)$ through an angle $\frac{\pi}{2} - \mu$ in the direction towards vertically up, unless this rotation carries $\alpha(p)$ past vertically up, when we define $\beta(p)$ to *be* vertically up. If $\alpha(p)$ is already vertically up, then we again define $\beta(p)$ to be vertically up. The rotation of α to β is called *upwards rotation*.

Figure 1 shows the extreme case when α is pointing as far down as possible.

This operation yields a *continuous* but not in general *smooth* vector field. However it may be altered to yield a smooth vector field by using a bump function to phase out the amount of rotation as the rotated vector approaches vertical. The properties of the resulting vector field for the proof (below) are not altered by this smoothing. Note that β is still normal (though not now perpendicular) to M and now has a positive vertical

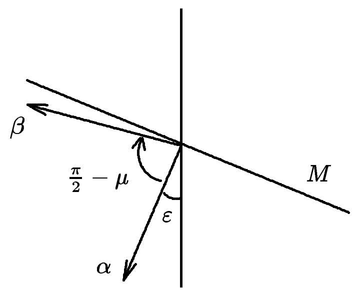


FIGURE 1. Upwards rotation

component of at least $\sin(\varepsilon - \mu)$. These facts can both be seen in figure 1; β makes an angle of at least $\varepsilon - \mu$ above the horizontal and at least μ with M. The line marked M in the figure is where the tangent plane to M at p might meet $P(p)$.

Now extend β to a global unit vector field γ on $Q \times \mathbb{R}$ by taking β on M and the vertically up field outside a tubular neighbourhood N of M in $Q \times \mathbb{R}$ and interpolating by rotating β to vertical along radial lines in the tubular neighbourhood. Call the flow defined by γ the *global flow* on $Q \times \mathbb{R}$.

Since the vertical component of γ is positive and bounded away from zero (by $\sin(\varepsilon-\mu)$) any point will flow upwards in the global flow as far as we like in finite time.

Now let M flow in the global flow. In finite time we reach a region where γ is vertically up. Since $\gamma|M = \beta$ is normal to M at the start of the flow, $\gamma|M$ remains normal to M throughout the flow and we have isotoped M together with its normal vector field to a compressible embedding. $\square$

3. The Multi-compression Theorem

We now explain why the isotopy constructed in the Compression Theorem can be assumed to be arbitrarily small in the C^0 sense. This implies that it can take place within an arbitrary neighbourhood of the given embedding. We can then apply the result to an immersion by working in an induced regular neighbourhood and prove the 'multi-compression theorem' which states that a number of vector fields can be straightened simultaneously.

What we shall do is to identify a certain submanifold of M which we call the *downset*. The closure of the downset is a manifold with boundary. We call the boundary the *horizontal set*. Nearly all the straightening process can be made to take place in an arbitrary neighbourhood of the downset and by choosing this neighbourhood to be sufficiently small, the straightening becomes arbitrarily small.

We start by defining the horizontal set. This is independent of the normal vector field and defined for embeddings in $Q \times \mathbb{R}^r$ for any $r \geq 1$.

The horizontal set

Suppose given a submanifold M^m in $Q^q \times \mathbb{R}^r$ where $m \leq q$. Think of Q as *horizontal* and $\mathbb{R}^r$ as *vertical.* Define the *horizontal set* of M, denoted $H(M)$, by $H(M) = \{(x,y) \in M \mid T_{(x,y)}(\{x\} \times \mathbb{R}^r) \subset T_{(x,y)}(M)\}$. Now suppose Q has a metric and give $Q \times \mathbb{R}^r$ the product metric, then $H(M)$ is the set of points in M which have *horizontal* normal fibres, ie (after the obvious identification) $H(M) = \{(x,y) \in M \mid \nu_{(x,y)} \subset T_x(Q)\}$ where $\nu_{(x,y)}$ denotes the fibre of the normal bundle at (x,y) defined by the metric.

The following result is given by appealing to transversality to an appropriate Grassmannian (for details of all the technical results needed for the proof see [5, section 8]).

Proposition 3.1. *After a small isotopy of M in $Q \times \mathbb{R}^r$ the horizontal set $H(M)$ can be assumed to be a submanifold of M.* □

The downset

After application of the proposition with $r = 1$ the corresponding horizontal set H is a submanifold of M of codimension $c = q - m + 1$. Let ψ be the unit perpendicular field on $M - H$ defined by choosing the $\mathbb{R}$ coordinate to be maximal, so ψ points 'upmost' in its normal fibre. Suppose M is equipped with a unit perpendicular field α. Call $D = \{x \in M \mid \alpha(x) = -\psi(x)\}$ the *downset* of M; so D is where α points *downmost* in the normal fibre. The following result also follows by a transversality argument:

Proposition 3.2. *After a small isotopy of α we can assume that α is transverse to $-\psi$, which implies that D is a manifold, and further we can assume that the closure of D is a manifold with boundary H.* □

Localisation

The following considerations explain why the downset is the key to the straightening process. Suppose that there is no downset (and no horizontal set). Then our perpendicular vector field α can be canonically isotoped to point to the upmost position in each normal fibre (ie to the section ψ) by isotoping along great circles. Once in this upmost position, upwards rotation straightens the field.

More generally, let $\overline{D}$ denote the closure of the downset after proposition 3.2 (so that $\overline{D}$ is a manifold with boundary H) and let W be a tubular neighbourhood of $\overline{D}$ in M. Then α can be canonically isotoped to ψ on $\overline{(M - W)}$ and this isotopy extended to W via a collar of ∂W in W. We call this isotopy *localisation* because it localises the straightening problem in W and we call the resulting field the *localised field.*

Now by carefully examinating the proof of the Compression Theorem given in section 2, it can be seen than the straightening isotopy is given by twisting the vector field out from underneath the down set. (In section 5 we shall draw some explicit pictures which show this clearly—see the remarks at the end of that section.) Now another technical result [5, lemma 4.3], which is again a form of general position, gives precise control over the extent to which flow lines can get "trapped" under the downset. With this control and with sufficient localisation, the isotopy which straightens the vector field can be made arbitrarily small in the C^0 sense. For full details here see [5, theorem 4.4]. We refer to this result as the "local compression theorem".

Straightening multiple vector fields

Now let M be embedded in $Q \times \mathbb{R}^n$ and suppose that M is equipped with n linearly independent normal vector fields. We say that M is *parallel* if the n vector fields are parallel to the n coordinate directions in $\mathbb{R}^n$.

Multi-Compression Theorem. *Suppose that M^m in embedded in $Q^q \times \mathbb{R}^n$ with n independent normal vector fields and that $q - m \geq 1$. Then M is isotopic (by a C^0–small isotopy) to a parallel embedding.*

Proof. Apply the local compression theorem to the first normal vector field. By the theorem M can be isotoped (by a small isotopy) to make this field parallel to the first coordinate axis. Then M lies over an immersion in $Q \times \mathbb{R}^{n-1}$ with $n-1$ independent normal fields. (This can be seen by thinking of the remaining $n-1$ fields as determining a embedding of $M \times D^{n-1}$ in $Q \times \mathbb{R}^n$ which is compressed into $Q \times \mathbb{R}^{n-1}$ by the straightening of the first field.) Now consider an induced neighbourhood of M pulled back by the immersion in $Q \times \mathbb{R}^{n-1}$. This neighbourhood is made of patches of $Q \times \mathbb{R}^{n-1}$ glued together and we can apply the local compression theorem to isotope M within this neighbourhood until the second vector field is parallel to the second coordinate axis. This isotopy determines a regular homotopy within $Q \times \mathbb{R}^{n-1}$ which lifts to an isotopy of M in $Q \times \mathbb{R}^n$ finishing with an embedding which has the first two normal fields parallel to the first two axes of $\mathbb{R}^n$. Continue in this way until all vector fields are parallel. $\square$

4. Immersion theory

Let M be an n–manifold. We shall explicitly describe a way of *rotating the fibres* of the tangent bundle TM into M. Regard the zero section M as 'vertical' and the fibres as 'horizontal'. Consider TM as a smooth $2n$–manifold, then its tangent bundle restricted to M is the Whitney sum $TM \oplus TM$. The two copies of TM are the vertical copy parallel to M and the horizontal copy parallel to the fibres of TM. Each vector $v \in TM$ then determines two vectors v_v and v_h in $TM \oplus TM$ which span a plane. In this plane we can 'rotate' v_h to v_v. Since we are not at this moment considering a particular metric 'rotation' needs to be defined: to be precise we consider the family of linear transformations of this plane given by

$$\begin{aligned} &v_h \mapsto c v_h + s v_v, \quad v_v \mapsto c v_v - s v_h \qquad \text{where} \\ &c = \cos \tfrac{\pi}{2} t, \quad s = \sin \tfrac{\pi}{2} t, \quad 0 \leq t \leq 1. \end{aligned}$$

This formula (applied to each such plane) determines a bundle isotopy (a 1–parameter family of bundle isomorphisms) which is the required rotation of the fibres of TM into M.

Basic Immersion Theorem. *Suppose that $q - m \geq 1$ and that we are given a bundle monomorphism $f\colon TM \to TQ$. Then the restriction $f|\colon M \to Q$ is homotopic to an immersion.*

Proof. Composing f with an exponential map for TQ we have a map $g\colon TM \to Q$ which embeds fibres into Q. Choose an embedding $q\colon M \to \mathbb{R}^n$ for some n and also denote by q the map $TM \to \mathbb{R}^n$ given by projecting the bundle TM onto M (the usual bundle projection) and then composing with q. We then have the embedding $g \times q\colon TM \to Q \times \mathbb{R}^n$. The fibres of TM are embedded parallel to Q and the n directions parallel to the axes of $\mathbb{R}^n$ determine n independent vector fields at M normal to the fibres of TM.

Now choose a complement for $T(TM)\,|\,M \cong TM \oplus TM$ in $T(Q \times \mathbb{R})\,|\,M$. Then the rotation of the fibres of TM into M (formula above) extends (by the identity on the complement) to a bundle isotopy of $T(Q \times \mathbb{R})\,|\,M$ which carries the these n fields normal to M to yield n independent normal fields. The result now follows from the multi-compression theorem. □

The proof just given is very explicit in contrast to the standard Hirsch–Smale approach [3, 6]. Given a particular bundle monomorphism $TM \to TQ$ the proof can be used to construct a homotopic immersion $M \to Q$. The only serious element of choice in the proof is the embedding of M in $\mathbb{R}^n$. See the next section for an explicit example.

Indeed the proof is sufficiently explicit that it can be extended with little change to give a relative and parametrised version. This extension has the full statement of immersion theory, see the introduction, as a corollary. For full details here see [5, theorem 5.3]

5. Pictures

The proofs given in previous sections hide a wealth of geometry. In this section we reveal some of this geometry. We start by drawing a sequence of pictures, where M has codimension 2 in $Q \times \mathbb{R}$, which are the end result of the isotopy given by the proof of the compression theorem in particular situations. These pictures contain all the critical information for constructing an isotopy of a general manifold of codimension 2 with normal vector field to a compressible embedding.

We then describe how the compression desingularises a map in a particular case (the removal of a Whitney umbrella) and we finish with an explicit compression of an (immersed) projective plane in $\mathbb{R}^4$. The image of the projection on $\mathbb{R}^3$ changes from a sphere with cross-cap to Boy's surface. This is the final (and the only non-trivial) step of the proof of the basic immersion theorem in an explicit example.

1 in 3

Consider the vector field on an angled line in $\mathbb{R}^3$ which rotates once around the line as illustrated in figure 2. Upwards rotation replaces this vector field by one which is vertically up outside an interval and rotates just under the line as illustrated in figure 3.

Seen from on top this vector field has the form illustrated in figure 4.

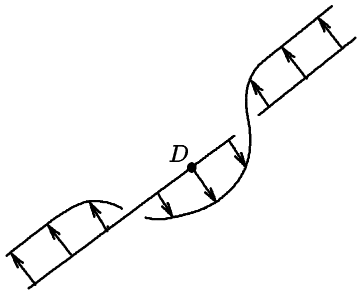


FIGURE 2. Perpendicular field

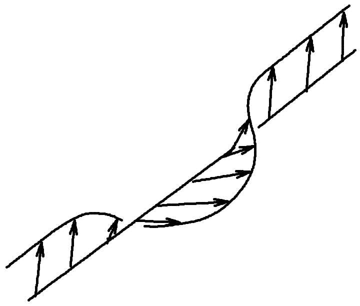


FIGURE 3. Upwards rotated field

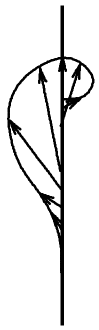


FIGURE 4. The upwards rotated field seen from on top

Now apply the global flow described in section 2. The interval where the vector field is not vertically up flows upwards more slowly and at the same time it flows under and to both sides on the original line. The result is the twist illustrated in figure 5.

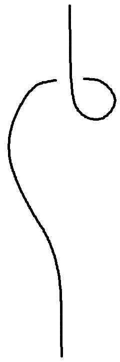
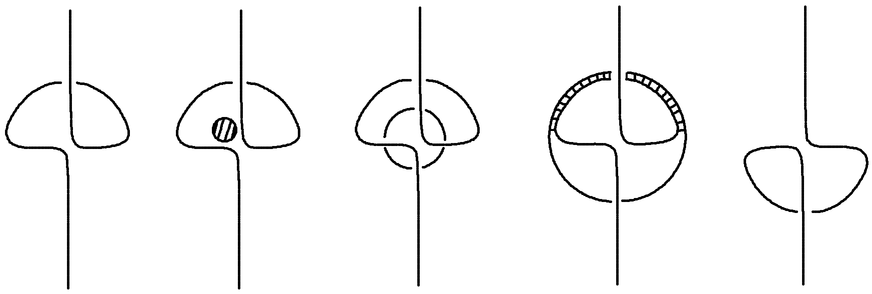


FIGURE 5. The effect of the global flow

2 in 4

Now observe that the twist constructed above has two possible forms depending on the slope of the original line. So now consider the surface in 4–space (with normal vector field) which is described by the moving line as it changes slope in 3–space from one side of horizontal to the other. The vector field so described is not grounded since in the middle of the movement the line is horizontal and one vector points vertically down. But a small general position shift moves this normal vector one side (ie into the past or the future) and makes the field grounded. We can then draw the end result of the isotopy provided by the compression theorem as the sequence of pictures in figure 6 which describe an embedded 2–plane in 4–space.



6.0 6.1 6.2 6.3 6.4

FIGURE 6. Embedding of 2 in 4

Notice that the pictures are not accurate but diagrammatic; the end result of the actual flow produces a surface which, described as a moving picture starts like figure 5 and ends with a rotation of figure 5. However the combinatorial structure of the pictures is the same as that produced by the flow. Read the figure as a moving picture from left to right. The picture is static before time 0 (figure 6.0); at time 1 a small disc appears (a 0–handle) and the boundary circle of this disc grows until it surrounds the main twist (figure 6.3). At this point a 1–handle bridges across from the circle to the twist. The 1–handle has been drawn very wide, because then the effect of the bridge can be clearly seen as the replacement of the upwards twist by the downwards twist in figure 6.4. Note that the 1–handle in figure 6.3 cancels with the 0–handle in figure 6.1 so the topology of the surface is unaltered and the whole sequence describes an embedded 2–plane in 4–space. Indeed we can see the small disc, whose boundary grows from times 1 to 3, as a little finger pushed into and under the surface pointing into the past.

Notice that there is a triple point in the projected immersion between 6.2 and 6.3 as the circle grows past the double point.

3 in 5

The embedding of 2 in 4 just constructed again has an asymmetry (corresponding to the choice of past or future for the general position shift). The pictures drawn were for the choice of shift to the past. The choice of shift to the future produces a similar picture with the finger pointing to the right. We can move from one set of pictures to the other by a similar construction illustrated in figure 7, which should be thought of as a moving sequence of moving pictures of 1 in 3 and hence describes an embedding of 3 in 5.

This figure should be understood as follows: Time passes down the page. At times before 0 the picture is static and is the same as figure 6. Just before time 1 (the top row of the figure) a small 2–sphere has appeared (a 0–handle). At time 1 this sphere has grown a little and now appears as the three little circles in the middle three slices (think of the sphere as a circle which grows from a point and then shrinks back down). At time 2 (the middle row of the figure) the sphere has grown to the point where it encloses the finger. At this point a 1–handle bridges between the finger and the sphere and this has the effect of flipping the finger over from a left finger to a right finger as shown in the bottom row (time 3).

It should now be clear how to continue this sequence of constructions to construct embeddings of n–space in $(n+2)$–space for all n.

Removal of a Whitney umbrella

We now look at the example described in figures 2 to 5, but from a different point of view. The compression isotopy is a sequence of embeddings of a line in $\mathbb{R}^3$ with normal vector fields. The whole sequence defines an embedding of a plane in $\mathbb{R}^4$ with a normal vector field. The projection on $\mathbb{R}^3$ of this plane has a singularity—in fact a Whitney umbrella. To see this, we have reproduced this sequence in figure 8 below. We have changed the isotopy to an equivalent one which shows the singularity clearly.

The left hand sequence in figure 8 is the view from the top, with the curve drawn out by the tip of the vector shown (in most places the vector is pointing up so this curve

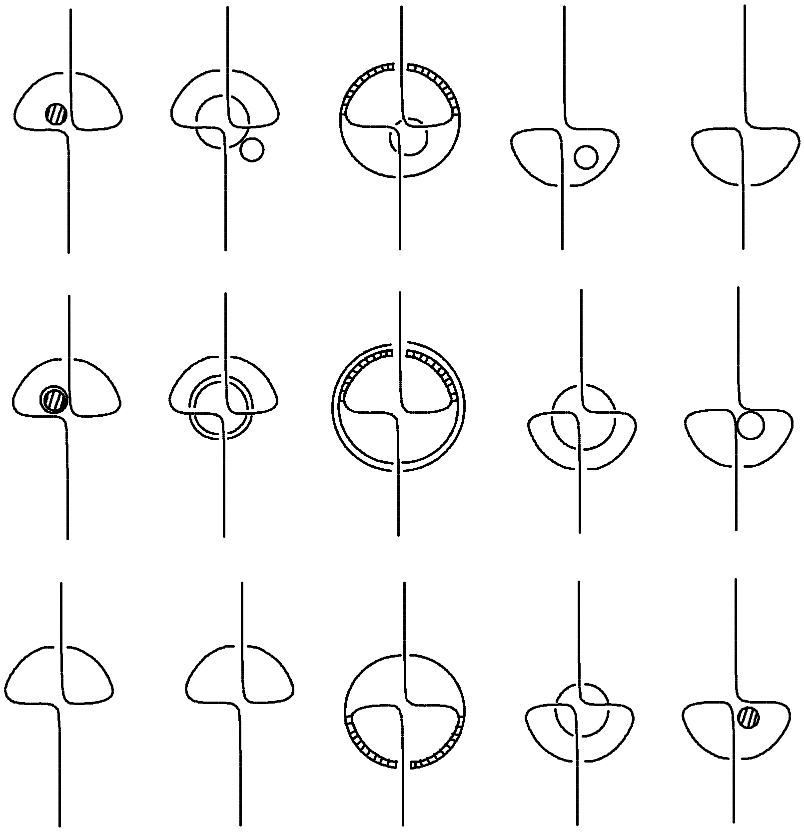


FIGURE 7. Embedding of 3 in 5

coincides with the submanifold), and the right hand sequence is the view from the side. The arrows in the bottom two pictures of the left sequence indicate slope. Arrows point *downhill.* Reading up the page gives an equivalent compression isotopy to that pictured in figures 2 to 5. The whole sequence defines an embedding of $\mathbb{R}^2$ in $\mathbb{R}^4$ with normal vector field which projects to a map of $\mathbb{R}^2$ to $\mathbb{R}^3$. The image has a line of double points at the top terminating in a singular point (the image of H) which can be seen to be a Whitney umbrella. The surface is flat at the bottom and has a 'ripple' at the top (ie a region of the form (curve shaped like the letter α)$\times I$). The ripple shrinks to a point at the umbrella.

If we now apply the compression theorem to this embedding with vector field, then the image changes into a sheet with a continuous ripple (a twist is created in each slice from the H–slice down). Thus the Whitney umbrella is desingularised by having a new ripple spliced into it.

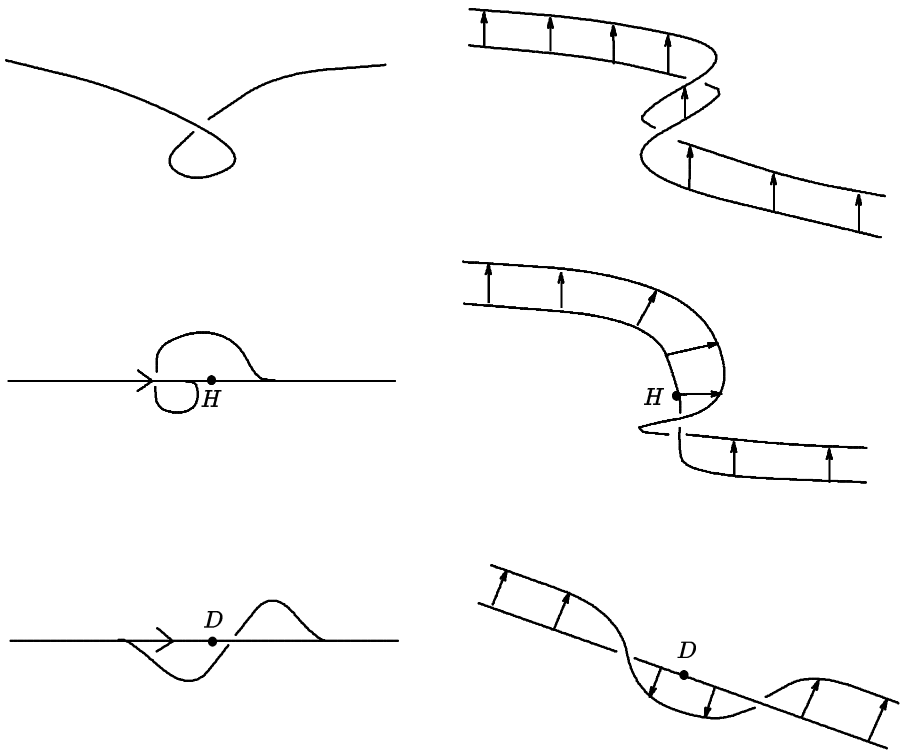


FIGURE 8. Surface above a Whitney umbrella

Cross-cap to Boy's surface

We finish by giving an explicit example of an immersion produced by the basic immersion theorem. Consider the map $P^2 \to \mathbb{R}^3$ which has image the well-known non-immersion of P^2 in $\mathbb{R}^3$ with a line of double points and two singularities (Whitney umbrellas) — a 2–sphere with cross-cap — illustrated in figure 9. This can be extended to a tangential map using the fact that $\pi_2(SO(3)) = 0$. The proof of the immersion theorem gives a covering embedding of P^2 in $\mathbb{R}^3 \times \mathbb{R}^5$ with five independent normal vector fields. The first four of these can be straightened without moving the image of P^2 and thus the covering embedding compresses to a covering immersion in $\mathbb{R}^3 \times \mathbb{R}^1 = \mathbb{R}^4$ with one normal vector field. We give an explicit description of this immersion (it has one double point) and the normal vector field in figure 10. The final immersion of P^2 in $\mathbb{R}^3$ is the result of compressing this immersion. We shall see that the final immersion is Boy's syrface.

In figure 10 we have sliced the sphere-with-cross-cap by planes which are roughly horizontal but titled a little into general position and then lifted to an immersion in $\mathbb{R}^4$. The vector information is given using the conventions defined in the left-hand pictures

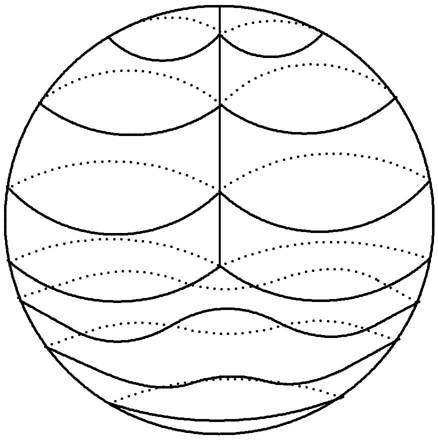


FIGURE 9. Non-immersion of P^2 in $\mathbb{R}^3$

in figure 8. Ignoring the vectors for the time being, the sequence starts with a 0–handle producing a small circle which passes through the top Whitney umbrella at H_1 (think of a plane tilted to the left slicing the sphere-with-cross-cap). The double point occurs at Q and then a 1–handle (a bridge) at B and the bottom Whitney umbrella at H_2 (think of the slicing plane titled backwards here). Then the resulting circle shrinks to a 2–handle at the bottom.

We now explain the vector field. At the top it is up (towards the eye) and then two opposite twist fields appear (think of a twist on a 'U' shaped curve). The right-hand twist field passes through H_1 to form a twist in the curve. The local sequence here is the same as figure 8 read upwards. The left hand twist field continues downwards, but notice that at X the slope (indicated by the arrows) changes. The field near X is the same as that used to create the figure 6 sequence described earlier. Another figure 8 sequence (reflected this time) changes the twist resulting from the bridge into an opposite twist field and the two twist fields cancel. We now draw the result of the compression.

Figure 11 comprises two views of the immersion obtained by projecting the resulting compressible immersion in $\mathbb{R}^4$ to an immersion in $\mathbb{R}^3$. The twist fields have all been replaced by ripples (as explained in connection with figure 8 above). This is indicated in the left view by a heavy black line near the top and the heavy dashed line near the middle. The heavy line represents a ripple on the *outside* of the top sheet and the dashed line represents a ripple on the *inside* of the top sheet. The square containing the point X should be imagined to be enlarged and contain a copy of the immersion in $\mathbb{R}^3$ obtained by projecting figure 6. Notice that figure 6 can be seen to start with a twist to the right of the line and ends with a twist to the left of the line. The projected immersion therefore starts with a ripple on one side of the sheet and ends with a ripple on the other side.

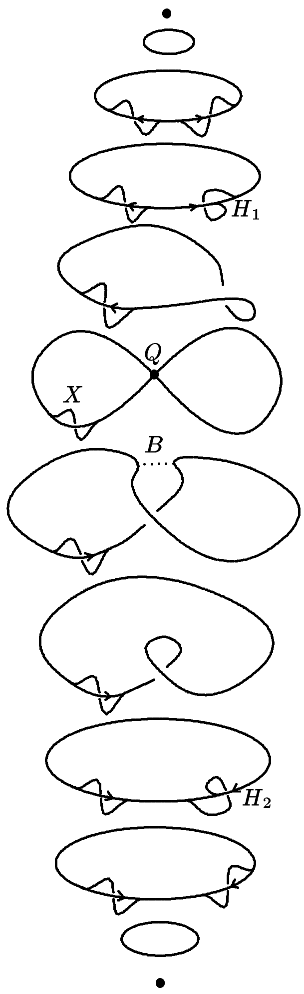


FIGURE 10. Covering immersion in $\mathbb{R}^4$

To the right in figure 11 is a sequence of cross-sections of the immersions, roughly on the same levels as the left-hand view. These cross-sections make clear what is happening near the singular points. Near the top the ripple turns around to join the 'big ripple' which is the right half of the cross-cap. Near the middle it joins into the back sheet of the Whitney umbrella. The small dotted line in the fourth section from the bottom at the right is a bridge (the 1–handle). The square marked X should again be imagined replaced by the sequence in figure 6 and notice that this sequence contains a triple point.

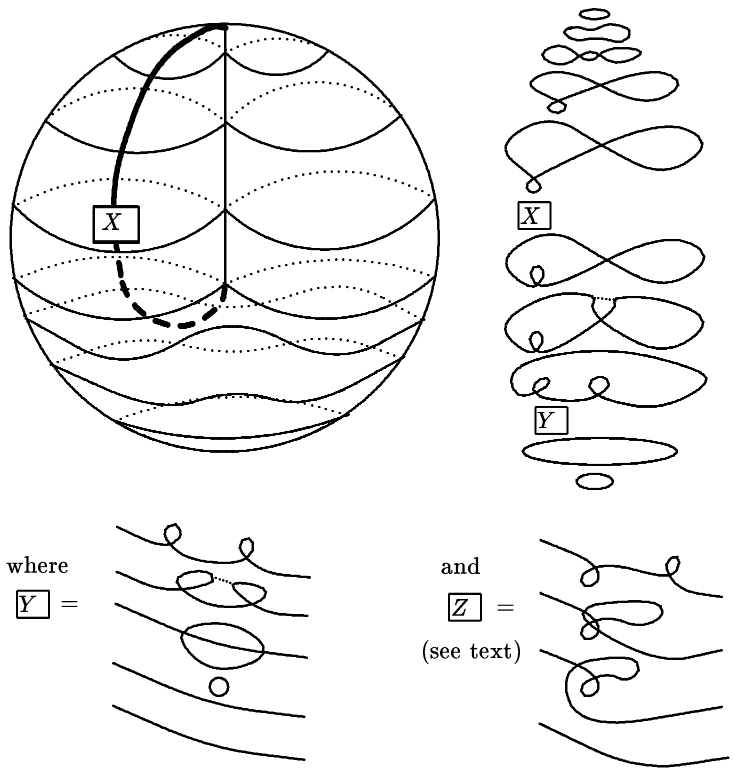


FIGURE 11. Immersion of P^2 in $\mathbb{R}^3$

The sequence labelled Y (to be inserted in the square marked Y) can be very simply visualised as an immersion: it is just an immersion in which a ripple makes a 'U' shape.

The final immersion is isotopic to Boy's surface. This is best seen by sliding the X sequence down to meet the Y sequence. The two sequences can then be combined in a single sequence (with no critical levels) detailed as Z in the figure. The immersion now has just three critical levels and can be seen to coincide with Philips' projection of Boy's surface [4].

Finally we remark that the pictures just given illustrate clearly the role of the downset in the straightening isotopy. In figure 2 the downset is one point, namely D and note that the initial vector field twists once around M in an interval centred on D. The straightening becomes smaller (and tighter) if the initial twist is made to take place in a smaller interval. In the surface which determines figure 6 the downset is a line and in figure 7 it is a plane. In figure 8 the downset is a half-open interval (comprising points like D in the middle of twist fields) with boundary at H, which is the horizontal set in this example.

In the final example (figures 9 to 11) the downset again comprises a point in the middle of all the twist fields. It thus forms an interval in the neighbourhood of which the ripple in figure 11 is constructed. It has as boundary the two singular points H_1 and H_2, which form the horizontal set in this example.

In all cases it can be seen that the straightening takes place in a neighbourhood of the downset and can be made small by choosing this neighborhood sufficiently small.

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