Characterisation of the three sphere following Haken

Colin Rourke

The purpose of this paper is to give an exposition of some work of Wolfgang Haken dating from the 1960's of which details have never been published. In [H; pages 135–138] Haken states and sketches a proof of a characterisation of S^3 in terms of clasp singularities. A detailed proof of this result, based in part on Haken's sketch, is given here.

The clasp characterisation will be stated later (theorem 2). The proof starts with some simple geometrical modifications which result in a Heegaard diagram with two disjoint families of embedded discs which fit another (more algebraic) characterisation of S^3 which we shall prove first (theorem 1 below).

Before stating this theorem we shall need to make some preliminary remarks on words of one or two letters.

Words of one or two letters

Let $\mathcal{S} = \{q_1, q_1^{-1}, q_2, q_2^{-1}, \ldots, q_t, q_t^{-1}\}$. The elements of $\mathcal{S}$ will be called **letters** and we shall be concerned with **words** w, w', w_1, w_2 etc. in these letters; the **empty word** is denoted e.

$F_t = F(q_1, q_2, \ldots, q_t)$ will denote the free group of rank t generated by $q_1, q_2, \ldots, q_t$.

An element of F_t is an equivalence class of words (under reduction); therefore a word w can be regarded in two ways (1) as an (unreduced) word in its own right (2) as the element of F_t which it determines. To avoid unnecessary notation the same notation will be used for both meanings; it should be clear from context which meaning is intended. Thus e denotes both the empty word and the identity element of F_t.

Definition *Simple consequences*

Let $\mathcal{L}$ be a list of words of one or two letters. A **simple consequence** of $\mathcal{L}$ is another word of one or two letters which is obtained from a word in $\mathcal{L}$ or the inverse of a word in $\mathcal{L}$ by making a sequence of substitutions of the following types:

Type 1: Suppose that $w = x$, $x \in \mathcal{S}$, is a one letter word in $\mathcal{L}$. Then we may substitute e for x or x^{-1} (i.e. delete x or x^{-1}) in a two letter word or substitute x or x^{-1} for e (i.e. insert x or x^{-1}) in a one letter word.

Type 2: Suppose that $w = xy$, $x, y \in \mathcal{S}$ is a two letter word in $\mathcal{L}$. Then we may substitute y^{-1} for x or x^{-1} for y in any word of $\mathcal{L}$.

It is not difficult to see that the simple consequences of $\mathcal{L}$ are in fact all the one or two letter words which are products of words of $\mathcal{L}$ or their inverses.

Moreover we can think of $\mathcal{L}$ as defining an equivalence relation on $\mathcal{S} \cup \{e\}$ generated by $x \sim y^{-1}$, $y \sim x^{-1}$ where $xy \in \mathcal{L}$ and $x \sim e \sim x^{-1}$ where $x \in \mathcal{L}$. Then the simple consequences are precisely the words of the form xy where $x \sim y^{-1}, x, y \in \mathcal{S} \cup \{e\}$.

We shall use the notation $\mathcal{L}^+$ for the set of simple consequences of $\mathcal{L}$.

Definition *Simple cancellation*

Let w be a word; we say that w **simply cancels to** w' written $w \searrow w'$ if w reduces to w' by cyclic permutation and the cancelling of pairs xx^{-1}, where $x \in \mathcal{S}$, and *the pair xx^{-1} occurs in w* (perhaps cyclically permuted).

For example $w \searrow q_i$ would mean that w is a cyclic permutation of $q_i x x^{-1} y y^{-1} z z^{-1} \ldots$ where each of $x, y, z \ldots \in \mathcal{S}$.

More generally, suppose that $\mathcal{L}$ is a list of words of one or two letters, then we say that w **simply cancels to** w' **rel** $\mathcal{L}$, written $w \searrow w'$ rel $\mathcal{L}$, if w reduces to w' by cyclic permutation and the cancelling of pairs xx^{-1}, or the removal of words in $\mathcal{L}$, or their inverses *all of which one or two letter words occur in w* (perhaps cyclically permuted).

Thus $w \searrow q_i$ rel $\mathcal{L}$ would mean that w is a cyclic permuation of $q_i w_1 w_2 w_3 \ldots$ where each w_i is a word of one or two letters, either of the form xx^{-1}, or from $\mathcal{L}$, or the inverse of a word from $\mathcal{L}$. Similarly $w \searrow q_i$ rel $\mathcal{L}^+$ would mean that w is a cyclic permuation of $q_i w_1 w_2 w_3 \ldots$ where each w_i is a word of one or two letters, either of the form xx^{-1}, or a simple consequence of $\mathcal{L}$.

The basic characterisation

Theorem 1 *Characterisation of the three sphere in terms of two families of discs*

Suppose that M is a 3–manifold with Heegaard decomposition $M = H \cup H'$ and that $H = H_0 \cup h_1 \cup h_2 \cup \ldots \cup h_t$ where H_0 has genus n and h_i are 1–handles.

Suppose further that there are two families of discs $\{D_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^t$ properly embedded in H' such that:

(1) $D_i \cap D_j = \emptyset$ and $Q_i \cap Q_j = \emptyset$ if $i \neq j$,

(2) $\partial D_i \subset \partial H_0$ each $i = 1, 2, \ldots, n$,

(3) $H_0 \cup \bigcup_i N(D_i) \cong B^3$, where $N(D_i)$ is a regular neighbourhood of D_i in H'.

Now let each h_i be oriented and have the letter q_i associated to it. Then given a closed curve C in ∂H, we can read the (cyclic) word $\omega(C)$ by following round C and writing q_i or q_i^{-1} for each crossing of h_i, according to direction.

Suppose finally :

(4) $w_i = \omega(\partial Q_i) \searrow q_i$ each $i = 1, 2 \ldots t$.

Then $M \cong S^3$.

Proof

Note that no assumption is made about the intersections $D_i \cap Q_j$. Indeed if the D_i and Q_j are all disjoint then the result follows quickly from Zieschang [Z]. This is seen as follows.

Let $K = H \cup \bigcup_i N(D_i)$ and $K' = \text{closure}(M - K)$. Then by condition (3) $M = K \cup K'$ is a Heegaard decomposition of M of genus t and we can naturally identify $F_t = F(q_1, q_2, \ldots, q_t)$ with $\pi_1(K)$. Now by condition (4) $\{w_i \,|\, i = 1, 2, \ldots, t\}$ is the standard generating set for F_t and Zieschang's theorem gives an automorphism of K carrying the curves ∂Q_i to the standard representatives of this generating set. Thus $M = K \cup K'$ is the standard Heegaard decomposition of S^3 of genus t.

Thus the proof consists in making the discs D_i and Q_j disjoint. We can suppose, by transversality, that the intersections $D_i \cap Q_j$ consist of a number of arcs and circles. Moreover standard "disc trading" arguments can be used to eliminate the circles. So we assume that the intersection set consists of a finite number of arcs.

We shall describe a process which modifies the discs D_i and Q_j and which results either in a new disjoint Q_j being identified or in at least one curve of intersection being removed from the intersection set. The process ends with the discs disjoint. During the process, properties (1), (2) and (3) are maintained whilst property (4) is successively modified. Moreover n is allowed to increase, whilst t remains fixed. In brief, one step of the process consists of choosing an arc α of the intersection set which is "outermost" in Q_j say, thickening α and enlarging H_0 by adjoining the thickened α. This cuts D_i say into two subdiscs, and D_i is replaced by these two discs. Finally the two halves of Q_j are inspected and either one of these halves or some other "Q–disc" is discarded.

At the end of the process the words w_i are no longer the standard generating set for F_t, but they still form a generating set of size t. Thus by Nielsen theory there is an automorphism of F_t carrying the w_i to the standard generating set, and Zieschang's result again implies that there is an automorphism of K carrying $\{\partial Q_i\}$ to the standard representatives of this generating set. Thus $M = K \cup K'$ is S^3.

At each step of the process the indices for the discs Q_i may be reordered and a Q–disc may be found which is disjoint from the D–discs. These disjoint Q–discs will be ordered first and labelled $\overline{Q}_1, \overline{Q}_2, \ldots$ and they are not subsequently modified during the rest of the process.

Description of the middle of the process

To be precise, in the middle of the process, the original sets of discs $\{D_i\}$ and $\{Q_i\}$ have been replaced by new sets $\{D_i\}$ and $\{\overline{Q}_1, \overline{Q}_2, \ldots, \overline{Q}_u, Q_{u+1}, \ldots, Q_t\}$ such that:

(A) Properties (1), (2) and (3) in the statement of the theorem hold.

(B) $\overline{Q}_i \cap D_j = \emptyset$ for $1 \le i \le u$ and all j.

(C) Let $\overline{w}_i = \omega(\partial \overline{Q}_i)$ then the words $\overline{w}_i$ possess a simple cancellation property described inductively as follows:

$\overline{w}_1 \searrow l_1$ where l_1 is a one or two letter word of the form $q_1 y$ where $y = e$ or $y = q_j$ or q_j^{-1} for $j > 1$.

Now let $\mathcal{L}_1 = \{l_1\}$ then $\overline{w}_2 \searrow l_2$ rel $\mathcal{L}_1^+$ where l_2 is a one or two letter word of the form $q_2 y$ where $y = e$ or $y = q_j$ or q_j^{-1} for $j > 2$.

In general, for $1 < i < u$, we define $\mathcal{L}_i = \{l_1, l_2, \ldots, l_i\}$ and then:

(C1) $\overline{w}_{i+1} \searrow l_{i+1}$ rel $\mathcal{L}_i^+$ where l_{i+1} is a one or two letter word such that $l_{i+1} \notin \mathcal{L}_i^+$.

(C2) If we define $\mathcal{L}_{i+1} = \mathcal{L}_i \cup l_{i+1}$ then $\mathcal{L}_{i+1}^+$ contains a word of the form $q_{i+1} y$ where $y = e$ or $y = q_j$ or q_j^{-1} for $j > i+1$. (Note that if $i+1 = t$ then this word must be q_t.)

(D) Let $\mathcal{L}_u = \{l_1, l_2, \ldots, l_u\}$ then $w_i = \omega(\partial Q_i) \searrow q_i$ rel $\mathcal{L}_u^+$ for $u < i \leq t$.

The generating set

We claim that properties (C) and (D) imply that at each stage of the process we have a generating set $G_u = \{\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_u, q_{u+1}, \ldots, q_t\}$, (of size t) for F_t, where the $\{\overline{w}_i\}$ here mean particular choices of (non-cyclic) words for $\{\omega(\partial \overline{Q}_i)\}$ (which were previously only defined up to cyclic permutation). In particular, at the end of the process, when $u = t$, $G_t = \{\overline{w}_i \,|\, 1 \leq i \leq t\}$ is a generating set and the theorem follows.

We first prove that the words $\{l_1, \ldots, l_u\}$ can be written as words in (particular choices of) $\{\overline{w}_1, \ldots, \overline{w}_u\}$. This is done by induction on i. Start by noting that $\overline{w}_1 \searrow l_1$ and hence for some particular choice of cyclic permutation, $\overline{w}_1 = l_1$ as elements of F_t. Now suppose that $\{l_1, \ldots, l_i\}$ can be written as words in (particular choices of) $\{\overline{w}_1, \ldots, \overline{w}_i\}$. By (C1) $\overline{w}_{i+1} \searrow l_{i+1}$ rel $\mathcal{L}_i^+$ i.e. for some particular choice we can write $\overline{w}_{i+1} = l_{i+1} v w z \ldots$ where $v, w, z \in \mathcal{L}_i^+$. But words in $\mathcal{L}_i^+$ are products of $\{l_1, \ldots, l_i\}$, which are in turn products of $\{\overline{w}_1, \ldots, \overline{w}_i\}$, by induction. So l_{i+1} is a product of $\{\overline{w}_1, \ldots, \overline{w}_{i+1}\}$, completing the induction step.

So it suffices to show that $H_u = \mathcal{L}_u \cup \{q_{u+1}, \ldots, q_t\}$ is a generating set for F_t. In other words that $q_i \in H_u^+$ for each i. We do this by reverse induction on i. If $u = t$, then as noted above in (C2), $q_t \in \mathcal{L}_u^+ = H_u^+$. If $u < t$ then $q_{u+1}, \ldots, q_t \in H_u^+$. So assume that $i < t$ and $i \leq u$ and $q_j \in H_u^+$ for $j > i$. Then by (C2) $\mathcal{L}_i^+$ contains a word of the form $q_i y$ where $y = e$ or $y = q_j$ or q_j^{-1} for $j > i$. Since $q_j \in H_u^+$ this implies $q_i \in H_u^+$ and completes the induction.

One step of the process

We now describe one step of the process. During this step either u increases by one or the number of arcs of intersection of the Q–discs with the D–discs decreases by at least one (or both); therefore the process must terminate in a finite number of steps.

We inspect Q_{u+1}. If $Q_{u+1} \cap D_j = \emptyset$ for each j, then we define $\overline{Q}_{u+1} = Q_{u+1}$ and then $l_{u+1} = q_{u+1}$ and property (C2) follows. Property (C1) holds because if $q_{u+1} \in \mathcal{L}_u^+$, then

q_{u+1} could be deleted from H_u which would then be a generating set for F_t of size $t-1$, which is impossible.

Now suppose that Q_{u+1} meets the D–discs in a number of arcs. By property (D), if we read round the boundary of Q_{u+1} we read a sequence $q_{u+1}vwz\ldots$ where each of v, w, z is a one or two letter word either of the form $xx^{-1}, x \in \mathcal{S}$ of in $\mathcal{L}_u^+$. Now one of these two letter words, $v = xy$ say, may be such that x (and therefore y also) is in $\mathcal{L}_u^+$. If this is the case, then we can replace v by xy in the above sequence. So we may assume that all the two letter words in the sequence are *essentially* two letter words in $\mathcal{L}_u^+$ or of the form $xx^{-1}, x \notin \mathcal{L}_u^+$. Notice that the ends of the arcs of intersection may split some of these two letter words. This is illustrated in figure 1. We choose a particular arc α which is "outermost" away from the letter q_{u+1}. See figure 1 again.

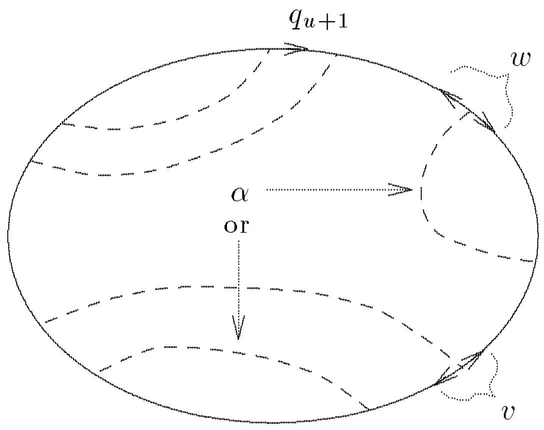

Figure 1

We thicken α and attach it to H_0. This increases n by one. The D–disc, D_r say, in which α lies gets cut into two pieces D_r^+ and D_r^-. We replace D_r by D_r^+ and D_r^-, then after reindexing the D–discs, properties (1), (2) and (3) are maintained.

The Q–disc Q_{u+1} is also cut into two pieces, Q_{u+1}^+ and Q_{u+1}^-, where Q_{u+1}^- is the outer piece (which is disjoint from the D-discs) and $\omega(\partial Q_{u+1}^+)$ contains the letter q_{u+1}. We inspect $\omega(\partial Q_{u+1}^-)$. There are two possibilities:

Possibility 1 $\omega(\partial Q_{u+1}^-) \searrow e$ rel $\mathcal{L}_u^+$.

This is the case if *either* (1) neither end of α splits a two letter word in the sequence, *or* (2) both ends of α split two letter words and the resulting new two letter word w (the two spare letters in $\omega(\partial Q_{u+1}^-)$ at the ends of α) lies in $\mathcal{L}_u^+$. In both these cases, $\omega(\partial Q_{u+1}^+) \searrow q_{u+1}$ rel $\mathcal{L}_u^+$. In case (1) this is clear, in case (2) note that if w' is the two letter word formed from the other spare letters, then $\omega(\partial Q_{u+1}^+) \searrow q_{u+1}w'$ rel $\mathcal{L}_u^+$. But since w' can be obtained from w by substitution (using the two letter words which are split) $w' \in \mathcal{L}_u^+$.

We discard the disc Q_{u+1}^- rename Q_{u+1}^+ as Q_{u+1} and we are back in the same situation as we were, but with one fewer arcs of intersection.

Possibility 2 $\omega(\partial Q_{u+1}^-) \searrow w$ rel $\mathcal{L}_u^+$ where w is a one or two letter word not in $\mathcal{L}_u^+$.

This is the case if either only one end of α splits a two letter word of the sequence (and w is the spare letter which, as remarked above, is not in $\mathcal{L}_u^+$) or if both ends split and w is the new two letter word formed by the two spare letters $x, y \notin \mathcal{L}_u^+$ and $w = xy \notin \mathcal{L}_u^+$.

In this case, we define $\overline{Q}_{u+1}$ to be Q_{u+1}^- and l_{u+1} to be w. Then property (C1) follows. We need to discard either Q_{u+1}^+ or one of the other Q–discs, and to check properties (C2) and (D). In order to do this we shall need to examine carefully the relationship of w with $\mathcal{L}_u^+$ and possibly reorder the indices $u+1, \ldots, t$.

Now let w' be the one or two letter word formed from the other spare letters, then $\omega(\partial Q_{u+1}^+) \searrow q_{u+1}w'$ rel $\mathcal{L}_u^+$. Since w' can be obtained from w by substitution (using the two letter words which are split) we have:

$$\omega(\partial Q_{u+1}^+) \searrow q_{u+1} \text{ rel } \mathcal{L}_{u+1}^+. \tag{$*$}$$

As remarked at the beginning of the section, we can regard $\mathcal{L}_u$ as defining an equivalence relation on $\mathcal{S} \cup \{e\}$. The equivalence classes are $C_0, C_1, C_2 \ldots$ where C_0 contains e. Notice that if C_i is a class then so is C_i^{-1}, for example $C_0^{-1} = C_0$. In fact C_0 is the only class such that $C_i^{-1} = C_i$, but we shall not need to use this fact explicitly. (This can be proved by a simple counting argument using the remark which follows. It also follows from the argument with which we finish the proof of the theorem.) Note also that C_0 consists precisely of the one letter words in $\mathcal{L}_u^+$ together with e.

Remark Each class $C_i \neq C_0$ contains a letter q_j or q_j^{-1} where $j > u$.

This follows from (C1). If q_k or $q_k^{-1} \in C_i$ and $k \leq u$ then by (C2) q_j or $q_j^{-1} \in C_i$ where $j > k$. This argument can be repeated until a q_j or q_j^{-1} is found in C_i for $j > u$.

Now w can have length one or two, and we examine these two case in turn.

Case 1 w is a one letter word x say.

In this case since $x \notin C_0$ the remark implies that if $x \in C_i$ say then q_j or $q_j^{-1} \in C_i$ where $j > u$.

Subcase (a) : $j = u+1$ In this case we discard Q_{u+1}^+ and make no changes to the order of the indices. Note that property (C2) follows since $q_j = q_{u+1} \in \mathcal{L}_{u+1}^+$. Property (D) is clear.

Subcase (b) : $j \neq u+1$ In this case we discard the disc Q_j and interchange the indices j and $u+1$. The disc previously labelled Q_{u+1}^+ becomes the new Q_j. Property (C2) follows necause the new q_{u+1} is the old $q_j \in \mathcal{L}_{u+1}^+$. Property (D) is clear for $i \neq j$. For $i = j$ it is equation $(*)$ above (note that the old $u+1$ is the new j).

Case 2 w is a two letter word xy say.

Recall that neither of x, y is in C_0, and note also that we cannot have $x \in C_i$ and $y \in C_i^{-1}$ where $i \neq 0$ because then $w \in \mathcal{L}_u^+$ which is not the case.

There are two subcases remaining.

Subcase (i) : $x \in C_i$ and $y \in C_{i'}$ where $0 \neq i$ or i' and $C_{i'} \neq C_i$ or C_i^{-1}. In this case the remark implies that each of C_i, $C_{i'}$ contains a q_j or q_j^{-1} for different j's, $j > u$. For definiteness suppose that $q_j \in C_i$ and $q_k \in C_{i'}$ where $u < j < k$. (The other possibilities are dealt with in a directly analogous way.) There are now two subsubcases very similar to the two subcases in main case 1 above.

Subsubcase (a) : $j = u + 1$ In this case we discard Q^+_{u+1} and make no changes to the order of the indices. Note that property (C2) follows since $q_j q_k^{-1} = q_{u+1} q_k^{-1} \in \mathcal{L}^+_{u+1}$. Property (D) is clear.

Subsubcase (b) : $j \neq u + 1$ In this case we discard the disc Q_j and interchange the indices j and $u+1$. The disc previously labelled Q^+_{u+1} becomes the new Q_j. Property (C2) follows necause the new q_{u+1} is the old q_j and old $q_j q_k^{-1} \in \mathcal{L}^+_{u+1}$. Property (D) is clear for $i \neq j$. For $i = j$ it is again equation $(*)$ above (note that the old $u+1$ is the new j).

Subcase (ii) : $x \in C_i$ and $y \in C_i$ where $i \neq 0$.

We shall show that this is impossible, and this completes the proof of the theorem. In fact we shall use Zieschang's theorem [Z] again to prove that in this case M contains an embedded projective plane which cannot happen, since M is a homotopy sphere.

The key observations are (1) $\{\overline{w}_1, \ldots, \overline{w}_u\}$ is part of a generating set and that if we factor F_t by the subgroup generated by this subset then the image of $\overline{w}_{u+1}$ is the same as the image of $w = l_{u+1}$ which is the square of a generator (in fact the square of the generator which is the image of the appropriate q_j or q_j^{-1} where $j > u$). (2) The words $\{\overline{w}_1, \ldots, \overline{w}_{u+1}\}$ are represented by the boundary curves of the discs $\{\overline{Q}_i\}$ for $1 \leq i \leq u+1$, which are embedded in K', where $K = H \cup \bigcup_i N(D_i)$ and $K' = \text{closure}(M - K)$, as before.

We apply Zieschang to find an automorphism of K carrying $\{\overline{w}_1, \ldots, \overline{w}_u\}$ to standard generating curves. The first u handles of K can now be cancelled by the first u of these embedded discs. Applying Zieschang again, the boundary of the final disc can be assumed to the standard representation l say of g^2 where g is a longidutinal curve on the resulting handlebody $\overline{K}$ say of genus $t - u$. The required embedded projective plane is formed by final embedded disc $\overline{Q}_{u+1}$ and the Möbius strip in $\overline{K}$ with boundary l.

End of proof of theorem 1

Clasp characterisation of the three sphere

We now turn to the theorem stated by Haken. The situation being considered is a 3–manifold M with a Heegaard decomposition $M = H \cup H'$, where H, H' are handlebodies (of genus n say) and such that the handles of H bound embedded discs D_i, $i = 1 \ldots n$ which are disjoint and contained in H' apart from a finite number of clasp singularities.

More precisely, we can consider H as a thickened bunch of circles such that the discs span these circles (think of a bunch of tennis rackets where the string part represents the discs). A **clasp** singularity is an intersection of two of these discs of the type illustrated in figure 2 below.

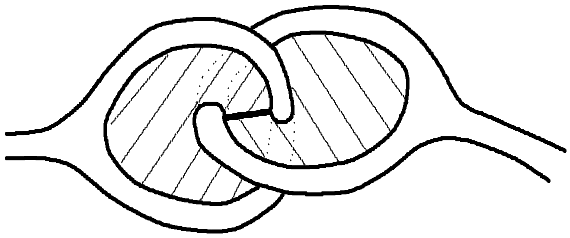

Figure 2

We suppose that the discs D_i meet only in a finite number of clasps and that the D_i meet the interior of H only in the meridional discs at the ends of the clasps. We shall call this arrangement a **standard picture with clasps**.

Theorem 2 *Clasp characterisation of the three sphere*

Suppose that the 3–manifold M^3 has a standard picture with clasps, and denote the clasps by α_i, $i = 1 \ldots t$ say.

Suppose that the α_i form a *trivial link* in H' (i.e. there are disjoint discs Q_i, $i = 1 \ldots t$ with $\alpha_i \subset \partial Q_i$ and $\partial Q_i - \alpha_i \subset \partial H'$).

Then $M^3 \cong S^3$.

Proof Thicken the clasps to form additional 1–handles and continue to denote by H the old handlebody H with these additional handles added (and by H' the complementary handlebody). Next excise the intersections of these new 1–handles from the D_i.

We now have two disjoint families of discs in H', namely the discs D_i, $i = 1 \ldots n$ and Q_i, $i = 1 \ldots t$ (of course the D_i may meet the Q_i).

Next we choose new meridional discs for the new 1–handles which are disjoint from the D_i, see figure 3 below, see also [H; figure 1].

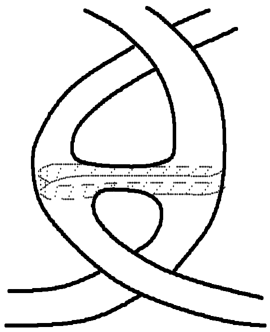

Figure 3

Alternatively, we can think of this new choice of meridional disc as the result of sliding the adjacent part of the handlebody over the new 1–handle as suggested in figure 4.

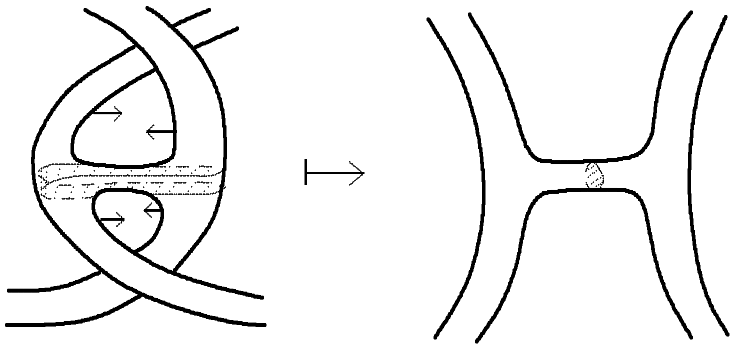

Figure 4

This has the effect that, if we write $H = H_0 \cup h_1 \cup h_2 \cup \ldots \cup h_t$ where H_0 has genus n and h_i are new 1–handles (after the slides), then the D_i cancel the handles of H_0 i.e. $H_0 \cup \bigcup_i N(D_i) \cong B^3$, where $N(D_i)$ is a regular neighbourhood of D_i in H'. It also changes the intersection of the ∂Q_i with these meridional discs by introducing several cancelling pairs of intersections. The situation now satisfies the hypotheses of theorem 1.

Concluding remarks

Theorem 2 is stronger than the basic characterisation of S^3 on which the Rêgo–Rourke programme for proving the Poincaré conjecture was based [R–R; theorem 4.1]. This latter characterisation had an extra hypothesis the *ordering condition*. Theorem 2 can moreover be extended to more general types of singularities than clasps. It can be extended to cover *ribbons*, *base intersections* and *tags* (see [R–R; section 5]). To be precise the proof can be extended to give a proof of [R–R; theorem 5.2], with no ordering condition. The proof is very similar (in particular theorem 1 needs no alteration) and we leave the interested reader to contruct it.

However the proof of theorem 2 is far less robust than the Rêgo–Rourke proof because the ordering condition also gives a strong measure of control over the order in which handles are cancelled which is lacking from the proof of theorems 1 and 2. This measure of control allows the proof to cope with more general singularities (for example triple points) by simplifying the neighbourhoods of these singularities first. A similarly robust proof of theorem 2 would lead at once to a proof of the Poincaré conjecture.

Given a homotopy 3–sphere, it is easy to fulfil all the hypotheses of theorem 2 *except* the hypothesis that the clasps should form a trivial link. It seems very difficult to achieve this final hypothesis so an attempt to prove the Poincaré conjecture using the methods of this paper will entail a substantial strengthening of theorem 1.

Now theorem 1 can in fact be readily strengthened in various ways : Hypothesis (4) can be weakened to allow $w_i = \omega(\partial Q_i) \searrow l_i$ where $l_1, l_2, \ldots, l_t$ is *any* generating set of one or two letters. Moreover the proof can be loosened to allow more than t Q–discs to be carried forward provided the $\overline{w}_i$ cancel (as in property (D)) to part of a generating set, the redundant discs being discarded at the end of the process rather than in the middle. Here again we leave the interested reader to contruct the proofs.

However to prove the Poincaré conjecture what is really needed is some extension of the methods of this paper to cope with generating sets with a limited number of *three* letter words in addition to words of length one and two. This seems difficult.

References

[H] **Wolfgang Haken**, *Algebraically trivial decompositions of homotopy 3–spheres*, Illinois J., 12(1968), 133–170

[R–R] **Eduardo Rêgo and Colin Rourke**, *A proof of the Poincaré conjecture*, Warwick University preprint, September 1986

[Z] **Heiner Zieschang**, *On simple systems of paths on complete pretzels*, A.M.S. Translations, Series 2, 92(1970), 127–137

Colin Rourke
Mathematics Institute
University of warwick
Coventry CV4 7AL UK