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Introduction to Piecewise-Linear Topology

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Preface

The first five chapters of this book form an introductory course in piecewise-linear topology in which no assumptions are made other than basic topological notions. This course would be suitable as a second course in topology with a geometric flavour, to follow a first course in point-set topology, and perhaps to be given as a final year undergraduate course.

The whole book gives an account of handle theory in a piecewise-linear setting and could be the basis of a first year postgraduate lecture or reading course. Some results from algebraic topology are needed for handle theory and these are collected in an appendix. In a second appendix are listed the properties of Whitehead torsion which are used in the s-cobordism theorem. These appendices should enable a reader with only basic knowledge to complete the book.

The book is also intended to form an introduction to modern geometric topology as a research subject, a bibliography of research papers being included.

We have omitted acknowledgements and references from the main text and have collected these in a set of "historical notes" to be found after the appendices.

We are planning eventually to write a further book which will include the topics of embedded handle theory, normal bundles, transversality and p.l. bordism and cobordism theory. For present reading on these topics, see the bibliography.

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Chapter 1. Polyhedra and P.L. Maps

In this chapter we introduce the main objects of study, polyhedra and p.l. maps. The chapter consists mostly of definitions, examples, and exercises. In a final section we introduce the main results of the book: the Poincaré conjecture and the h-cobordism theorem. This section may be omitted until after Chapter 5 if the reader wishes; we have included it here to give a taste of deeper results.

Basic Notation

A *map* is a continuous function. $\mathrm{cl}(X)$ denotes the closure of X. $\mathbb{R}$ denotes the real numbers and $\mathbb{R}^n$ (Euclidean n-space) the space of n-vectors $\{x=(x_1, x_2, \ldots, x_n)\}$ of real numbers. We will use the product metric on $\mathbb{R}^n$ given by $d(x,y)=\sup|x_i-y_i|$. "Linear" always means linear in the affine sense; thus a *linear subspace* (or just subspace) $V\subset\mathbb{R}^n$ is a translated vector subspace, or equivalently: for each finite set $\{a_i\}\subset V$ and real numbers λ_i with $\sum\lambda_i=1$ we have $\sum\lambda_i a_i\in V$. A map $f\colon V\to\mathbb{R}^m$ is *linear* if $f(\sum\lambda_i a_i)=\sum\lambda_i f(a_i)$.

Joins and Cones

Let $A, B\subset\mathbb{R}^n$ be subsets. Define their *join* AB to be the subset $AB=\{\lambda a+\mu b\,|\,a\in A,\ b\in B\}$ where $\lambda,\mu\in\mathbb{R}$, $\lambda,\mu\geqq 0$ and $\lambda+\mu=1$. Then AB consists of all points on straight-line segments "arcs" with endpoints in each of A and B. If $A=\emptyset$ we define $AB=B$.

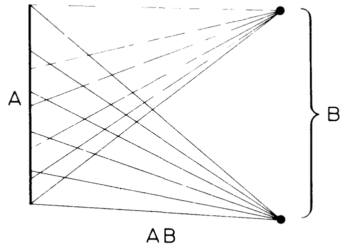


Fig. 1

If $A=\{a\}$ is a one-point set then we often abbreviate $\{a\}$ to a. We say that aB is a *cone* with *vertex* a and *base* B (or simply that aB is a cone) if each point is expressed *uniquely* as $\lambda a+\mu b$ with $b\in B$, $\lambda, \mu \geqq 0$ and $\lambda+\mu=1$. Equivalently $a\notin B$ and the arcs ab_1 and ab_2, for each pair of distinct points $b_1, b_2 \in B$, meet only at a.

Example

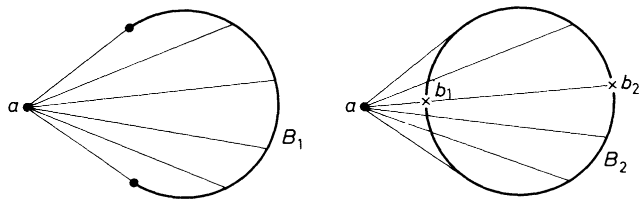


Fig. 2

aB_1 is a cone while aB_2 is not. The example makes it clear that the property of being a cone depends on the presentation of the set aB.

Polyhedra

1.1 A subset $P\subset \mathbb{R}^n$ is a *polyhedron* if each point $a\in P$ has a cone neighbourhood $N=aL$ in P, where L is compact; N is called a *star* of a in P and L a *link* and we write $N=N_a(P)$, $L=L_a(P)$. Note that the case $L=\emptyset$ is not excluded so that a point is a polyhedron.

Examples of Polyhedra

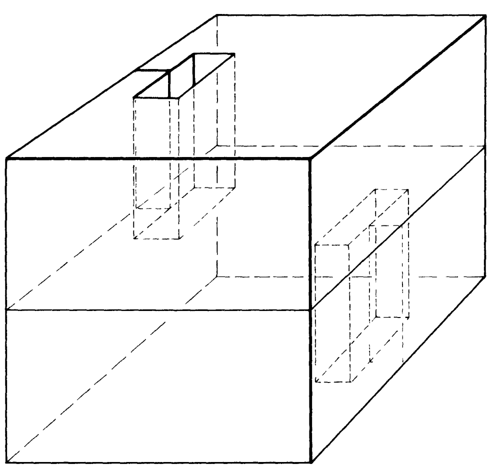


Fig. 3. A house with 2 rooms, each having one entrance

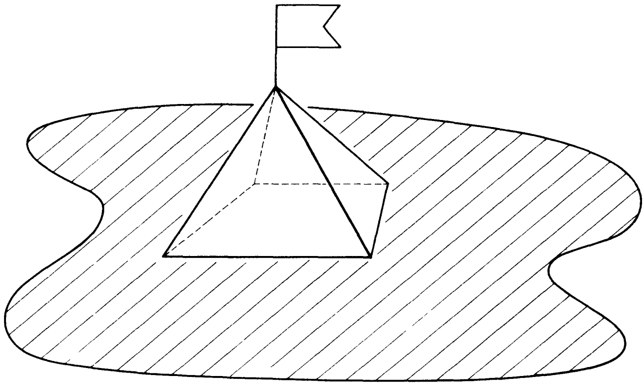


Fig. 4. A pyramid with a flag sitting on an infinite plane

Examples of Non-Polyhedra

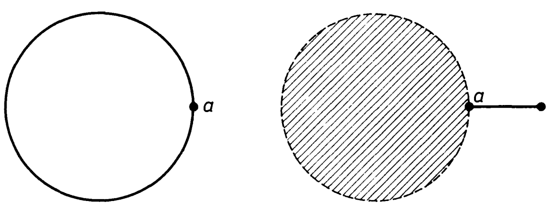


Fig. 5

A circle C X = an open disc with a tail

In the first example a has no cone neighbourhood in C. In the second example a has a cone neighbourhood $aL \subset X$ but L is non-compact. However $X-a$ is a polyhedron! More examples are given in 1.3, below.

1.2 *Remark.* In 1.1 we could take N to be the ε-neighbourhood $N_\varepsilon(a, P)=\{x \mid x \in P,\ d(a, x) \leqq \varepsilon\}$ and L to be $\dot{N}(a, P)=\{x \mid x \in P,\ d(a, x)=\varepsilon\}$ for some suitably small $\varepsilon>0$. For given any cone neighbourhood $N=aL$ of a in P, use compactness of L to find an $\varepsilon>0$ such that $d(a, L) \geqq \varepsilon$ then it is easy to see that $N_\varepsilon(a, P)=a\dot{N}_\varepsilon(a, P)$ is a cone.

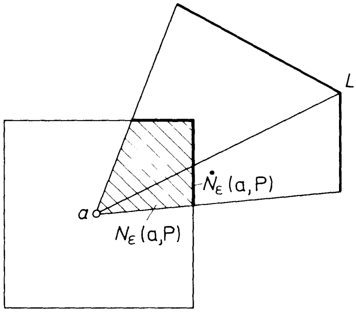


Fig. 6

1.3 *Examples and exercises*

(1) $\mathbb{R}^n$ is a polyhedron. The subset $\mathbb{R}^n_+ \subset \mathbb{R}^n$, defined by $x_n \geqq 0$, is a polyhedron. Subspaces of $\mathbb{R}^n$ are polyhedra.

(2) An open subset of a polyhedron is a polyhedron.

(3) The intersection of finitely many polyhedra is a polyhedron. (Use 1.2.)

(4) Let $P_1, P_2 \subset \mathbb{R}^n, \mathbb{R}^m$ be polyhedra and identify $\mathbb{R}^n \times \mathbb{R}^m$ with $\mathbb{R}^{n+m}$ by $(x, y) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_m)$ then $P_1 \times P_2 \subset \mathbb{R}^{n+m}$ is a polyhedron. For if $a_1 L_1$, $a_2 L_2$ are cone ε-neighbourhoods then so is $a_1 L_1 \times a_2 L_2$.

(5) Let $P = \bigcup P_\alpha$ where $P_\alpha \subset \mathbb{R}^n$ are compact polyhedra and the union is *locally finite* in the sense that each point $p \in P$ has a neighbourhood meeting only finitely many of the P_α. Then P is a polyhedron. (Use 1.2.)

(6) *Cubes.* Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then $N_\varepsilon(a, \mathbb{R}^n) = [a_1 - \varepsilon, a_1 + \varepsilon] \times \cdots \times [a_n - \varepsilon, a_n + \varepsilon]$ is a polyhedron by (4), called a "cube". A *face* of $N_\varepsilon(a, \mathbb{R}^n)$ is obtained by replacing each factor $[a_i - \varepsilon, a_i + \varepsilon]$ either by itself or by $\{a_i - \varepsilon\}$ or $\{a_i + \varepsilon\}$, and then the faces are also polyhedra by (4) and hence $\dot{N}_\varepsilon(a, \mathbb{R}^n)$ which is the union of the *proper* faces (i.e. the faces not equal to the cube) is a polyhedron by (5).

We write I^n for the *unit n*-cube $[-1, 1]^n = N_1(0, \mathbb{R}^n)$ and $\dot{I}^n = \dot{N}_1(0, \mathbb{R}^n)$ for its *boundary*. $I^1 = [-1, 1] \subset \mathbb{R}$ should not be confused with the *unit interval* $I = [0, 1] \subset \mathbb{R}$.

(7) A cone aP on a compact polyhedron P is itself a compact polyhedron. For let $x \in aP$, then if $x = a$ we can take $N_x(aP) = aP$ and if $x \neq a$ we can take $N_x(aP) = a N_y(P)$ where $x = \lambda a + \mu y$, $y \in P$; since we have $a N_y(P) = x(N_y(P) \cup a L_y(P))$ when $x \neq y$, and $a N_y(P) = y(a L_y(P))$ when $x = y$. See Fig. 7.

(8) Suppose $P \subset V$ is a polyhedron in a subspace and $f\colon V \to \mathbb{R}^m$ is linear and injective then $f(P)$ is a polyhedron.

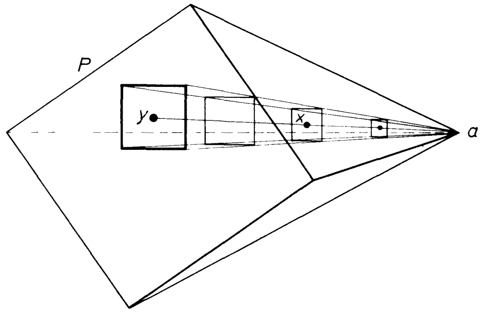


Fig. 7

By 1.2 and examples (3) and (6) we can assume that all links and stars are polyhedra. This we do from now on.

Piecewise-Linear Maps

1.4 A map $f\colon P \to Q$ between polyhedra is *piecewise-linear* (abbreviated p.l.) if each point $a \in P$ has a star $N = aL$ such that $f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$ where $x \in L$ and $\lambda, \mu \geqq 0$, $\lambda + \mu = 1$. In other words, f is locally conical, in the sense that it maps rays of the local cone structure linearly.

1.5 *Examples*

(1) A linear map is p.l.

(2) The restriction of a p.l. map to a subpolyhedron is p.l. A *subpolyhedron* is a subset which is itself a polyhedron.

(3) Define $f\colon P \to Q$ to be *linear* if it is the restriction of a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Then, combining (1) and (2), f is p.l.

(4) Let $P = \bigcup P_\alpha$ be a locally finite decomposition of P into compact subpolyhedra. If $f\colon P \to Q$ is a map such that $f|P_\alpha$ is p.l. for each α, then f is p.l.

Remark. Combining examples (3) and (4), we see that a map which is linear in pieces is p.l. In Chapter 2 we prove that all p.l. maps are obtained in this way, and this explains the terminology.

1.6 *Exercises*

(1) The cartesian product of two p.l. maps is p.l.

(2) The composition of two p.l. maps is p.l.

(3) *The cone construction.* Let aP, bQ be cones and $f\colon P \to Q$ a map.

Define the *cone* on f, f': $aP \to bQ$ by $f'(\lambda a+\mu x)=\lambda b+\mu f(x)$ where $x \in P$. Prove that the cone on a p.l. map or homeomorphism is itself a p.l. map or homeomorphism.

(4) A map f: $P \to Q$ is p.l. if and only if the *graph* of f

$$\Gamma(f)=\{(x, f(x)) \in \mathbb{R}^{n+m} \mid x \in P\}$$

is a polyhedron.

Hint: $\lambda(x, f(x))+\mu(y, f(y))=(z, f(z))$ for some z if and only if $f(\lambda x+\mu y)=\lambda f(x)+\mu f(y)$.

(5) Show that the inverse of a p.l. homeomorphism is again p.l.

P.l. homeomorphism is the principal equivalence relation of p.l. topology, and properties preserved under p.l. homeomorphism are called p.l. *invariants.* We will often use the symbol $\cong$ for a p.l. homeomorphism.

1.7 *Exercises*

(1) Give examples to show

(a) The union of two polyhedra is not necessarily a polyhedron.

(b) The infinite union of compact polyhedra is not necessarily a polyhedron.

(c) The image of a non-compact polyhedron under an injective p.l. map need not be a polyhedron. What about compact polyhedra, and general p.l. maps? (See 2.5 for answers.)

(2) Show by radial projection that the (topological) homeomorphism class of $L_a(P)$ is a p.l. invariant of the pair (a, P).

The Standard Mistake

The last exercise prompts the observation that *projection maps are not necessarily p.l.* For example the graph of a projection of one arc into another is part of a hyperbola.

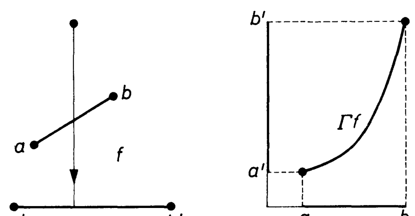


Fig. 8

A p.l. version of exercise (2) will be given in Chapter 2, using "pseudo-radial projection".

P.L. Embeddings

Exercise 1.7(c) shows that we have to be careful about defining p.l. embeddings. We say that a p.l. map $f: P \to Q$ is a *p.l. embedding* provided $f(P)$ is a subpolyhedron of Q and $f: P \to f(P)$ a p.l. homeomorphism.

***Convention.* From now on we will usually omit the prefix p.l.**

Thus map, embedding, homeomorphism will mean p.l. map etc. When we have need to use non p.l. maps we will use the phrase "topological map" in order to avoid confusion.

Manifolds

1.8 A polyhedron M is an *unbounded p.l. manifold of dimension n* (or simply an n-manifold) if each point $x \in M$ has a neighbourhood in M, which is (p.l.) homeomorphic to an open set in $\mathbb{R}^n$; such a neighbourhood is called a *coordinate neighbourhood.* We often indicate the dimension of an n-manifold M by writing M^n.

M is an *n-manifold with boundary* if each point has a neighbourhood homeomorphic to an open subset of either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Define the *boundary* of M, ∂M, an unbounded $(n-1)$-manifold, to consist of points corresponding to $\mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^n_+$. The boundary is well-defined by 1.7(2) and elementary algebraic topology. This also follows by an easy induction using p.l. invariance of links (2.21(2)).

Terminology. A manifold M is *closed* provided $\partial M = \emptyset$ and M is compact. If M is any manifold, define the *interior* of M, $\operatorname{int} M$, to be $M - \partial M$.

1.9 *Examples and exercises*

(1) $\mathbb{R}^n$, $\mathbb{R}^n_+$ and subspaces of $\mathbb{R}^n$ are manifolds.

(2) An open subset of a manifold is a manifold.

(3) The product of an n-manifold with a q-manifold is an $(n+q)$-manifold.

Hint: Define a homeomorphism of $\mathbb{R}^2_{++} = \{x \in \mathbb{R}^2, x_1 \geqq 0, x_2 \geqq 0\}$ onto $\mathbb{R}^2_+$ by using a linear homeomorphism of $\mathbb{R}^2_{+++} = \{x \in \mathbb{R}^2, x_1 \geqq x_2 \geqq 0\}$ onto $\mathbb{R}^2_{++}$. Use this on suitable coordinates to define a homeomorphism of $\mathbb{R}^n_+ \times \mathbb{R}^q_+$ onto $\mathbb{R}^{n+q}_+$.

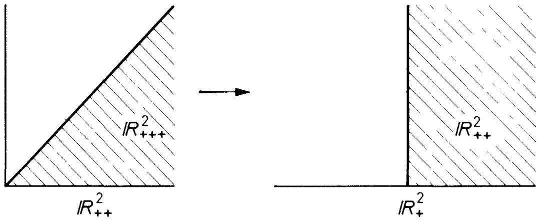


Fig. 9

(4) It follows from (3) that I^n is an n-manifold with boundary.

(5) $\mathbb{R}^n \cong \partial I^{n+1}$ – one point. (This is difficult, see 3.20 for a proof using machinery.)

Balls and Spheres

A manifold homeomorphic with I^n is called an *n-ball* or *n-disc* often written B^n or D^n. A manifold homeomorphic with ∂I^{n+1} is called an *n-sphere*, usually written S^n.

1.10 Lemma. *Let B^n, D^n be n-balls and $h: \partial B^n \to \partial D^n$ a homeomorphism. Then h extends to a homeomorphism h_1 of B^n with D^n.*

Proof. We can assume $B^n = D^n = I^n$ and then define $h_1(\lambda x) = \lambda h(x)$ for $x \in \dot{I}^n$ and $0 \leqq \lambda \leqq 1$. This is the cone construction applied to $I^n = 0\dot{I}^n$.

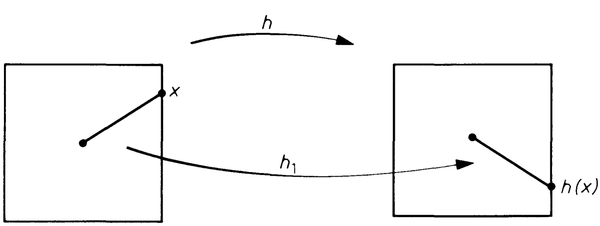


Fig. 10

The Poincaré Conjecture and the *h*-Cobordism Theorem

We now state the main theorems for which we are heading.

Poincaré conjecture. Let M^n be a closed manifold having the homotopy type of an n-sphere, then M is an n-sphere.

Theorem A. *The conjecture is true for* $n \geqq 6$.

In fact the conjecture is true for $n=5$, but the proof at the moment is beyond the scope of an elementary treatment. For $n=3,4$ the conjecture is still, at the time of writting, unsolved.

We will deduce Theorem A from the h-cobordism theorem (Theorem B below). A *cobordism* (W^w, M_0, M_1) consists of a compact manifold W with ∂W the disjoint union of manifolds M_0 and M_1. When M_0 and M_1 are understood, we refer to W itself as a cobordism. W is an h-cobordism if both inclusions $M_0 \subset W$ and $M_1 \subset W$ are homotopy equivalences.

Theorem B. *Suppose* W^w *is a simply connected h-cobordism and* $w \geqq 6$. *Then* $W \cong M_0 \times I$ *and hence* $M_0 \cong M_1$.

Remark. If M_0, M_1 and W are all simply-connected, then by Whitehead's theorem (see Appendix A) it is enough to assume that all the relative homology groups $H_*(W, M_0)$ and $H_*(W, M_1)$ vanish. But by Lefschetz duality (see appendix and proof given in Chapter 5) it is enough to assume this for one end only. Consequently we can state Theorem B in the following form, which is the form in which it will be proved.

Theorem B′. *Suppose* (W^w, M_0, M_1) *is a cobordism and that*

(1) $\pi_1(M_0)=\pi_1(M_1)=\pi_1(W)=0$

(2) $H_*(W, M_0)=0$

(3) $w \geqq 6$.

Then $W \cong M_0 \times I$.

We shall also prove a relative version of the theorem (for cobordisms between manifolds with boundary) and a version for non-simply

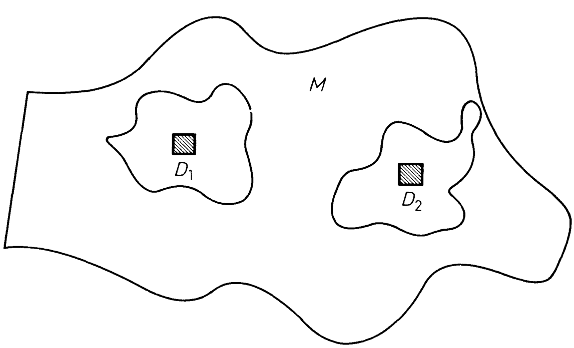


Fig. 11

connected manifolds (the s-cobordism theorem). We conclude this chapter by showing that Theorem A follows from Theorem B′:

In M choose two disjoint standard n-cubes inside coordinate neighbourhoods. Call them D_1 and D_2.

Denote $W_1 = \mathrm{cl}(M - D_1)$ and $W = \mathrm{cl}(W_1 - D_2)$. Then W_1 and W are manifolds since $\mathrm{cl}(\mathbb{R}^n - I^n)$ is a manifold by an exercise on the lines of 1.9(3). We claim that W is an h-cobordism between ∂D_1 and ∂D_2. First of all $\pi_1(\partial D_1) = \pi_1(\partial D_2) = 0$ and $\pi_1(W) = \pi_1(M) = 0$ since W has the homotopy type of $M - \{\text{two points}\}$.

Now

$$\begin{aligned} H_*(W, \partial D_2) &\cong H_*(W_1, D_2) && \text{(excision)} \\ &\cong \tilde{H}_*(W_1) && \text{(since } D_2 \text{ is contractible).} \end{aligned}$$

But

$$\begin{aligned} H_*(W_1) &\cong H^{n-*}(W_1, \partial D_1) && \text{(Lefschetz duality)} \\ &\cong H^{n-*}(M, D_1) && \text{(excision)} \\ &\cong \tilde{H}^{n-*}(M) \\ &\cong \left.\begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}\right\} && \text{since } M \text{ is a homotopy sphere.} \end{aligned}$$

It follows that $\tilde{H}_*(W_1) = 0$ and hence that W is an h-cobordism.

By Theorem B′ there is a homeomorphism $h\colon W \to \dot{I}^n \times I^1$ and we extend h to a homeomorphism of M with $\dot{I}^{n+1}$ by two applications of 1.10.

Chapter 2. Complexes

In this chapter we introduce the principal tools of p.l. topology: simplicial complexes and simplicial maps. The connections between these and polyhedra and p.l. maps is the major concern of the chapter. The rest of the chapter deals with other useful tools: pseudo-radial projection, joins and collars. The results on convex cells which we need are given in an appendix to the chapter.

Simplexes

2.1 Proposition. *The join operation is associative and commutative and*

$$A_0 A_1 \dots A_n = \{\sum \lambda_i a_i \mid \lambda_i \geqq 0, \sum \lambda_i = 1, a_i \in A_i\}.$$

Proof. Define $A_0 A_1 \dots A_n$ inductively to be $(A_0 \dots A_{n-1}) A_n$ and prove the identity inductively. Associativity and commutativity then follow. The induction step follows from the equation

$$\sum \lambda_i a_i = (1-\lambda_n)\left(\left(\frac{\lambda_0}{1-\lambda_n}\right) a_0 + \dots + \left(\frac{\lambda_{n-1}}{1-\lambda_n}\right) a_n\right) + \lambda_n a_n.$$

Now define a finite set $\{v_0, v_1, \dots, v_n\} \subset \mathbb{R}^m$ to be *independent* if it is not contained in any subspace of dimension $<n$, or equivalently if the vectors $\{v_i - v_0\}$ are linearly independent. Then define an *n-simplex* $A \subset \mathbb{R}^m$ to be the repeated join $v_0 v_1 \dots v_n$ of $n+1$ independent points. We call the points v_i the *vertices* of A and say that they *span* A. A simplex spanned by a subset of the vertices is called a *face* of A. If B is a face of A we write $B < A$. B is a *proper* face if also $B \neq A$. The vertices are also regarded as faces. The point $\hat{A} = \sum \frac{1}{n+1} v_i$ is the *barycentre* of A. Note that a simplex is a compact polyhedron by induction since it is the cone on an $(n-1)$-simplex (see 1.3(7)). The empty set is regarded as a (-1)-simplex, has no vertices, and is thus a face of all simplexes.

Exercises

Let A be an n-simplex in $\mathbb{R}^m$.

(1) Show that A is contained in a unique minimal subspace V of $\mathbb{R}^m$ of dimension n. We write $V = \langle A \rangle$ and say A *spans* V. Note that if A is a 0-simplex $A = \langle A \rangle$.

(2) Define $\mathring{A}$, $\dot{A}$ to be the interior and frontier of A in $\langle A \rangle$ and show that $\mathring{A} \neq \emptyset$ and $\dot{A} = \bigcup \{B \mid B < A, B \neq A\}$.

(3) Show that $v \in A$ is a vertex if and only if $L \cap A$ is not a neighbourhood of v in L for every line L through v in $\mathbb{R}^m$.

2.2 Theorem. *A compact polyhedron is a finite union of simplexes. In general a polyhedron is a locally finite union of simplexes.*

Proof. Let P be a compact polyhedron in $\mathbb{R}^m$ and define the subspace $\langle P \rangle \subset \mathbb{R}^m$ *spanned* by P to be the intersection of all subspaces V with $P \subset V$. The proof is by induction on $n = \text{dimension}\,\langle P \rangle$. Without loss of generality we may assume (cf. 1.3(8)) that $P \subset \mathbb{R}^n$. Let $a \in P$ and suppose aL is a star of a in P, which is also an ε-neighbourhood (1.2). Now let F be a proper face of the cube $N_\varepsilon(a, \mathbb{R}^n)$ (cf. 1.3(6)) then $\dim \langle F \cap L \rangle < n$ and hence $F \cap L$ is a finite union of simplexes. Since $L \subset \dot{N}_\varepsilon(a, \mathbb{R}^n)$ it follows that L is a finite union of simplexes, say $L = \bigcup_i A_i$. Then $aL = \bigcup_i aA_i$ is also a finite union. The result now follows by compactness.

In the general case the idea is to use local compactness to decompose P into a locally finite union of cone ε-neighbourhoods, each of which is a finite union of simplexes by the first half. We will list the steps in the proof and leave the details to the reader:

(1) Find a countable base $\{U_i\}$ of ε-neighbourhoods for $\mathbb{R}^n$.

(2) Define $U_{i_1}, U_{i_2}, \ldots$ by U_j is a U_{i_k} if $U_j \cap P$ is non-empty and compact.

(3) By taking suitable increasing unions of the U_{i_k} find compact polyhedra $A_1, A_2, A_3, \ldots$ so that $P = \bigcup_i A_i$ and $A_i \subset$ interior of A_{i+1} in P for each i.

(4) Show that a finite cover of A_i extends to one of A_{i+1} so that no new neighbourhood meets A_{i-1}.

Exercise (Dimension). Define the dimension of an n-simplex to be n and in general define the dimension of a polyhedron by $\dim(P) = \max. \dim(A_i)$, where $P = \bigcup A_i$ is the decomposition of 2.2. Check that dimension is well-defined.

2.3 Corollary. *Let $f\colon P \to Q$ be p.l., then there is a locally finite decomposition of P into simplexes, $P = \bigcup A_i$, such that $f \mid A_i$ is linear for each i.*

Proof. Apply 2.2 to Γf.

2.4 Lemma. *The linear image of a simplex is a polyhedron.*

Proof. Let $A \subset \mathbb{R}^m$ be an n-simplex and let $f: \mathbb{R}^m \to \mathbb{R}^p$ be linear. Then either $f|\langle A\rangle$ is injective, in which case $f(A)$ is an n-simplex, or else $f(\langle A\rangle)$ is a subspace of lower dimension, in which case $f(A)=f(\dot{A})$. [For if $x \in f(A)$ then $f^{-1}(x) \cap \langle A\rangle$ is a subspace of dimension >0 which must meet the frontier of A in $\langle A\rangle$, namely $\dot{A}$.] In this case the result follows by induction since $\dot{A}$ is a union of simplexes of dimension $<n$.

2.5 Corollary (cf. 1.7). *The image of a compact polyhedron under a p.l. map is a compact polyhedron.*

Proof. By 2.3 and 2.4 it is a finite union of compact polyhedra.

Cells

A subset $C \subset \mathbb{R}^m$ is *convex* if for each pair of points $a, b \in C$, the arc $ab \subset C$.

A compact convex polyhedron which spans a subspace of dimension n is called a *linear n-cell* or just a *cell*.

2.6 *Examples and remarks*

(1) An n-simplex is an n-cell; I^n is an n-cell. A 0-cell is a point and a 1-cell is an arc.

(2) The linear image of a cell is a cell (convexity is obvious, polyhedron by 2.5).

(3) Let $\{a_0, a_1, \ldots, a_r\} \subset \mathbb{R}^m$ be any finite set, then their join $a_0 a_1 \ldots a_r = \{\sum \lambda_i a_i \mid \lambda_i \geqq 0, \sum \lambda_i = 1\}$ is a cell, called the cell *spanned* by $\{a_0, \ldots, a_r\}$. This follows from (2) since there is a linear map from an r-simplex onto the cell.

The converse to (3) is also true, see 2.7 below.

(4) The intersection or product of two cells is a cell.

(5) The intersection of a cell with a subspace or a half space is a cell.

(6) If C is an n-cell then $\dim C = n$. For, since $\dim \langle C\rangle = n$, C contains an independent set of $n+1$ points, and by convexity, it contains the simplex spanned by this set, which could be taken to be one of the simplexes in the decomposition of C given by 2.2.

(7) Define $\mathring{C}$, $\dot{C}$ to be the interior and frontier of C in $\langle C\rangle$. Then $\mathring{C} \neq \emptyset$ by the last remark.

We now define the faces of a cell. Let C be a cell and $x \in C$ an arbitrary point. Define $\langle x, C\rangle$ to be the union of lines L through x in $\mathbb{R}^m$ such that $L \cap C$ (which is a 1-cell or 0-cell and hence either an arc or a point) is an arc with x in its interior. It follows from convexity that $\langle x, C\rangle$ is a subspace of $\mathbb{R}^m$ (the proofs of this fact and of 2.7 below are given in an appendix to this chapter). If there are no such lines then define $\langle x, C\rangle = x$ and call x a *vertex* of C. In general call the cell $\langle x, C\rangle \cap C$ a *face* of C denoted C_x and written $C_x < C$. Thus a vertex is a face and, by 2.6(7), $C < C$. If $D < C$ and $D \neq C$ then say D is a *proper* face of C and then $\dim D < \dim C$ since $\langle x, C\rangle \subsetneqq \langle C\rangle$ where $D = C_x$. The empty set is defined to be a face of all cells.

2.7 Proposition. *Suppose C is an n-cell, then:*

(1) *C has finitely many vertices $v_0, v_1, \ldots, v_r$ which span C.*

(2) *If $F < C$ then F is spanned by a subset of the vertices and hence C has only finitely many faces.*

Warning: Not all subsets of vertices span faces, for example two opposite corners of a square.

(3) $C = \textit{disjoint} \bigcup \{\mathring{F} \mid F < C\}$,
$\dot{C} = \textit{disjoint} \bigcup \{\mathring{F} \mid F < C, F \neq C\}$.

(4) *If $F < D < C$ then $F < C$.*

(5) *If $F, D < C$ then $F \cap D < C$.*

(6) *Let $x \in C$ then $C =$ the cone xB where $B = \bigcup \{F \mid F < C, x \notin F\}$.*

Exercise. Check that the definition of a face of a general cell is compatible with that for a simplex or a cube.

Cell Complexes

A *cell complex* K is a finite collection of cells in some $\mathbb{R}^n$ satisfying

(1) If $C \in K$ and $B < C$ then $B \in K$.

(2) If $B, C \in K$ then $B \cap C$ is a face of both B and C.

Define the *underlying polyhedron* $|K|$ to be the union of the cells of K.

2.8 *Examples and exercises*

(1) A cell C determines two complexes $\{B \mid B < C\}$ and $\{B \mid B < C, B \neq C\}$ (by 2.7). With abuse of notation, we denote these C and $\dot{C}$ respectively.

(2) If K is a cell complex in $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear homeomorphism then $fK = \{fC \mid C \in K\}$ is a cell complex.

(3) Let $G \subset \mathbb{R}^m$ be compact then there is a cell complex K with $G \subset |K|$ and typical m-cell of the form

$$[n_1, n_1 + 1] \times [n_2, n_2 + 1] \times \cdots \times [n_m, n_m + 1]$$

where n_i are integers. In other words K is part of the cube "lattice" in $\mathbb{R}^m$.

(4) If K is a cell complex then $|K| = \text{disjoint} \bigcup \{\mathring{C} \mid C \in K\}$.

(5) If K and L are cell complexes then their *intersection* $M=\{A\cap B|A\in K, B\in L\}$ and their *product* $K\times L=\{A\times B|A\in K, B\in L\}$ are cell complexes.

Hint: First prove $\langle x, A\cap B\rangle=\langle x, A\rangle\cap\langle x, E\rangle$, which implies $(A\cap B)_x=A_x\cap B_x$ and a similar result with $\times$ replacing $\cap$.

(6) If f: $|K|\to|L|$ is linear on each cell of K then $\{A\cap f^{-1}(B)|A\in K, B\in L\}$ is a cell complex.

(7) If $a|K|$ is a cone then $aK=\{a, aB, B|B\in K\}$ is a cell complex, called the *cone* on K.

Now define $L\subset K$ to be a *subcomplex* if L is also a cell complex.

(8) The *r-skeleton* of K, $K^r=\{C|C\in K, \dim C\leqq r\}$, is a subcomplex.

(9) If $C\in K$ then the *star* of C in K, $\mathrm{st}(C,K)=\{B|B<D>C, D\in K\}$ is a subcomplex.

Subdivisions

L is a *subdivision* of K, written $L\lhd K$, if $|L|=|K|$ and each cell of L is contained in a cell of K.

Let $a\in|K|$, we say $K'\lhd K$ is obtained by *starring* at a if K' is obtained from K by replacing each cell $C\in K$ with $a\in C$ by the complex aB where $B=\{F|F<C, a\notin F\}$ (cf. 2.7(6)). The result of starring at points $a_1, a_2, \ldots, a_r\in|K|$ in order is called a *stellar* subdivision of K.

Examples

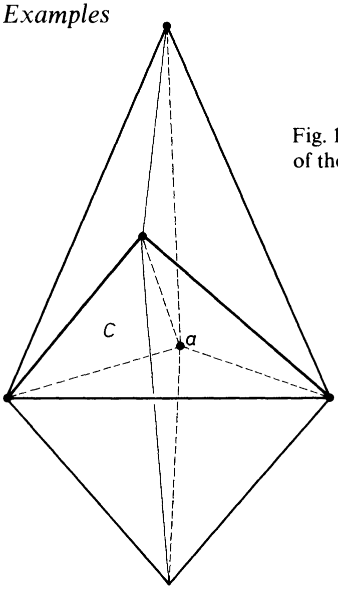


Fig. 12. A stellar subdivision of the union of two tetrahedra

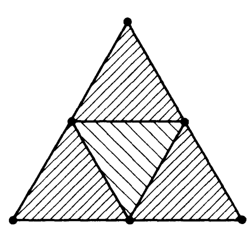


Fig. 13. A subdivision of a 2-simplex which is not stellar

Simplicial Complexes

A cell complex K is *simplicial* if each $C \in K$ is a simplex.

2.9 Proposition. *A cell complex can be subdivided to a simplicial complex without introducing any new vertices.*

Proof. Let K be the cell complex. Order the vertices of K and suppose inductively we have constructed a simplicial subdivision of K^{r-1}. Let C be an r-cell of K and x its first vertex. Let B be the union of faces of C which do not contain x. By induction, B has been subdivided to a simplicial complex and C is the cone xB. This shows how to subdivide C; the ordering ensures compatibility with the subdivision of $\dot{C}$.

Exercise. The subdivision of 2.9 may be described as starring at each vertex of K in turn using the given ordering.

2.10 Corollary. *Given any simplex $A \subset \mathbb{R}^n$ and compact set $G \subset \mathbb{R}^n$, there is a simplicial complex K with $A \in K$ and $G \subset |K|$.*

Proof. Let L be the simplicial subdivision of the cube $[0, 1]^n$ given by 2.9. Since complexes are preserved by linear homeomorphisms (2.8(2)) we may assume $A \in L$. L extends to the required K by 2.8(3) and 2.9.

2.11 Theorem. *Any compact polyhedron is the underlying polyhedron of some simplicial complex.*

Proof. Write $P \subset \mathbb{R}^n$ as the finite union of simplexes $A_1, \ldots, A_r$ by 2.2. By 2.10 find complexes K_i with $A_i \in K_i$ and $P \subset |K_i|$. Then the intersection of the K_i is a cell complex M which contains subcomplexes corresponding to each A_i and hence one corresponding to P. Finally use 2.9 to replace this by a simplicial complex.

2.12 Addendum. *If $|K| \supset |L_i|$ $i = 1, \ldots, r$ then there are simplicial subdivisions $K' \lhd K$ and $L_i' \lhd L$ such that $L_i' \subset K'$, each i.*

Proof. By 2.9 we can assume that K and each L_i are simplicial and then in the proof of 2.11 take the simplexes A_i to be the simplexes of $K, L_1, \ldots, L_r$.

Simplicial Maps

Let K, L be cell complexes and $f\colon |K| \to |L|$ a map. We say (f, K, L) is *cellular* or simply f is cellular if, for each $C \in K$, $f|C$ is linear and $f(C)$ is a cell of L. If K and L are simplicial then say that f is *simplicial.* Note that a cellular map is automatically p.l. by 1.5(4). A cellular homeomorphism is called a *cellular isomorphism* or just an isomorphism. The inverse of an isomorphism is also an isomorphism.

Exercise. A simplicial map is determined by its values on vertices: i.e. if $f\colon K^0 \to L^0$ carries the vertices of each simplex of K into some simplex of L, then f is the restriction of a unique simplicial map.

2.13 Lemma. *Let $f\colon |K| \to |L| \subset \mathbb{R}^n$ be a map which is linear on cells of K. Then there are simplicial subdivisions $K' \lhd K$ and $L' \lhd L$ such that $f\colon |K'| \to |L'|$ is simplicial.*

Proof. Each $f(A_i)$, $A_i \in K$, is a cell by 2.6(2) and by 2.12 we can find simplicial $L' \lhd L$ such that $f(A_i) = |\tilde{A}_i|$ for $\tilde{A}_i \subset L'$. Then

$$K'' = \{A \cap f^{-1} B \mid A \in K,\ B \in L'\}$$

is a cell complex by 2.8(6). Let K' be the simplicial subdivision of K'' given by 2.9 then $f\colon K' \to L'$ is simplicial.

2.14 Theorem. *Let $f\colon |K| \to |L|$ be p.l. then there are simplicial subdivisions $K' \lhd K$, $L' \lhd L$ such that $f\colon |K'| \to |L'|$ is simplicial.*

Proof. By 2.3 we can decompose $|K|$ into a finite union of simplexes $A_1, \dots, A_r$ such that $f|A_i$ is linear. By 2.12 we can find $K'' \lhd K$, $A_i'' \lhd A_i$ such that $A_i'' \subset K''$. f is then linear on simplexes of K'' and the result follows by 2.13.

Convention. From now on "complex" means simplicial complex and letters J, K, L, K', L' etc. denote simplicial complexes. We sometimes write $f\colon K \to L$ for $f\colon |K| \to |L|$ is simplicial.

Triangulations

The last two theorems (2.11 and 2.14) have shown the intimate connection between polyhedra and simplicial complexes – every compact polyhedron underlies some simplicial complex and every p.l. map between compact polyhedra is a simplicial map between suitable complexes.

We now introduce a more general relation between complexes and polyhedra which has the advantage of being p.l. invariant:

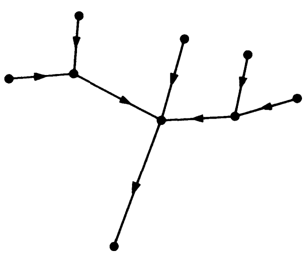
A *triangulation* of a compact polyhedron P is a pair (K, t) where $t\colon |K| \to P$ is a (p.l. as always) homeomorphism. We identify two triangulations of P if they differ by (simplicial) isomorphism, that is if

$$\begin{tikzcd} {|K_1|} \arrow[d, "i"'] \arrow[dr, "t_1"] & \\ {|K_2|} \arrow[r, "t_2"'] & P \end{tikzcd}$$

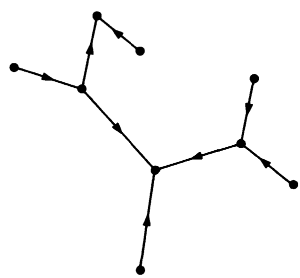
commutes, where i: $K_1 \to K_2$ is an isomorphism. Notice that (K, t) corresponds to a complex $\tilde{K}$ with $|\tilde{K}| = P$ if and only if t is linear on simplexes, and we therefore call such triangulations *linear*. If $K' \lhd K$ then we call the triangulation (K', t) a *subdivision* of (K, t). Any two triangulations of P have a common subdivision by 2.14 applied to the homeomorphism $t_2^{-1} \circ t_1 \colon |K_1| \to |K_2|$. When considering one particular triangulation of P, we will often identify $|K|$ with P via t, and only be more precise when confusion is possible. For example,Theorem 2.14 can now be reinterpreted as a theorem about maps between triangulated polyhedra, and similarly for the general subdivisionTheorem 2.15 below.

Subdividing Diagrams of Maps

A *diagram* D is a finite directed 1-complex, in which each vertex is labelled by a space and each edge by a map between the spaces at its ends. A diagram is a *tree* if it is simply-connected (as a 1-complex) and is a *one-way* tree if each space is the domain of at most one map.



A one-way tree



A tree

Fig. 14

Exercise. Any tree may be constructed by starting with a vertex and inductively adjoining directed edges by identifying one vertex with an existing vertex, and any such construct is a tree.

We will consider diagrams labelled by triangulated polyhedra and p.l. maps and use the convention mentioned above, so that we identify a triangulated polyhedron with the complex which triangulates it. Let D be such a diagram, a subdivision $D' \lhd D$ is obtained by relabelling each vertex by a subdivision of the original label. A diagram D is *simplicial* if each map in D is simplicial.

2.15 Theorem. *Let T be a one-way tree of triangulated polyhedra and p.l. maps then T has a simplicial subdivision T'. If all the maps in T are injective then the one-way condition may be omitted.*

Proof. By induction on the number of maps in T. Find a map f: $|K| \to |L|$ in T such that K is not involved in any other map of T and choose a simplicial subdivision f: $K' \to L'$. Let T_* denote T with K and f omitted and with L' replacing L. By induction T_* has a simplicial subdivision T_*' and $L'' \in T_*'$ where $L'' \lhd L'$. We can find $K'' \lhd K'$ such that f: $K'' \to L''$ is simplicial by Lemma 2.16 below and this gives a simplicial subdivision of T, as required. In the injective case we use the same proof except that we might have f: $K \leftarrow L$ instead of f: $K \to L$. In this case use Lemma 2.17 instead of 2.16.

2.16 Lemma. *Suppose that f: $K \to L$ is simplicial and $L' \lhd L$. Then there is $K' \lhd K$ such that f: $K' \to L'$ is simplicial.*

Proof. The cell complex $K'' = \{A \cap f^{-1} B | A \in K,\ B \in L'\}$ (see 2.8(6)) subdivides K and f: $K'' \to L'$ is cellular. Let $K' \lhd K''$ be the simplicial subdivision of 2.9 then f: $K' \to L'$ is simplicial.

2.17 Lemma. *Suppose that f: $L \to K$ is a simplicial injection and $L' \lhd L$. Then there is $K' \lhd K$ such that f: $L' \to K'$ is simplicial.*

Proof. Identify L with $f(L)$ and choose a point $a_i \in \mathring{A}_i$ for each $A \in K$ such that $A \notin L$. Now define K' inductively over skeleta by the formulae

$$A_i' = \begin{cases} a_i \dot{A}_i' & A_i \notin L, \\ A_i' & A_i \in L,\ A_i' \subset L'. \end{cases}$$

Remark. A more economical subdivision of K extending L' is given later (see 3.4).

Examples and exercises

(1) The "dual" of 2.16 is false. The unit interval I is a simplex and hence can be also considered as a complex. Let $I_{\frac{1}{3}}$ be I with a new vertex at $\frac{1}{3}$ and let $I' \lhd I_{\frac{1}{3}}$ have a further vertex at $\frac{2}{3}$. Then $f(0)=0$, $f(\frac{1}{3})=1$, $f(1)=0$ determines a simplicial map f: $I_{\frac{1}{3}} \to I$ and there is no $I'' \lhd I$ so that f: $I' \to I''$ is simplicial.

(2) The one-way condition in 2.15 cannot be dropped for consider

$$\begin{tikzcd}[row sep=small] & I \\ I \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & I \end{tikzcd}$$

Here f is defined as in (1) and g is similar except that $g(\frac{2}{3})=1$. This tree cannot be triangulated.

(3) Let $f\colon I_{\frac{1}{3}} \to I_{\frac{2}{3}}$ be defined by $f(0)=0$, $f(1)=1$ and $f(\frac{1}{3})=\frac{2}{3}$. Then there is no $I' \lhd I$ such that $f\colon I' \to I'$ is simplicial. Thus loops, even of homeomorphisms, cannot in general be triangulated.

Derived Subdivisions

If in the proof of 2.17 we have $L' = L$ and allow L, K to be cell complexes then $K' \lhd K$ is called a *derived* subdivision of K, obtained by *deriving* K *away from* L. If $L = \emptyset$ then K' is a *first derived*, usually denoted $K^{(1)}$, and an *r-th derived* $K^{(r)}$ is defined inductively by $K^{(r)} = (K^{(r-1)})^{(1)}$. A derived is *barycentric* if each $a_i = \hat{A}_i$. Note that $K^{(1)}$ is always a simplicial complex.

Exercises

(1) Show that $K^{(1)} = \{a_{i_0} a_{i_1} \dots a_{i_r} \mid A_{i_0} < \dots < A_{i_r} \in K\}$.

(2) Show that deriving may be described as starring A_i at a_i in order of decreasing dimension of A_i.

Abstract Isomorphism of Cell Complexes

Cell complexes K, L are *abstractly isomorphic* if there is a bijection $j\colon K \to L$ such that $A < B \in K$ implies $j(A) < j(B)$.

2.18 Lemma. *If $j\colon K \to L$ is an abstract isomorphism of cell complexes then there is a homeomorphism $f\colon |K| \to |L|$ such that $f(A) = j(A)$ for each $A \in K$.*

Proof. Choose deriveds $K^{(1)}$ and $L^{(1)}$ and define the simplicial isomorphism $f\colon K^{(1)} \to L^{(1)}$ by $f(a_i) = b_k$ where $j(A_i) = B_k$.

Notice that f may be regarded as built up by inductive use of the cone construction.

Pseudo-Radial Projection

As promised in Chapter 1 we now prove p.l. invariance of links and stars. Let K be a complex and let $a \in K$ be a vertex. Define

$$\operatorname{lk}(a, K) = \{A \mid A \in K,\ aA \in K,\ a \notin A\}$$

then it is easy to see that $\operatorname{st}(a, K)$ is the cone $a \operatorname{lk}(a, K)$. Thus $|\operatorname{st}(a, K)|$ and $|\operatorname{lk}(a, K)|$ are an example of a link and star of a in $|K|$. Conversely, given a compact polyhedron P with $a \in P$ and a star $N = aL$ of a in P, triangulate $P-(N-L)$ with L a subcomplex and extend to N by taking the cone on L from a. Then $N = |\mathrm{st}(a, K)|$ and $L = |\mathrm{lk}(a, K)|$ in this triangulation. Therefore p.l. invariance of links and stars is a consequence of the following lemma (in the non-compact case, consider a compact neighbourhood of a in P):

2.19 Lemma. *Suppose that $f: (|K|, a) \to (|L|, b)$ is a homeomorphism with $a \in K$, $b \in L$. Then there is a homeomorphism $|\mathrm{lk}(a, K)| \to |\mathrm{lk}(b, L)|$.*

Proof. Let $f_1: (K', a) \to (L', b)$ be a simplicial subdivision of f then $\mathrm{lk}(a, K')$ is isomorphic to $\mathrm{lk}(a, L')$. Therefore it suffices to show $|\mathrm{lk}(a, K')|$ homeomorphic to $|\mathrm{lk}(a, K)|$ for $K' \lhd K$. Let the simplexes of $\mathrm{lk}(a, K')$ be A_i, $i = 1, \ldots, r$, and let A_i^+ be the extended cone on A_i from a defined by

$$A_i^+ = \{\lambda a + \mu b \mid b \in A_i,\ \lambda \leqq 1,\ \mu \geqq 0 \text{ and } \lambda + \mu = 1\}.$$

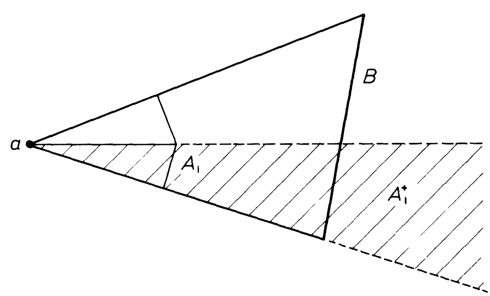


Fig. 15

Then $M = \{A_i^+ \cap B \mid B \in \mathrm{lk}(a, K)\}$ is a simplicial subdivision of $\mathrm{lk}(a, K)$. Moreover (topological) radial projection $|\mathrm{lk}(a, K')| \to |M|$ maps simplexes homeomorphically onto simplexes and hence determines a simplicial isomorphism by restricting to vertices. This isomorphism is referred to as a *pseudo-radial projection.*

2.20 Corollary. *A linear n-cell is an n-ball.*

Proof. Let C be an n-cell then without loss we may suppose $\langle C \rangle = \mathbb{R}^n$. Choose $a \in \mathring{C}$ and $N = aL$ a cone ε-neighbourhood of a in C. Now $C = a\dot{C}$ is also a star of a in C by 2.7(6); it follows that C is homeomorphic to N which is linearly homeomorphic to I^n.

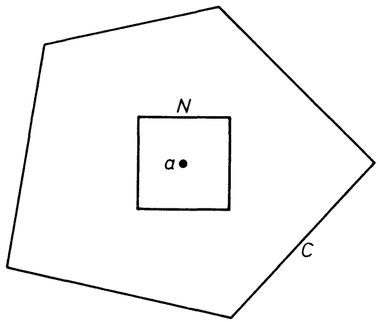


Fig. 16

2.21 *Exercises*

(1) Suppose J is a simplicial complex then $|J|$ is an n-manifold if and only if $|\mathrm{lk}(x, J)|$ is an $(n-1)$ sphere or ball for each vertex $x \in J$.

(2) Prove that $I^n \not\cong \dot{I}^{n+1}$ by induction using 2.19. Deduce that the boundary of a manifold is well-defined.

Remark. By 2.20 a cell C is a manifold with int $C = \mathring{C}$ and $\partial C = \dot{C}$. This means that the two notations for interior and boundary are consistent and we will, from now on, use them interchangeably. For example if M is any manifold then we will write either int M or $\mathring{M}$ for its interior.

External Joins

Let $P, Q \subset \mathbb{R}^n$ be compact polyhedra then PQ is a union of joins of simplexes by 2.1 and hence a union of cells (2.6(3)) and thus also a compact polyhedron. However PQ is not a p.l. invariant of P and Q since it depends on the geometric relationship of P and Q as subsets of $\mathbb{R}^n$; but in the special case that P and Q are *independent* in $\mathbb{R}^n$ we shall see that PQ is a p.l. invariant:

Subsets $A, B \subset \mathbb{R}^n$ are *independent* if each point in AB may be written uniquely in the form $\lambda a + \mu b$, $\lambda, \mu \geqq 0$, $\lambda + \mu = 1$, $a \in A$, $b \in B$. Equivalently $A \cap B = \emptyset$ and the interiors of the arcs $a_1 b_1$ and $a_2 b_2$ are disjoint unless $a_1 = a_2$ and $b_1 = b_2$ where $a_1, a_2 \in A$, $b_1, b_2 \in B$. In particular aB is a cone if and only if a, B are independent. We also define $\emptyset$ and any A to be independent.

2.22 *Exercises and remarks*

(1) If A and B are simplexes then they are independent if and only if their vertices form an independent set. Hence in this case AB is a simplex of dimension $\dim A + \dim B + 1$.

(2) If $|K|, |L|$ are independent then define the *simplicial join* KL to consist of simplexes A, B, AB for $A \in K$, $B \in L$. KL is then a complex of dimension $\dim K + \dim L + 1$.

(3) Let $A \in K$. Define $\mathrm{lk}(A, K) = \{B | AB \in K, A \cap B = \emptyset\}$. Then $|A|$ and $|\mathrm{lk}(A, K)|$ are independent and $\mathrm{st}(A, K) = A\,\mathrm{lk}(A, K)$.

(4) If $f, g\colon A, B \to C, D$ are maps between independent pairs then define the *join* $t\colon AB \to CD$ by $t(\lambda a + \mu b) = \lambda f(a) + \mu g(b)$. Then the join of simplicial maps is simplicial and hence the join of two (p.l.) maps is a (p.l.) map.

(5) Suppose given homeomorphisms $P_0 \cong P_1$, $Q_0 \cong Q_1$, where P_i, Q_i are independent, $i = 0, 1$, then by (4) we have a homeomorphism $P_0\,Q_0 \cong P_1\,Q_1$.

We now define the *external join* of polyhedra $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^m$ denoted $P * Q \subset \mathbb{R}^{n+m+1}$. Let $i_1\colon P \to \mathbb{R}^{n+m+1}$ be defined by

$$i_1(x) = (x_1, \ldots, x_n, 0, 0, \ldots, 0) \quad \text{and} \quad i_2\colon Q \to \mathbb{R}^{n+m+1}$$

by

$$i_2(x) = (0, \ldots, 0, x_1, \ldots, x_m, 1).$$

Then $i_1(P)$ and $i_2(Q)$ are independent and we define $P * Q = i_1(P)\,i_2(Q)$. By (5) above $P * Q$ is homeomorphic to any independent join PQ. Given $f, g\colon P, Q \to P_1, Q_1$ define $f * g\colon P * Q \to P_1 * Q_1$ by (4) above.

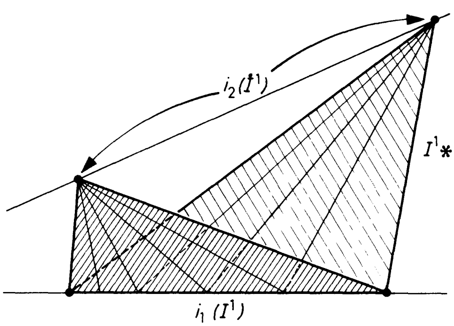


Fig. 17. The external join of I^1 and $\dot{I}^1$

2.23 Proposition. *Joins of balls and spheres obey the rules*

$$B^p * B^q = B^{p+q+1}$$

$$S^p * B^q = B^{p+q+1}$$

$$S^p * S^q = S^{p+q+1}.$$

Proof. The first part follows from the fact that the join of two cells is a cell. For the second half consider the independent subsets $\dot{I}^{p+1} \times 0$,

$0 \times I^q \subset \mathbb{R}^{p+q+1}$. Their join is a cell. For the last part consider the boundary of this example with q replaced by $q+1$.

2.24 *Exercises*

(1) The join operation $*$ is associative and commutative up to a linear isomorphism.

(2) Define independence for a finite number of subsets $A_1, \ldots, A_n$ by uniqueness of the formula of 2.1. Check that this generalises the notion of independence for finite sets of points (defined below 2.1) and that each join $(A_{i_1} \ldots A_{i_r})(A_{i_{r+1}} \ldots A_{i_q})$ is independent.

(3) Show that $|\mathrm{lk}((x, y), P \times Q)| \cong |\mathrm{lk}(x, P)| * |\mathrm{lk}(y, Q)|$ and deduce from 2.21(1) that $X \times \mathbb{R}^1$ is a manifold if and only if X is a manifold.

(4) Show that $|J|$ is an n-manifold implies that $\mathrm{lk}(A, J)$ is a ball or a sphere of dimension $n - \dim A - 1$.

Hint: Use induction and the fact that $\mathrm{lk}(a, \mathrm{lk}(B, J)) = \mathrm{lk}(A, J)$ where B is a top dimensional face of A and a is the opposite vertex.

(5) Show that $A * B = S^n$ implies that both A and B are spheres.

Collars

Let $P \subset Q$ be polyhedra then a *collar* on P in Q is an embedding $c: P \times I \to Q$ such that $c(x, 0) = x$ and such that $c(P \times [0, 1))$ is an open neighbourhood of P in Q; we also call $c(P \times I)$ a collar. We are interested in the existence of collars. An obviously necessary condition is that the collar should exist *locally* i.e. for each $a \in P$ there exist neighbourhoods $N(a, P)$, $N(a, Q)$ with $N(a, Q) = N(a, P) \times I$ where $N(a, P)$ is identified with $N(a, P) \times 0$. This condition is also sufficient; we will prove this in the case when P is compact:

2.25 Theorem. *Suppose $P \subset Q$ is locally collared and compact. Then there is a collar on P in Q.*

Proof. Suppose $Q \subset \mathbb{R}^n$ and define $Q_+ = Q \times 1 \cup P \times I \subset \mathbb{R}^n \times \mathbb{R}^1$. Then Q_+ can be regarded as Q with a collar added to P "on the outside".

We will construct a homeomorphism $h: Q \to Q_+$ by "pushing" along the I-lines of $P \times I$ such that $h|P: P \to P \times 0$ is the identity. Then h^{-1} of the natural collar on $P \times 0$ in $P \times I$ gives a collar on P in Q. We will now describe one local "push": Let $a \in P$ and let $N(a, Q) = N(a, P) \times [1, 2]$ be the neighbourhoods given by local collaring, and assume without loss that $N(a, P) = N_a(P)$ and $N_a(P) \times [1, 2]$ are stars. Then we have $N_a(P) \times [0, 2]$ embedded in Q_+ with $N_a(P)$ identified with $N_a(P) \times 1$. Define a self homeomorphism of $N_a(P) \times [0, 2]$ by regarding it as a cone with base

$L_a(P) \times [0, 2] \cup N_a(P) \times \partial[0, 2]$ and vertex $(a, 1)$. Move the vertex from $(a, 1)$ down to $(a, \frac{1}{2})$ and extend by the cone construction. Call the resulting homeomorphism of Q_+, given by extending by the identity, h_a.

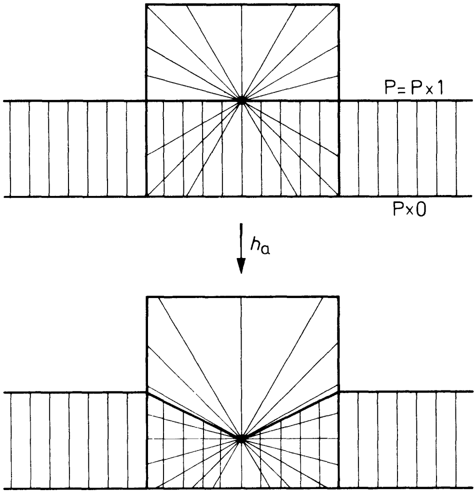


Fig. 18

Now, using local collarability and compactness of P, find a set $N_{a_i}(P)$ of stars, $i = 1, 2, \ldots, t$, for each of which the local push described above exists, and such that $\bigcup_i (N_{a_i}(P) - L_{a_i}(P)) = P$. Then define the homeomorphism h': $Q_+ \to Q_+$ to be the composition $h_{a_t} \circ h_{a_{t-1}} \circ \cdots \circ h_{a_1}$, i.e. the result of doing each push in order.

Notice that each push h_a carries the point (x, s) to (x, s') where $s' \leqq s$ and $s' < s$ if $s \neq 0$ or 2 and $x \in N_a(P) - L_a(P)$. Therefore for each $x \in P = P \times 1$ we have $h'(x) = (x, t)$ where $0 < t < 1$.

Now let $T = h'(Q) \cap P \times I$, i.e. the part of $P \times I$ "above" $h'(P)$; we show how to "stretch" T onto $P \times I$ by a homeomorphism g which is the identity on $P \times 1$ and carries $h'(P)$ to $P \times 0$ and then, after extending g by the identity to Q, we can define $h = g \circ h'$. Consider the projection $p: h'(P) \subset P \times I \to P$ and triangulate P by a complex K so that p is simplicial. Then for each $A \in K$ we have the cell $A_+ = T \cap A \times I$; and T becomes a cell complex by taking cells A_+ with their faces. T is then abstractly isomorphic with $K \times I$; so the required homeomorphism is given by 2.18, and we observe that the proof of 2.18 allows us to assume $g|P \times 1 = \text{id}$ and g: $h'(P) \to P \times 0$ is the obvious map.

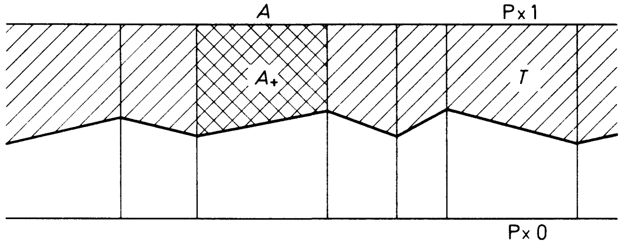


Fig. 19

Remark. More general collaring theorems will be proved in Chapter 4. We leave it as an exercise to remove the compactness condition on P.

Hint: Define h' similarly, using a locally finite set of pushes. Then construct g inductively over compact pieces.

2.26 Corollary. *Let M be a manifold with ∂M compact then ∂M may be collared in M.*

Proof. Local collaring is implied by the definition of a manifold.

2.27 Final exercises

(1) *Abstract simplicial complexes.* Given a simplicial complex K we can "abstract" the information

(i) vertex set of K
(ii) the subsets of this set which span simplexes.

This suggests defining an abstract simplicial complex to consist of

(i) a finite set K^0
(ii) a family K of subsets of K^0 (the simplexes)

such that

(a) if $\tau \subset \sigma \in K$ then $\tau \in K$. It then follows that
(b) if $\sigma, \tau \in K$ then $\sigma \cap \tau \in K$.

Prove that K can be *realised* as a simplicial complex in $\mathbb{R}^{|K^0|-1}$ and that any two realisations are isomorphic. (*Hint:* Realise the vertices independently and use the exercise above 2.13 for the second half.)

(2) *Gluing.* Let $P_0 \subset P$, $Q_0 \subset Q$ be polyhedra and h: $P_0 \to Q_0$ a homeomorphism. Define $P \cup_h Q$ to be the (topological) space obtained by identifying P_0 with Q_0 by h. Prove that $P \cup_h Q$ can be embedded as a polyhedron in $\mathbb{R}^n$ for some n so that the natural maps $P \to P \cup_h Q$ and $Q \to P \cup_h Q$ are p.l. embeddings. (*Hint:* Triangulate everything and use exercise (1).)

(3) *Abstract polyhedra.* Let P be a topological space and e_α: $P_\alpha \to P$ topological embeddings where P_α are polyhedra and the e_α are p.l. related

in the sense that $e_\alpha^{-1} \circ e_\beta$ is p.l. whenever it is defined; suppose further that P is the identification space of $\{P_\alpha\}$ under e_α. Then provided P is compact it embeds as a polyhedron in $\mathbb{R}^n$ for some n so that each e_α is p.l. (*Hint:* Generalise the method of (2).)

(4) *Periodic homeomorphisms.* Let $f\colon |K| \to |K|$ be *periodic* (i.e. $f^n = \mathrm{id}$ for some n) then there is a subdivision $K' \lhd K$ so that $f\colon K' \to K'$ is simplicial. (*Hint:* Consider the abstract polyhedron obtained by identifying each $a \in K$ with $f(a)$). (Compare examples below 2.17.)

(5) *Ball complexes.* Suppose that K is a finite collection of balls and write $|K| = \bigcup \{B \mid B \in K\}$, then K is a *ball complex* if

(i) $|K| = \text{disjoint} \bigcup \{\mathring{B} \mid B \in K\}$

(ii) if $A, B \in K$ then $A \cap B$ is a union of balls of K.

Show that

(iii) ∂A is a union of balls of K for each $A \in K$ and prove a generalisation of 2.18 for ball complexes.

(6) *Dual cones.* Let K be a simplicial complex and let $K^{(1)}$ be a first derived and $A \in K$. Define the *dual cone*

$$A^*(K) = \{a_1 a_2 \dots a_r \mid A < A_1 < A_2 < \dots < A_r \in K\}$$

and then $A^*(K) = a A^{\check{}}(K)$, where

$$A^{\check{}}(K) = \{a_1 a_2 \dots a_r \mid A < A_1 < A_2 < \dots < A_r \in K,\ A \neq A_1\}.$$

Show that $A^{\check{}}(K) \cong (\mathrm{lk}(A, K))^{(1)}$ by pseudo-radial projection from a.

(7) *The dual complex.* Let $|J|$ be a manifold. Use (6) and 2.24(4) to show that $J^* = \{|A^*(J)|, |A^*(\partial J)| \mid A \in J\}$ is a ball complex.

Appendix to Chapter 2. On Convex Cells

Lemma A. *$\langle x, C \rangle$ is a subspace.*

Proof. Let L, L' be lines through x in $\langle x, C \rangle$ and let π be the plane defined by L and L'. We show that $\pi \subset \langle x, C \rangle$ and the result follows. Now by definition x is in the interior of arcs $ab, a'b'$ in $L \cap C$, $L' \cap C$ respectively. Then, by convexity, C contains the quadrilateral $aa'bb'$ in π. Any line through x in π meets this quadrilateral in an interval containing x in its interior and hence lies in $\langle x, C \rangle$, as required.

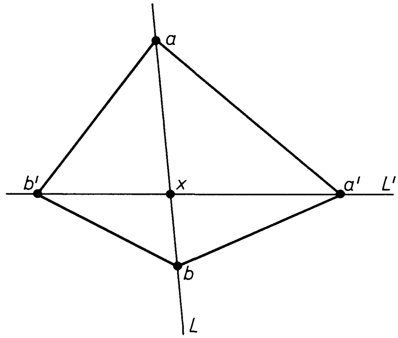


Fig. 20

Remark. Observe that the proof shows $x \in \mathring{C}_x$ and $\dim(C_x) = \dim\langle x, C\rangle$.

Lemma B. *Let L be any line in $\mathbb{R}^n$ meeting C in the arc ab and let x, y be any two interior points of ab then*

$$\langle a, C\rangle \subsetneqq \langle x, C\rangle = \langle y, C\rangle \supsetneqq \langle b, C\rangle.$$

Proof. $L \subset \langle x, C\rangle$ and $\langle y, C\rangle$ by definition. Let $L' \subset \langle x, C\rangle$ be a line through x; if we show that L'', the line parallel to L' through y, lies in $\langle y, C\rangle$ then the middle equality follows easily. Now $x \in \operatorname{int} cd$ with $cd \subset L' \cap C$ and by convexity C contains the triangle bcd which meets L'' in an arc with y interior, showing $L'' \subset \langle y, C\rangle$, as required.

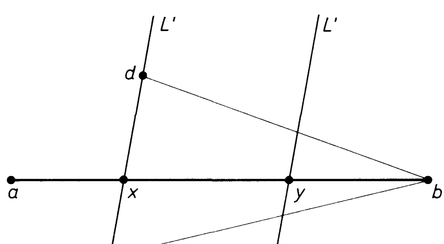


Fig. 21

Now a similar proof shows that $\langle a, C\rangle \subset \langle x, C\rangle$ but $L \not\subset \langle a, C\rangle$ which establishes the lemma.

Corollary 1. $C_a \subsetneqq C_x = C_y \supsetneqq C_b$.

Corollary 2. *If* $F < C$, $x \in \mathring{F}$ *then* $F = C_x$.

Proof. $F = C_y$ say and $y \in \mathring{F}$, then the line $\langle x y \rangle$ meets F in an arc with both x and y interior so that $C_x = C_y = F$ by Corollary 1.

Corollary 3. *If* $F, D < C$, $\mathring{F} \cap \mathring{D} \neq \emptyset$ *then* $F = D$.

Proof. Let $x \in \mathring{F} \cap \mathring{D}$ then $F = C_x = D$ by Corollary 2.

Corollary 4. *If* $D < C$ *and* $x \in D$ *then* $C_x = D_x$.

Proof. $D = C_y$ say and we have

(1) $\langle x, D \rangle = \langle x, C \rangle \cap \langle y, C \rangle$

by definitions. But we also have

(2) $\langle x, C \rangle \subset \langle y, C \rangle$

by Lemma B, since $y \in \mathring{D}$ and $x \in D$, therefore

$$\begin{aligned} D_x = \langle x, D \rangle \cap D &= \langle x, D \rangle \cap \langle y, C \rangle \cap C \\ &= \langle x, C \rangle \cap \langle y, C \rangle \cap C \quad \text{by (1)} \\ &= \langle x, C \rangle \cap C = C_x \qquad\quad \text{by (2)}. \end{aligned}$$

We now prove 2.7.

Part (4). This follows at once from the last corollary since $F = D_x$ for some x.

Part (3). For each $x \in C$ we have $x \in \mathring{C}_x$ so $C = \bigcup \{\mathring{F} \mid F < C\}$ but this is a disjoint union by Corollary 3. Now $\dot{C} = C - \mathring{C}$ which proves the second half.

Parts (1) and (2). Part (2) follows from Part (1), since a vertex of F is one of C by Part (4). We prove (1) by induction on n. First of all vertices are isolated since each point $a \in C$ has a cone neighbourhood aL and there are no vertices in $aL - L$ other than a; so by compactness there are only finitely many vertices. Now $C \supset v_0, v_1, \ldots, v_r$ by convexity and we have to prove the inclusion the other way. Let $x \in C$ and ab be the arc $L \cap C$ containing x for some line L. Then $a, b \in \dot{C}$ and hence lie in proper faces, for which our induction hypothesis holds. Therefore $a, b \in v_0 v_1 \ldots v_r$ since the vertices of the faces of C are vertices of C by (4). Then $ab \subset v_0 v_1 \ldots v_r$ by convexity of the latter and $x \in v_0 v_1 \ldots v_r$, as required.

Part (5). Let $x \in (F \cap D)^\circ$ then by Corollary 4 $F_x = C_x = D_x$ so that $C_x \subset F \cap D$. But $\langle x, C \rangle \supset \langle x, F \cap D \rangle$ since $C \supset F \cap D$ and this implies $C_x \supset F \cap D$.

Part (6). By convexity $C \supset xB$ so let $y \in C$, $y \neq x$ and continue the arc xy in C as far as possible past y and let the end point be z. Then $x \notin \langle z, C\rangle$ for otherwise z is in the interior of $\langle xy\rangle \cap C$. Therefore $x \notin C_z$ and $z \in C_z \subset B$. We have shown $xB \supset C$ and it remains to show xB is a cone. But if xyz is an arc with $y, z \in B$ then $x \in B_y$, by definition, so $x \in B$ which is a contradiction.

Chapter 3. Regular Neighbourhoods

Full Subcomplexes

Suppose $L \subset K$ are simplicial complexes. Define the simplicial map $f_L \colon K \to I$ by setting $f_L(v)=0$ for vertices $v \in L$ and $f_L(v)=1$ for other vertices. We then have $L \subset f_L^{-1}(0) \subset K$ and we say that L is *full* in K if $L = f_L^{-1}(0)$. We write $L \ominus K$ if L is a full subcomplex of K. As immediate consequences of the definition we observe:

3.1 (a) $f_L^{-1}(1) \ominus K$

(b) $L \ominus K$ implies $T \cap L \ominus T$ for any $T \subset K$.

We will need the following easy criteria for fullness.

3.2 *Exercise.* Suppose $L \subset K$ then the following are equivalent:

(a) $L \ominus K$

(b) each simplex of K meets L in a face, possibly empty

(c) no simplex of $K - L$ meets L in its whole boundary.

3.3 Lemma. (a) *If $L \subset K$ then there is a subdivision $K' \lhd K$ such that $L \ominus K'$.*

(b) *If $L \ominus K$ and $K' \lhd K$ inducing $L' \lhd L$ then $L' \ominus K'$.*

Proof. (a) Form K' by starring each simplex $A \in K - L$, which meets L in its whole boundary, at any interior point. The result then follows from 3.2(c) since if $A \in K' - L$ and $\dot{A} \subset L$ then $\dot{A} \subset K$ implying $A \in K$ which contradicts $A \in K'$ since A should have been starred (see Fig. 22).

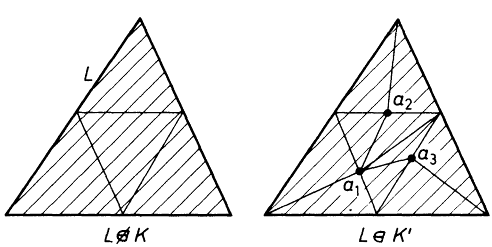


Fig. 22

(b) follows easily from 3.2(b).

3.4 *Exercise.* $L \ominus K$ and $L' \lhd L$. Then there is $K' \lhd K$ with $L' \subset K'$ and no new vertices in $K' - L'$.

Remark. 3.3 and 3.4 give an alternative proof of 2.17 with a considerably more economical subdivision.

Derived Neighbourhoods

Suppose $L \subset K$. Define the *simplicial neighbourhood* of L in K

$$N(L, K) = \{A \mid A \in K,\ A < B,\ B \cap |L| \neq \emptyset\}$$

i.e. the smallest subcomplex of K which is also a topological neighbourhood of L in K. Define the *simplicial complement* of L in K

$$C(L, K) = \{A \mid A \in K,\ A \cap |L| = \emptyset\}.$$

Then $C(L, K) = f_L^{-1}(1)$ and $K = N(L, K) \cup C(L, K)$. Define $\dot{N}(L, K) = N(L, K) \cap C(L, K)$ and then:

3.5 $\dot{N}(L, K) \ominus N(L, K)$ by 3.1.

A subdivision $K' \lhd K$ obtained by deriving K away from $L \cup C(L, K)$ is said to be a *derived of* K *near* L. Then K' is obtained from K by deriving simplexes which meet $|L|$ but are not in L.

Exercise. $L \ominus K'$ where K' is derived near L.

Now suppose $L \ominus K$ and K' is a derived of K near L. Then $N(L, K')$ is a *derived neighbourhood* of L in K. Given two deriveds of K near L, K_1 and K_2, then:

3.6 the canonical isomorphism $s: K_1 \to K_2$ carries $N(L, K_1)$ onto $N(L, K_2)$ and is the identity on $L \cup C(L, K)$.

Next define $I_\varepsilon \lhd I$ by introducing a vertex at ε where $0 < \varepsilon < 1$. Then the cell complex

$$N_\varepsilon(L, K) = \{A \cap f_L^{-1} B \mid A \in K,\ B < [0, \varepsilon]\}$$

is called the *ε-neighbourhood* of L in K. If we define a derived K' of K near L by choosing the new vertices on $f^{-1}(\varepsilon)$ then it is easy to see that $N(L, K') \lhd N_\varepsilon(L, K)$ (see Fig. 23).

3.7 Lemma. *Suppose $L \ominus K$ and $K_1 \lhd K$ inducing $L_1 \lhd L$. Then there are deriveds K', K_1' of K, K_1 near L, L_1 so that $|N(L, K')| = |N(L_1, K_1')|$.*

Proof. Choose $\varepsilon > 0$ sufficiently small that $f_L^{-1}[0, \varepsilon]$ contains no vertices of $K_1 - L_1$. Define K' and K_1' by choosing all the new vertices on $f_L^{-1}(\varepsilon)$ and then we have

$$N(L_1, K_1') \lhd N_\varepsilon(L, K) \rhd N(L, K').$$

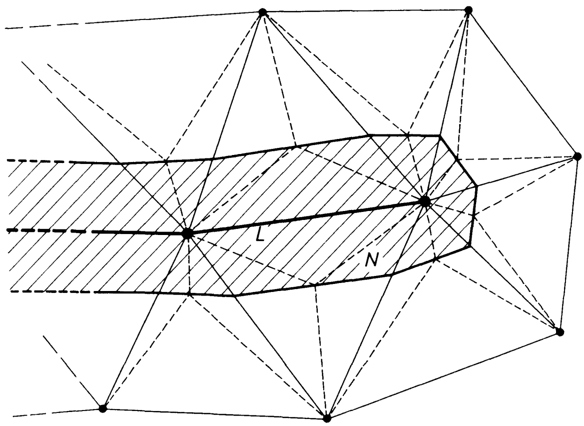


Fig. 23

Regular Neighbourhoods

Now suppose $X \subset Y$ are polyhedra, with X compact, and that K triangulates a neighbourhood of X in Y with $|L| = X$ where $L \ominus K$, and that K' is a derived of K near L. We then have a derived neighbourhood $N(L, K')$ and the underlying polyhedron $N = |N(L, K')|$ is said to be a *regular neighbourhood* of X in Y. Existence of regular neighbourhoods follows from 2.2 (for finding a compact neighbourhood of X in Y) and 3.3(a). Uniqueness is proved in the next theorem; a stronger result (uniqueness up to isotopy) will be proved later.

3.8 Theorem. *If N_1, N_2 are regular neighbourhoods of X in Y then there is a homeomorphism h: $Y \to Y$ which carries N_1 onto N_2 and is the identity on X and outside some compact subset of Y.*

Proof. By definition $N_i = |N(L_i, K_i')|$ for $i = 1, 2$ where $L_i \ominus K_i$ and K_i triangulates a neighbourhood of X in Y. By 2.15 there is a triangulation K_0 of $|K_1| \cup |K_2|$ which contains subdivisions of both K_1 and K_2. Then $L_0 \ominus K_0$ by 3.3(b) and $N(L_0, K_0')$ is a derived neighbourhood. But by Lemma 3.7 and the canonical uniqueness of derived neighbourhoods (3.6) we have $|N(L_0, K_0')| \cong |N(L_i, K_i')| = N_i$ for $i = 1, 2$ and it only remains to observe that each homeomorphism, being a composition of isomorphisms (3.6), keeps X and the complement of a compact neighbourhood of X in Y fixed and therefore extends by the identity to the required homeomorphism of Y.

3.9 Corollary. *Suppose $X \subset Y$ is locally collarable and X is compact then a regular neighbourhood of X in Y is a collar.*

Proof. By 3.8 and 2.25 it suffices to consider $L = K \times 0 \subset K \otimes I$ where $K \otimes I$ denotes $K \times I$ subdivided as in 2.9. But

$$|N_\varepsilon(L, K \otimes I)| = |K| \times [0, \varepsilon].$$

Regular Neighbourhoods in Manifolds

Now suppose $X \subset M$ is a compact polyhedron in the manifold M.

3.10 Proposition. *A regular neighbourhood N of X in M is a compact manifold with boundary. If $X \subset \operatorname{int} M$ then $\partial N = |\dot{N}(L, K')|$.*

Proof. It suffices to consider an ε-neighbourhood $N_\varepsilon(L, K)$. Let $x \in N$, then $x \in \mathring{A}$ for $A \in K$ and A meets L; choose a vertex $v \in A \cap L$ and consider $B_v = |\operatorname{st}(v, K)| \cap N$, then since $x \in$ interior of $|\operatorname{st}(v, K)|$ in $|K|$ we have $x \in$ interior of B_v in N. But $B_v = |\operatorname{st}(v, N_\varepsilon)|$ is a star of v in M and hence a ball. It follows that there is a coordinate neighbourhood for N at x.

For the last part observe that $N \subset \operatorname{int} M$ and $\dot{N}$ is the frontier of N in M.

Exercise. Use exercise 2.24(3) to give an alternative proof for 3.10 after observing that $f_L^{-1}[\varepsilon, \tau]$ for $0 < \varepsilon < \tau < 1$ is a cell complex abstractly isomorphic with $\dot{N}_\varepsilon \times I$.

We now come to the crucial *simplicial neighbourhood theorem* (3.11) which enables one to recognise regular neighbourhoods in the absence of a triangulation extending beyond the neighbourhood itself.

3.11 Theorem (S.N.T.). *Suppose X is a compact polyhedron in the interior of the manifold M and that N is a neighbourhood of X in* $\operatorname{int} M$. *Then N is a regular neighbourhood if and only if*

(i) *N is a compact manifold with boundary*

(ii) *there are triangulations (K, L, J) of $(N, X, \partial N)$ with $L \circleddash K$, $K = N(L, K)$ and $J = \dot{N}(L, K)$.*

Proof. If N is a regular neighbourhood then conditions (i) and (ii) follow at once from definition and 3.10. The converse is proved by a short induction on $n = \dim M$ together with Corollaries 3.12 to 3.14. Assume S.N.T. in dimension n.

3.12$_n$ Corollary. *Suppose $B^n \subset \operatorname{int} M^n$ is a ball and $x \in \operatorname{int} B^n$. Then B^n is a regular neighbourhood of x in M.*

Proof. We can take B to be an n-simplex and define K by starring at x.

3.13_n Corollary. *Suppose $B^n \subset S^n$ is a ball in a sphere then* $\mathrm{cl}(S^n - B^n)$ *is a ball.*

Proof. We can take $S^n = \dot{A}$ where A is an $(n+1)$-simplex. Then $\mathrm{int}\, B$ meets $\mathring{C}$ for some n-simplex $C < A$; choose $x \in \mathring{C} \cap \mathrm{int}\, B$ then B and C are both regular neighbourhoods of x in A by 3.12 and so by uniqueness (3.8) we can take $B = C$. But $\mathrm{cl}(\dot{A} - C) = \mathrm{st}(a, \dot{A})$, where a is the vertex opposite C, is a ball by 2.23.

3.14_{n+1} Corollary. *If $Q \subset \mathrm{int}\, M$ are $(n+1)$-manifolds then* $\mathrm{cl}(M - Q)$ *is an $(n+1)$-manifold.*

Proof. For $p \in \partial Q$ we have $\mathrm{lk}(p, \mathrm{cl}(M - Q)) = \mathrm{cl}(\mathrm{lk}(p, Q) - \mathrm{lk}(p, M))$ which is a ball by 3.13_n.

Finally to complete the induction we show

$3.14_{n+1} \Rightarrow \text{S.N.T.}_{n+1}$:
Let K' be a derived of K near L and $N_1 = |N(L, K')|$. Then K' is also derived near J and $J \ominus K$ by 3.5, so that $C_1 = |N(\dot{J}, K')|$ is a regular neighbourhood of $\dot{N}$ in N and hence a collar by 3.9; moreover $C_1 = C(L, K')$ so that $C_1 = \mathrm{cl}(N - N_1)$. Let K'' be K' derived near L and $N_2 = |N(L, K'')|$, $C_2 = \mathrm{cl}(N_1 - N_2)$ which is a collar for similar reasons. Finally $\mathrm{cl}(Q - N)$ is a manifold by 3.14 and hence there is a collar C_3 on $\dot{N}$ in $\mathrm{cl}(Q - N)$ by 2.25 (see Fig. 24).

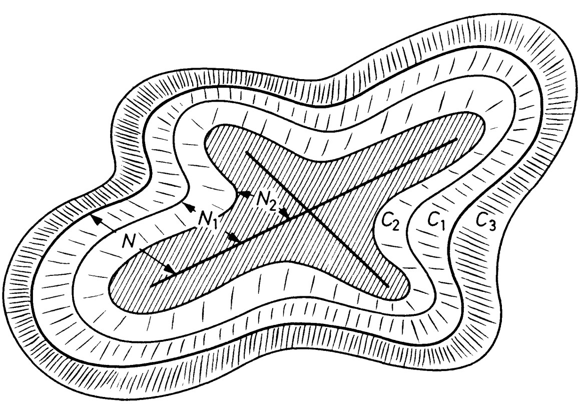


Fig. 24

Then $C = C_2 \cup C_1 \cup C_3 \cong \dot{N} \times [0, 3]$ and using a homeomorphism λ of $[0, 3]$ to itself such that $\lambda | \{0, 3\} = \mathrm{id}$ and $\lambda(1) = 2$ we have a homeomorphism h of C such that $h | \partial C = \mathrm{id}$ and $h(C_2) = C_2 \cup C_1$. h extends by the identity to a homeomorphism of M which throws N_1 onto N hence showing that N is a regular neighbourhood.

Exercise. Generalise the simplicial neighbourhood theorem to the case when M is a polyhedron. In place of condition (i), assume that N is a compact polyhedron with $\dot{N}$ a collarable subpolyhedron and that $\mathrm{cl}(M-N)$ is collarable at $\dot{N}$ as well.

3.15 Corollary. *Suppose $B_i^{n-1} \subset \partial B_i^n$ are balls for $i=1,2$ (say B_i^{n-1} is a* face *of B_i^n) then any homeomorphism of B_1^{n-1} with B_2^{n-1} extends to a homeomorphism of B_1^n with B_2^n.*

Proof. By 3.13 $\mathrm{cl}(\partial B_i^n - B_i^{n-1})$ is a ball and the result follows by two applications of 1.10; first extend to ∂B_1^n then to B_1^n itself.

3.16 Corollary. *The union of two balls which meet in a common face is a ball.*

Proof. By 3.15 applied twice, the union is homeomorphic to $S^0 * B^{n-1} \cong B^n$.

3.17 Corollary. *Let M be a manifold with compact boundary then a collar on ∂M in M is a regular neighbourhood.*

Proof. Consider the double of M, DM which is obtained by gluing a copy M_0 of M to M along ∂M. Then $M_0 \subset \mathrm{int}\,DM$ and we can apply the S.N.T. But the collar determines a neighbourhood of M_0 in DM which can be triangulated by $J \cup K \otimes I$ where $(M_0, \partial M) = (|J|, |K \times 0|)$; by the S.N.T. this is a regular neighbourhood and restricting to M we see that the collar is a regular neighbourhood of ∂M in M.

3.18 Corollary (Regular neighbourhood collaring theorem). *Suppose $N_1 \subset \mathrm{int}\,N_2$ are two regular neighbourhoods of X in $\mathrm{int}\,M$. Then* $\mathrm{cl}(N_2 - N_1) \cong \dot{N}_1 \times I$.

Proof. There is a regular neighbourhood N_1' of X in M so that $\mathrm{cl}(N_2 - N_1')$ is a collar, by the proof of the S.N.T. Then, by the S.N.T., N_1 and N_1' are both regular neighbourhoods of X in $\mathrm{int}\,N_2$ and hence there is a homeomorphism of $\mathrm{int}\,N_2$, which is the identity outside a compact set, carrying N_1 to N_1'. This extends by the identity to N_2 and hence carries $\mathrm{cl}(N_2 - N_1)$ onto $\mathrm{cl}(N_2 - N_1')$.

3.19 Corollary (Combinatorial annulus theorem). *Given n-balls A and B with $A \subset \mathrm{int}\,B$ then* $\mathrm{cl}(B-A) \cong S^{n-1} \times I$.

Proof. By 3.12 A and B are both regular neighbourhoods of $x \in \mathrm{int}\,A$ in any manifold M with $B \subset \mathrm{int}\,M$.

3.20 *Exercise.* Use 3.19 to prove that $\mathbb{R}^n \cong S^n -$ one point (Exercise 1.9(5)) by writing both $\mathbb{R}^n$ and $S^n -$point as a union of nested n-balls.

Isotopy Uniqueness of Regular Neighbourhoods

The idea of sliding Y over itself gives rise to the notion of "an isotopy of Y". Composing with an embedding of X in Y we have an "ambient isotopy of X in Y". An "isotopy of X in Y" corresponds to the idea of sliding X about in Y without moving Y. The problem of determining when an isotopy of X in Y is ambient (i.e. when a given movement of X in Y can be realised by moving Y) is discussed in the next chapter.

3.21 *Definitions*

(1) A map $F\colon X\times I\to Y\times I$ is *level-preserving* if $F(X\times t)\subset Y\times t$ for each $t\in I$. We can then define $F_t\colon X\to Y$ by $F(x,t)=(F_t(x),t)$.

(2) An *isotopy of* Y is a level preserving homeomorphism $H\colon Y\times I\to Y\times I$ such that $H_0=\mathrm{id}$. We say that H_1 is the *finishing* homeomorphism of the isotopy and that H_1 is *ambient isotopic* to the identity.

(3) An *isotopy of* X *in* Y is a level-preserving embedding $F\colon X\times I\to Y\times I$ and we say that the embeddings F_0 and F_1 are *isotopic*. We say that H *covers* F if $F=H\circ(F_0\times\mathrm{id})$ in other words if

$$\begin{tikzcd} X\times I \arrow[rr, "F_0\times \mathrm{id}"] \arrow[dr, "F"'] & & Y\times I \arrow[dl, "H"] \\ & Y\times I & \end{tikzcd}$$

commutes.

(4) An *ambient isotopy* is an isotopy which is covered by some isotopy of Y and we say F_0, F_1 are *ambient isotopic*. (This extends the usage of "ambient" in (2).) We also say that the subsets $F_0(X)$ and $F_1(X)$ are ambient isotopic.

(5) An isotopy, ambient isotopy, etc., *fixes* a subset $V\subset X$ if $F|V\times I=F_0\times\mathrm{id}|V\times I$ and we say F has *support* in U, or is supported by U, if F fixes $X-U$. We also say F is *mod* V if F fixes V.

Remark. An isotopy between homeomorphisms is ambient if and only if it is itself a homeomorphism.

Exercise. "Isotopy" and "ambient isotopy" are equivalence relations on the set of embeddings of X in Y.

3.22 Proposition. (i) *Let B^n, C^n be balls and $h_0, h_1\colon B^n\to C^n$ homeomorphisms which agree on $\dot{B}^n$, then h_0, h_1 are ambient isotopic mod $\dot{B}^n$.*

(ii) *Suppose M is a manifold with compact boundary, then any isotopy of ∂M extends to one of M with support in a collar of ∂M.*

Proof. (i) (Alexander trick). We can take $B^n=C^n=I^n$ and construct the required homeomorphism H of $I^n\times I$ as follows.

$$H_0 = h_0, \qquad H_1 = h_1$$
$$H|\dot{I}^n \times I = (h_0|I^n) \times \mathrm{id}$$
$$H(x) = x \qquad \text{where } x = (0, \tfrac{1}{2}) \in I^n \times I$$

and $H|I^n \times I$ is defined by conical extension from x (see Fig. 25).

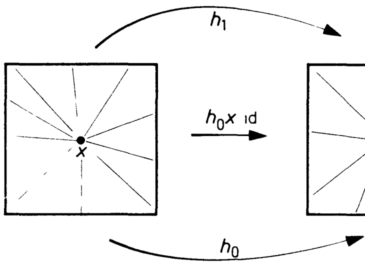


Fig. 25

(ii) Choose a collar $c\colon \partial M \times I \to M$ and extend H to $\mathrm{im}(c)$ by

$$H'_t(x, s) = \begin{cases} (H_{t-s}(x), s) & \text{for } s \leqq t \\ (x, s) & \text{for } s \geqq t \end{cases}$$

where s is the coordinate for the collar and t the coordinate for the isotopy. Extend to the rest of M by the identity.

3.23 Corollary. *Let K be a cell complex and $f\colon |K| \to |K|$ a homeomorphism which carries each cell of K into itself. Then f is ambient isotopic to the identity keeping fixed any subcomplex L on which f is already the identity.*

Proof. Isotope $f|C$ for $C \in K$ to the identity by induction on dimension of C using 3.22(i); extend each isotopy to higher dimensional cells by repeated use of 3.22(ii) with M a ball.

3.24 Regular neighborhood theorem. *Suppose N_1 and N_2 are regular neighbourhoods of X in Y then there is an isotopy H of Y fixed on X and of compact support carrying N_1 onto N_2 $(H_1(N_1) = N_2)$. Moreover if Y is a manifold and $X \subset \mathrm{int}\, Y$ then we can assume further that H is fixed on any regular neighbourhood $N \subset (\mathrm{int}\, N_1 \cap \mathrm{int}\, N_2)$ and outside any open neighbourhood U of $N_1 \cup N_2$.*

Proof. For the first part observe that the uniqueness Theorem 3.9 provided a homeomorphism which was a composition of isomorphisms of deriveds and the required isotopy is provided by 3.23. For the second half we have $C_i = \mathrm{cl}(N_i - N)$ a collar for $i = 1, 2$ by 3.18 and hence C_i is a regular neighbourhood of N in $U - \mathrm{int}\,N$ by 3.17. So by the first part there is an isotopy of $U - \mathrm{int}\,N$ of compact support and fixed on N carrying C_1 to C_2; extending by the identity gives the required isotopy of M.

Exercise. Prove the stronger part of 3.24 for polyhedra in polyhedra using the S.N.T. for polyhedra.

Collapsing

We now turn to the classical treatment of regular neighbourhoods based on collapsing. For most applications the treatment we have given so far, based on the simplicial neighbourhood theorem, is all that is needed (for instance the final sections of this chapter); however collapsing is a very useful tool and has strong connections with torsion (see Appendix B).

Definition. Suppose $X \supset Y$ are polyhedra and that $X = Y \cup B^n$ and $Y \cap B^n =$ a face B^{n-1}. Then we say that there is an *elementary collapse* of X on Y, and write $X \searrow^{e} Y$. The collapse is *across B^n onto B^{n-1} from* the complementary face $C^{n-1} = \mathrm{cl}(\partial B^n - B^{n-1})$, see Fig. 26.

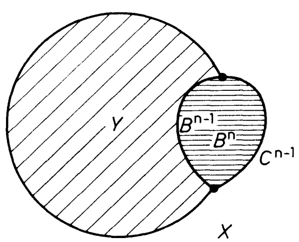


Fig. 26

We say X *collapses* on Y and write $X \searrow Y$ if there is a sequence of elementary collapses $X = X_0 \searrow^{e} X_1 \searrow^{e} \cdots \searrow^{e} X_n = Y$. If Y is a point we say X is *collapsible* and write $X \searrow 0$.

Remarks on Simple Homotopy Type

If $X \searrow Y$ then $Y \subset X$ is a homotopy equivalence since there is a deformation retraction $r\colon X \to Y$ given by deforming each of the balls B^n onto the face B^{n-1}. Therefore a sequence of collapses and their inverses

$$X_0 \searrow X_1 \swarrow X_2 \searrow X_3 \swarrow \cdots \searrow X_n$$

determines a homotopy equivalence $X_0 \to X_n$ which is called a *simple* homotopy equivalence. Simple homotopy equivalence is then an equivalence relation on polyhedra and the equivalence classes are called *simple homotopy types*. For example the house with two rooms (see Chapter 1 for a picture) has the simple homotopy type of a point—first thicken all the walls in $\mathbb{R}^3$ (this is the inverse of a collapse) and observe that the result is a 3-ball, which collapses in 3 steps. In general a homotopy equivalence $h: X \to Y$ determines a torsion element $\tau(h) \in \mathrm{Wh}(\pi_1(X))$ which is zero if and only if h is simple; see Appendix B.

Examples of collapses

(1) B^n collapses in n steps since it collapses on B^{n-1} which collapses inductively in $(n-1)$ steps.

(2) Let X be compact and $C(X)$ denote the cone on X, then $C(X) \searrow 0$. For write $X=|K|$ and collapse $C(A)$ from A for $A \in K$ inductively in order of decreasing dimension. A similar proof shows $C(X) \searrow C(X_0)$ for any $X_0 \subset X$.

(3) If $X \searrow 0$ then $C(X) \searrow X$ for if $X \mathrel{\searrow\!\!\!\!\searrow} Y$ then $C(X) \mathrel{\searrow\!\!\!\!\searrow} C(Y) \cup X$ by collapsing $C(B^n)$ from $C(C^{n-1})$.

(4) If X is compact and $Y \searrow Y_0$ then $X \times Y \searrow X \times Y_0$. For write $X=|K|$ and assume without loss of generality that $Y \mathrel{\searrow\!\!\!\!\searrow} Y_0$ across B^n from C^{n-1}; then $X \times Y \searrow X \times Y_0$ by inductively collapsing $A \times B^n$ from $A \times C^{n-1}$ for $A \in K$.

(5) *Trails.* Let $X \searrow Y$ and suppose $P \subset X$ is a compact polyhedron then there is a compact polyhedron $P_+ \supset P$ such that $X \searrow P_+ \cup Y$ and $\dim P_+ \leqq \dim P + 1$ called a *trail* of P under the collapse. P_+ is constructed inductively as follows. Suppose $X_i \mathrel{\searrow\!\!\!\!\searrow} X_{i+1}$ across B^n onto B^{n-1} and P_i has been constructed. Choose a homeomorphism h: $(B^n, B^{n-1}) \to (B^{n-1} \times I, B^{n-1} \times 0)$ by 3.15 and define $P_{i+1} = P_i \cup h^{-1}$ (shadow $h(P_i \cap B^n)$), where shadow(T) for $T \subset B^{n-1} \times I$ is defined by $(x,t) \in$ shadow(T) if and only if $(x,s) \in T$ for some $s \geqq t$. Then

$$\dim P_{i+1} \leqq \dim P_i + 1 \quad \text{and} \quad \dim P_{i+1} \cap X_{i+1} \leqq \dim P_i;$$

moreover $X_i \cup P_i \searrow X_{i+1} \cup P_{i+1}$ since $B^{n-1} \times I \searrow B^{n-1} \times 0 \cup \mathrm{shadow}(T)$ by the proof of (4).

Shelling

Now suppose that $M_1 \subset M$ are n-manifolds and $M \mathrel{\searrow\!\!\!\!\searrow} M_1$ across B^n from C^{n-1} onto B^{n-1}. Then we must have $B^{n-1} \subset \partial M_1$ and $C^{n-1} \subset \partial M$. A collapse of this type is an *elementary shelling* and a sequence of such collapses is a shelling.

3.25 Lemma. *If M shells to M_1 then there is a homeomorphism $h: M \to M_1$ which is the identity outside an arbitrary neighbourhood of $M - M_1$.*

Proof. It is sufficient to prove this for an elementary shelling. Let $M = M_1 \cup B^n$, $M_1 \cap B^n = B^{n-1}$. Choose a collar c on ∂M_1 in M_1 then $c(B^{n-1} \times I)$ is a ball D^n and B^n and D^n meet in the common face B^{n-1}. By a suitable choice of c we may suppose that D^n is in the neighbourhood. Let D^{n-1} be the complementary face of D^n. Then $\mathrm{id}|D^{n-1}$ extends to a homeomorphism of $B^n \cup D^n$ with D^n by 3.15. This extends by the identity to the required homeomorphism of M with M_1.

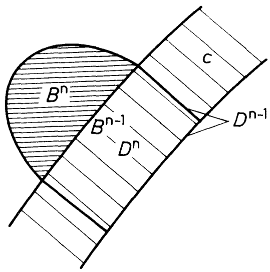


Fig. 27

The connection between collapsing and regular neighbourhoods is contained in the next theorem. We postpone the proof until after the corollaries.

3.26 Theorem. *Suppose $X \subset M$ is a compact polyhedron and that $X \searrow Y$. Then a regular neighbourhood of X in M shells to a regular neighbourhood of Y in M.*

3.27 Corollary. *If $X \searrow 0$ then a regular neighbourhood of X is a ball.*

Proof. The regular neighbourhood of a point is a ball.

3.28 Corollary. *A collapsible manifold is a ball.*

Proof. It is a regular neighbourhood of itself in itself and is therefore a ball by 3.27.

3.29 Corollary. *If $X \subset \mathrm{int}\, M$ and $X \searrow Y$ then a regular neighbourhood of X is a regular neighbourhood of Y.*

Proof. They are homeomorphic mod Y by 3.25 and so the result follows from the S.N.T.

3.30 Corollary (Collapsing criterion for regular neighbourhoods). *Let N be a neighbourhood of X in* int M. *Then N is regular if and only if*

(i) *N is a compact manifold with boundary,*

(ii) $N \searrow X$.

Proof. Suppose N is regular then we have to prove (ii). Take N to be an ε-neighbourhood and collapse each cell $A \cap f^{-1}[0, \varepsilon]$ from the face $A \cap f^{-1}(\varepsilon)$, for $A \in K$, in order of decreasing dimension.

Conversely, suppose $N \searrow X$ and let N_1 be a regular neighbourhood of N in $\operatorname{int} M$, then N_1 is a regular neighbourhood of X by 3.28 and $C_1 = \operatorname{cl}(N_1 - N)$ is a collar by 3.9. Choose another regular neighbourhood N' of X in $\operatorname{int} N_1$ then $C_2 = \operatorname{cl}(N_1 - N')$ is a collar by 3.18. Then C_1 and C_2 are both regular neighbourhoods of $\dot{N}_1$ in $N_1 - X$ and the uniqueness theorem gives a homeomorphism of $N_1 \operatorname{mod} X$ throwing C_1 onto C_2 and hence N onto N' proving that N is regular, as required.

Proof of Theorem 3.25. Suppose the result true if the collapses are across balls of dimension $< n$. By induction on the length of the collapse we may assume $X \searrow^{e} Y$, and that $X = Y \cup B^{n-1} \times I$ with $B^{n-1} \times I \cap Y = B^{n-1} \times 0$. Now choose triangulations J, K, L of M, X, Y so that $L \ominus K \ominus J$. Now subdivide K further so that the projection p: $B^{n-1} \times I \to I$ is simplicial with respect to some linear triangulation of I, having vertices $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_q = 1$; see Fig. 28.

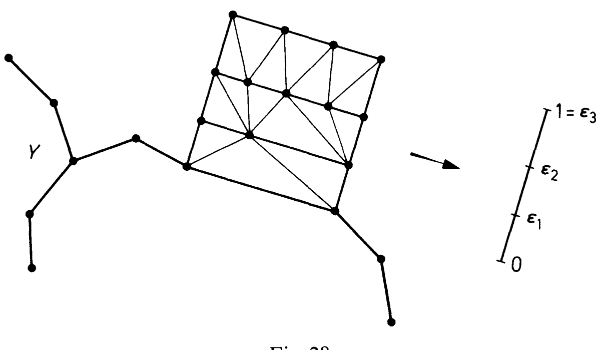


Fig. 28

By 3.4 we can extend this triangulation to a subdivision of J without destroying the fullness properties. Now the collapse $X \searrow^{e} Y$ decomposes into q collapses across the balls $p^{-1}[\varepsilon_i, \varepsilon_{i+1}]$, so that without loss of generality we may assume $L \ominus K \ominus J$ and K has no vertices in $B^{n-1} \times I - B^{n-1} \times \dot{I}$ (i.e. $q = 1$). Now choose a first derived subdivision of J so that simplexes which meet $p^{-1}\mathring{I}$ are derived along $p^{-1}(\frac{1}{2})$, see Fig. 29.

Now it is easy to see that $N(K', J') = N(L', J') \cup N(L'_1, J')$ where L_1 is the subcomplex triangulating $B^{n-1} \times 1$. From Corollary 3.27 and induction we see that $N(L'_1, J')$ is an m-ball. We claim that $W = N(L', J') \cap N(L'_1, J')$ is a regular neighbourhood of $p^{-1}(\frac{1}{2})$ in $N(L'_1, J')$ and is there-

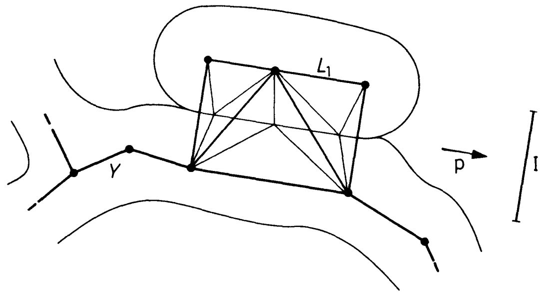


Fig. 29

fore an $(m-1)$-ball, again by 3.27 and induction. The theorem then follows.

To see the claim, consider the simplicial map $f\colon J \to \Delta^2$ defined by $f(L_1)=0$, $f(L)=1$ and f (other vertices)$=2$. Here Δ^2 is a 2-simplex with vertices $\{0, 1, 2\}$. Derive Δ^2 as shown in Fig. 30; then we can assume that J' was chosen so that $f\colon J' \to (\Delta^2)'$ is simplicial. Then $W=f^{-1}(b\,c)$, and W is obtained from $f^{-1}(a\,c)$ by deriving on $f^{-1}(b)$. It follows that W is a regular neighbourhood, as required.

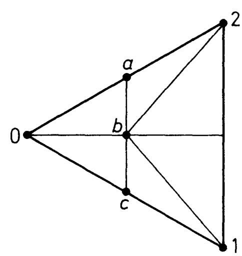


Fig. 30

Orientation

In this section we use regular neighbourhood theory to give a geometric treatment of orientation. It is convenient to use a result from algebraic topology (in fact this dependence on algebraic topology can be eliminated, see 3.35(7)). Let $r_n\colon I^n \to I^n$ be reflection in the x_1-direction i.e.

$$r_n(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).$$

3.31 (See Appendix A). $r_n\colon \partial I^n \to \partial I^n$ is not homotopic to the identity.

3.32 Theorem. *Let $h\colon \partial I^n \to \partial I^n$ be a homeomorphism. Then h is ambient isotopic to one of* id *or* r_n.

Combining the last two results we see that there are exactly two ambient isotopy classes of homeomorphisms of an n-sphere. To prove the theorem we need to know how to move points around in a manifold:

3.33 Lemma (homogeneity of manifolds). *Let M be connected and $p, q \in \operatorname{int} M$ then there is an isotopy of $M \bmod \partial M$ carrying p to q.*

Proof. If $M = I^n$ then the cone construction provides a homeomorphism of $M \bmod \partial M$ carrying p to q and the result follows from 3.22(i). For the general case let U be the set of points in $\operatorname{int} M$ which can be reached from p by an isotopy of $M \bmod \partial M$. Since each point in $\operatorname{int} M$ has a ball neighbourhood, U is open in M. For similar reasons $\operatorname{int} M - U$ is open in M. Therefore $U = \operatorname{int} M$.

Proof of Theorem 3.32. The proof is by induction on n. The result is obvious for $n=1$. Let $F < I^n$ be the face $x_n = 1$ and $a \in \mathring{F}$. Then we can assume $h(a) = a$ by 3.33 and that $h(F) = F$ by the regular neighbourhood theorem. Now F is a translate of I^{n-1} and $h | \partial F$ is ambient isotopic to either id or r_{n-1} by induction. This isotopy extends to ∂I^n by two applications of 3.22(ii) and the result now follows by 3.22(i) applied to each of F, $\operatorname{cl}(\partial I^n - F)$.

3.34 Disc theorem. *Let M be a connected n-manifold and $h_1, h_2 : I^n \to \operatorname{int} M$ embeddings. Then h_1 is ambient isotopic to one of h_2 or $h_2 \circ r_n$.*

Proof. By 3.33 we may assume that $h_1(0) = h_2(0)$ and, by the regular neighbourhood theorem, that $h_1(I^n) = h_2(I^n)$. Then $h_2^{-1} \circ h_1 | \partial I^n$ is ambient isotopic to one of id or r_n by 3.32. Composing with h_2 gives an ambient isotopy of $h_1(\partial I^n)$ which extends to M by two applications of 3.22(ii). h_1 now agrees with one of h_2 or $h_2 \circ r_n$ on ∂I^n and the result follows from 3.22(i).

The disc theorem shows that in a connected manifold M there are either one or two ambient isotopy classes of embeddings of I^n in $\operatorname{int} M$, and for $n > 1$ we define M^n to be *orientable* if there are two classes and *non-orientable* if there is only one. For $n = 0$, a connected 0-manifold (a point) is regarded by convention as having two orientations $+$ and $-$. An *orientation* for an orientable manifold M is a choice of isotopy class and if h: $I^n \to \operatorname{int} M$ is in this class then we say *h orients M*. An *oriented* manifold is a manifold with a choice of orientation. If g: $M \to M$ is a homeomorphism then g is *orientation-preserving* if $g \circ h$ is isotopic to h for each h: $I^n \to \mathring{M}$; otherwise h is *orientation-reversing*.

3.35 *Examples and remarks*

(1) To show that M is non-orientable it is sufficient to find one embedding h: $I^n \to \operatorname{int} M^n$ such that h is ambient isotopic to $h \circ r_n$, for by the disc theorem any embedding is isotopic to one of h or $h \circ r_n$.

(2) Spheres are orientable for if the identity on $F < \dot{I}^n$ is ambient isotopic to $r_n|F$ then by 3.22(i) applied to $\mathrm{cl}(\dot{I}^n - F)$ we have $\mathrm{id}|\partial I^n$ isotopic to $r_n|\partial I^n$ contradicting 3.31. The inclusion $F \subset \dot{I}^n$ defines a *standard* orientation for $\dot{I}^n$.

(3) A homeomorphism of ∂I^n is isotopic to the identity if and only if it preserves orientation. For r_n clearly reverses it.

(4) If M^n is orientable and $M_0^n \subset M^n$ then M_0 is orientable, moreover any orientation of M *restricts* to one of M_0 by considering those embeddings whose images lie in $\mathrm{int}\, M_0$. For example, I^n has a standard orientation by (2).

(5) If $M = U \cup V$, where U and V are open with $U \cap V \neq \emptyset$, and U and V are oriented so that the restricted orientations agree on $U \cap V$, then M is oriented by the orientations of U and V. We leave the proof as an exercise, to show that no embedding is isotopic to its reflection, split the isotopy into parts each of which takes place in either U or V.

(6) If M is orientable then so is ∂M for consider $h\colon I^{n-1} \to \partial M$. Then using a collar of ∂M we can define $\bar{h}\colon I^n \to \mathrm{int}\, M$ by

$$\bar{h}(x_1, x_2, \ldots, x_n) = c\left(h(x_1, \ldots, x_{n-1}), \frac{3+x_n}{4}\right).$$

In other words, $\bar{h}$ is h pushed in along c.

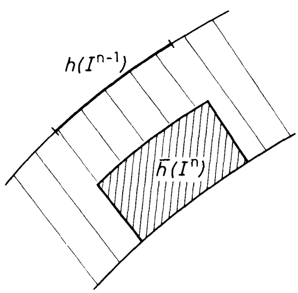


Fig. 31

Then if h is ambient isotopic to $h \circ r_{n-1}$ then $\bar{h}$ is ambient isotopic to $\bar{h} \circ r_n$ by taking the product isotopy on $\mathrm{im}(c)$ and extending to M by 3.22(ii).

If u is an orientation for M then one of $\bar{h}$ or $\overline{h \circ r_{n-1}} \in u$ and the class of h such that $\bar{h} \in u$ is the *induced* orientation for ∂M. For $n=1$ there is only one choice for $\bar{h}$ and induced orientation on ∂M is given by the convention that orientation is $+$ if and only if $\bar{h} \in u$.

(7) We have used 3.31 essentially only once in our treatment of orientation (to show that spheres and balls are orientable). However there is a

direct proof of this fact provided by remark (5) above and exercise (6) below. Thus orientation makes sense without any appeal to algebraic results.

Exercises

(1) Prove that if M is a manifold with boundary and h_1, h_2: $(I^{n-1} \times I, I^{n-1} \times 0) \to (M, \partial M)$ are two embeddings, then h_1 is ambient isotopic to one of h_2 or $h_2 \circ (r_{n-1} \times \mathrm{id})$.

Hint: Examine the proof of 3.32 and relativise each step (see Chapter 4 for more general methods).

(2) Deduce from (1) that the induced orientation for ∂M is independent of the collar used, since $\bar{h}$ is determined by $\tilde{h}$: $I^{n-1} \times I \to M$ defined by $\tilde{h}(x, t) = c(hx, t)$.

(3) Define a local orientation at $x \in \mathring{M}$ to be an orientation for a co-ordinate neighbourhood of x. Use the proof of 3.33 to show how to "transport" a local orientation along an arc α in M.

(4) Prove that the end result of (3) depends only on the homotopy class of α rel endpoints. Then define a homomorphism w: $\pi_1(M) \to \mathbb{Z}_2$ by transporting an orientation around a loop and comparing the result with the original orientation. Deduce that a simply-connected manifold is orientable.

(5) Show that M is orientable if and only if $M \times \mathbb{R}^1$ is orientable.

Hint: Cover M by balls so that orientations agree on overlaps.

(6) Give a proof that $\mathbb{R}^n$ is orientable as follows:

(a) $GL(p, \mathbb{R})$ has at least two path components detected by sign of determinant.

(b) Let f: $I^n \to \mathbb{R}^n$ be an embedding and suppose that I^n is triangulated so that f is linear on simplexes. Consider the differential of f on each n-simplex of I^n and show that the sign of $\det(df)$ does not alter at an $(n-1)$-simplex.

(c) Deduce the result by considering an isotopy $I^n \times I \to \mathbb{R}^n \times I$ as an embedding of I^{n+1} in $\mathbb{R}^{n+1}$.

Connected Sums

Suppose M_1, M_2 are connected oriented n-manifolds. We form an oriented n-manifold $M_1 \# M_2$ called the connected sum of M_1 and M_2 as follows.

Choose embeddings h_i: $I^n \to M_i$ $i = 1, 2$ in the given orientation classes. Then $M_1 \# M_2$ is formed by identifying $M_1 - h_1(\mathrm{int}\, I^n)$ with $M_2 - h_2(\mathrm{int}\, I^n)$ along $h_1(\dot{I}^n)$ and $h_2(\dot{I}^n)$ by the homeomorphism $h = h_2 \circ r_n \circ h_1^{-1}$: $h_1 \dot{I}^n \to h_2 \dot{I}^n$. It is easy to see that $M_1 \# M_2$ is a manifold which could also have been obtained by identifying collars on $h_i(\dot{I}^n)$ with one of the directions reversed. Then the orientations of M_1 and M_2 agree on the overlap, since h reverses orientation, and we have a well defined orientation on $M_1 \# M_2$ by

3.35(5). The disc theorem shows that this construction is independent of the choice of the embeddings h_i.

3.36 *Exercises*
(1) Show that if one of M_1 or M_2 is not orientable then there is a well defined connected sum $M_1 \# M_2$ which is not orientable.
(2) Show that $\#$ is associative and commutative up to homeomorphism and that S^n is a unit.

We now give an application of connected sums.

Schönflies Conjecture

Suppose $S^{n-1} \subset S^n$ are spheres, then the closures of the components of $S^n - S^{n-1}$ are n-balls.

Remark. It follows from duality (see Appendix A) that $S^n - S^{n-1}$ has precisely two components. However, an elementary proof can be given. We leave this to the reader.

Now let T be the closure of a component; then we have two problems:
(1) Is T a manifold?
(2) Given that T is a manifold, is T a ball?

From 3.14, T is a manifold if and only if the other closure is a manifold and looking at the link of a point in S^{n-1} we see that (1) is equivalent to the Schönflies conjecture in dimension $n-1$. To avoid this inductive dependence we define $S^{n-1} \subset S^n$ to be *locally flat* if the closures of the components are manifolds (we have more general notions of local flatness in Chapter 4) and restate the problem as follows.

3.37 *Problem.* Suppose $S^{n-1} \subset S^n$ is locally flat then are the closures of the components of $S^n - S^{n-1}$ n-balls?

In this form the answer is known to be "yes" for $n \neq 4$. For $n \leqq 3$ there is a direct geometrical argument (see bibliography) and for $n \geqq 5$ it follows from the Poincaré theorem, since $T \underset{\partial}{\cup} B^n$ is a homotopy sphere (in fact a topological sphere by 3.39 below). This leaves the case $n=4$ still unsolved at the time of writing. This shows, by the inductive proof sketched above, that the Schönflies conjecture is true for $n \leqq 3$ but unsolved for $n \geqq 4$ and that the only obstruction to the solution of the conjecture lies in dimension 4.

We give a partial solution to 3.37:

3.38 **Weak Schönflies theorem.** *Let T be the closure of a component of the complement of a locally flat S^{n-1} in S^n and let $p \in \operatorname{int} T$. Then $T - p \cong S^{n-1} \times \mathbb{R}_+$.*

3.39 **Corollary.** *T is topologically an n-ball.*

Proof. Identify $S^{n-1}\times\mathbb{R}_+$ with I^n-0 and define the topological homeomorphism $h\colon T\to I^n$ to be the given homeomorphism on $T-p$ and define $h(p)=0$.

Proof of 3.38. Choose $q\in S^n-T$ then by 3.20 we can identify S^n-q with $\mathbb{R}^n$ and we have $T\subset\mathbb{R}^n$. Now let εT be T shrunk linearly towards p by a factor $\varepsilon>0$ chosen so small that $\varepsilon T\subset\operatorname{int}T$. We will show that $\operatorname{cl}(T-\varepsilon T)$ is a collar and then

$$\begin{aligned}T-p&=\operatorname{cl}(T-\varepsilon T)\cup\operatorname{cl}(\varepsilon T-\varepsilon^2 T)\cup\cdots\\&\cong S^{n-1}\times[0,1]\cup S^{n-1}\times[1,2]\cup\cdots\\&=S^{n-1}\times\mathbb{R}_+\end{aligned}$$

as required.

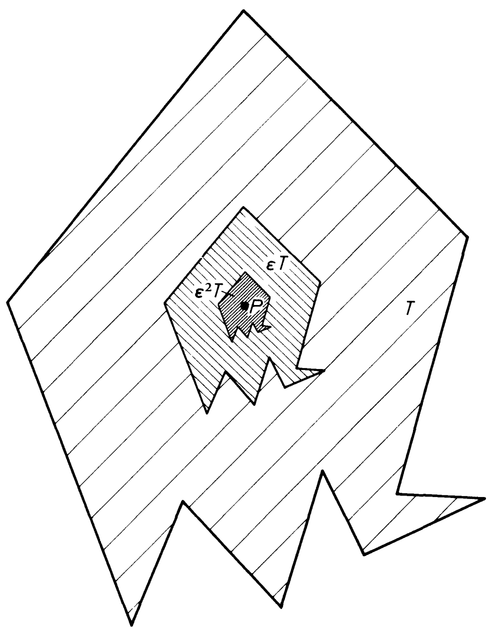


Fig. 32

Now define manifolds $M_1=B_1^n\cup T$, $M_2=T'\cup B_2^n$, and $W=B_1^n\cup\operatorname{cl}(T-\varepsilon T)\cup B_3^n$, where ∂B_1 is identified with ∂T, ∂B_2 with $\partial T'$ and ∂B_3 with $\partial\varepsilon T$. Then clearly

$$M_1\# M_2=S^n.$$

Now $W\# M_1$ can be thought of as removing B_3^n and replacing by T. But T and εT are canonically homeomorphic. So $W\# M_1$ can also be thought of as replacing B_3^n by εT this yields $B_1^n \cup T$ i.e. M_1. We have proved

$$W \# M_1 = M_1 .$$

Add M_2 to both sides:

$$(W \# M_1) \# M_2 = M_1 \# M_2 .$$

Then by exercise 3.36 we have $W \# S^n = S^n$, which implies $W = S^n$. It follows that $\mathrm{cl}(T - \varepsilon T)$ is obtained from S^n by removing disjoint balls and hence is a collar by 3.13 and 3.19.

Chapter 4. Pairs of Polyhedra and Isotopies

In this chapter we recast the last two chapters for pairs of polyhedra and manifolds. The proofs of the extended results will often be essentially the same as those of the original results and in this case we will refer back and merely sketch the changes; if the changes are obvious the proof will be omitted. We give two applications to isotopies. The first concerns "cellular moves" and will be used in the next chapter to prove basic unknotting theorems. The second application is to the general isotopy extension theorem, and is given in the final section of the chapter; this theorem will not be used again in the book and this section may be omitted or read at any later stage if the reader wishes.

Definitions

A *pair* of polyhedra (P, P_0) is a polyhedron P with a subpolyhedron $P_0 \subset P$. A *map of pairs* $f\colon (P, P_0) \to (Q, Q_0)$ is a (p.l.) map $f\colon P \to Q$ such that $f(P_0) \subset Q_0$. If P and P_0 are manifolds of dimension q and n respectively then (P, P_0) is a (q, n)*-manifold pair* denoted (Q, M), (Q^q, Q^n), $Q^{q,n}$ etc. The *codimension* of $Q^{q,n}$ is $q-n$. If Q^q and Q^n are both spheres then $Q^{q,n}$ is a *sphere pair* and if both are balls then it is a *ball pair*. A manifold pair $Q^{q,n}$ is *proper* if $Q^n \cap \partial Q^q = \partial Q^n$, and then the *boundary* $(\partial Q^q, \partial Q^n)$ or $\partial Q^{q,n}$ is a $(q-1, n-1)$-manifold pair. A proper manifold pair is *locally flat* if each point $p \in Q^n$ has a neighbourhood in $Q^{q,n}$ homeomorphic as a pair with an open set in $\mathbb{R}^{q,n}_+$, where $\mathbb{R}^{q,n}_+$ is the pair $\mathbb{R}^n_+ \times 0 \subset \mathbb{R}^q_+$, and then it is clear that $\partial Q^{q,n}$ is also locally flat. The *standard* (q, n)-ball pair is $I^{q,n} = (I^q, I^n \times 0)$ and $\partial I^{q+1,n+1}$ is the standard (q, n)-sphere pair. A ball or sphere pair is *unknotted* if it is homeomorphic with the appropriate standard pair.

Links and Stars

Joins and cones of pairs are defined in the obvious way. A *star pair* of $a \in P_0$ is a pair (N, N_0) of stars in (P, P_0) such that (N, N_0) is a cone pair (aL, aL_0) and then (L, L_0) is a *link pair*. Existence of star and link pairs follows from 1.2 (choose the smaller ε) and p.l. invariance from the proof of 2.19 which provides a homeomorphism of pairs. As corollaries we have

4.1 Corollary (cf. 2.20). *A proper cell pair is an unknotted ball pair.*

Proof. Let $C^{q,n}$ be the cell pair and $a \in \mathring{C}^n$. Without loss assume $C^{q,n} \subset \mathbb{R}^{q,n}$. Choose a cone ε-neighbourhood $(N, N_0) = (aL, aL_0)$ then $C^{q,n}$ and (N, N_0) are both star pairs of a in $C^{q,n}$, while the latter pair is linearly homeomorphic to the standard pair.

4.2 Corollary (cf. 2.21(1)). *Suppose $J \supset J_0$ are simplicial complexes then $(|J|, |J_0|)$ is a proper locally unknotted manifold pair if and only if $(|\mathrm{lk}(x, J)|, |\mathrm{lk}(x, J_0)|)$ is an unknotted ball or sphere pair for each vertex $x \in J$.*

4.3 Proposition (cf. 2.23). *Joins of sphere and ball pairs obey the rules*

$$B^{p,q} * B^{p',q'} = B^{p+p'+1, q+q'+1}$$

$$B^{p,q} * S^{p',q'} = B^{p+p'+1, q+q'+1}$$

$$S^{p,q} * S^{p',q'} = S^{p+p'+1, q+q'+1}$$

(where $q = -1$ or $q' = -1$ means the pair $(B^p, \emptyset)$ etc.). Moreover if both pairs on the left hand side are unknotted then so is the pair on the right hand side.

Proof. The first half follows by a double application of 2.23. For the second half use the proof of 2.23 and 4.1.

Exercise. Prove the converse to the second half of 4.3 by looking at a link and using induction.

4.4 Proposition (cf. 1.10). *A homeomorphism between the boundaries of unknotted ball pairs extends to the interiors. Moreover we can choose the extension to agree with any given extension on the subball.*

Proof. The first half follows from the cone construction. For the second half let $h: B^{p,q} \to D^{p,q}$ be the extension given by the first half and $g: B^q \to D^q$ the given extension on the subballs. Consider $t_1 = g \circ h_1^{-1}: D^q \to D^q$, where $h_1 = h | D^q$, then since g and h_1 agree on $\dot{D}^q$ we have $t_1 | \dot{D}^q = \mathrm{id}$. Now write $D^p = D^q * S^{p-q-1}$ by 4.3 and use the join construction with $\mathrm{id} | S^{p-q-1}$ to extend t_1 to $t: D^{p,q} \to D^{p,q}$ with $t | \dot{D}^p = \mathrm{id}$. Then $t \circ h: B^{p,q} \to D^{p,q}$ is the required extension.

Remark. For the rest of this section we will deal only with proper locally flat manifold pairs and "*manifold pair*" *will mean proper locally flat manifold pair.* From 4.2 we see that the problem of whether an arbitrary proper pair is locally flat depends on whether ball and sphere pairs unknot. This in turn depends on codimension. For codimension 1 this is the unsolved Schönflies conjecture (cf. 3.38). In codimension 2 knots are easily constructed by suspending a knotted arc in B^3. However in codimension $\geqq 3$ all pairs unknot by a general unknotting theorem (which we will prove in Chapter 7) hence all proper pairs of codimension $\geqq 3$ are automatically locally flat.

Collars

We generalise the treatment of Chapter 2 to pairs. Let

$$\begin{array}{ccc} Q & \supset & Q_0 \\ \cup & & \cup \\ P & \supset & P_0 \end{array}$$

be a pair (Q, Q_0) with subpair (P, P_0). Then the latter pair is *locally collared* in the former, if for each $a \in P$ there are neighbourhood pairs satisfying

$$(N(a, Q), N(a, Q_0)) = (N(a, P), N(a, P_0)) \times I.$$

In other words the natural generalisation of the absolute definition holds. The proof of 2.25 using these neighbourhoods provides a collar on (P, P_0) in (Q, Q_0); that is to say

4.5 Theorem. *Let $(P, P_0) \subset (Q, Q_0)$ be locally collared with (P, P_0) compact. Then (P, P_0) is collared in (Q, Q_0).*

4.6 Corollary. *Let $M^{q,n}$ be a manifold pair with compact boundary. Then $\partial M^{q,n}$ is collared in $M^{q,n}$.*

Regular Neighbourhoods

Now let $(X, X_0) \subset (P, P_0)$ where both X and X_0 are compact P_0 is a closed subset of P, and $X_0 = X \cap P_0$. Then *derived neighbourhoods* of X in (P, P_0) are constructed by triangulating a neighbourhood of X in P by the complex J so that X and $P_0 \cap |J|$ both correspond to subcomplexes K and J_0 with $K \trianglelefteq J$. Then both $N(K, J')$ and $N(K_0, J_0')$ are derived neighbourhoods where $K_0 = K \cap J_0$ and the pair $N(K_0, J_0') \subset N(K, J')$ is a *derived neighbourhood* of X in (P, P_0).

ε-neighbourhoods are similarly constructed and the underlying polyhedron pair corresponding to a derived or ε-neighbourhood is a *regular neighbourhood.* The proof of 3.8, unchanged, shows that regular neighbourhoods are unique up to a homeomorphism of (P, P_0) with compact support and fixed on X.

Simplicial Neighbourhood Theorem for Pairs

4.7 Theorem. *Let* (N, N_0) *be a neighbourhood of* (X, X_0) *in the interior of the manifold pair* $M^{q,n}$. *Then* (N, N_0) *is a regular neighbourhood if and only if*

(1) (N, N_0) *is a manifold pair*

(2) *there is a triangulation* (K, K_0) *of* (N, N_0) *with subcomplexes* (L, L_0), (J, J_0) *corresponding to* (X, X_0), $(\dot{N}, \dot{N}_0)$ *such that* $L \trianglelefteq K$, $K = N(L, K)$, $J = \dot{N}(L, K)$ *and similar formulae with* J_0, K_0, L_0 *replacing* J, K, L.

Proof. "Only if" follows from definitions and a similar proof to 3.10, using 4.2. "If" is proved by induction together with the following corollaries (the induction starts with $n = -1$, i.e. the absolute case).

4.8$_{q,n}$ Corollary (cf. 3.12). *Let* $B^{q,n} \subset M^{q,n}$ *be an unknotted ball pair, then* $B^{q,n}$ *is a regular neighbourhood of any point* $x \in \mathring{B}^n$ *in* $M^{q,n}$.

Proof. We have $(B^{q,n}, x) = (I^{q,n}, 0)$ by the cone construction and we may triangulate $I^{q,n}$ as the cone from 0 on a triangulation of $\dot{I}^{q,n}$. The result now follows from $4.7_{q,n}$.

4.9$_{q,n}$ Corollary (cf. 3.13). *Let* $B^{q,n} \subset S^{q,n}$ *be an unknotted ball pair in an unknotted sphere pair. Then* $\mathrm{cl}(S^{q,n} - B^{q,n})$ *is an unknotted ball pair.*

Proof. We can take $S^{q,n} = \dot{I}^{q+1,n+1}$ and $B^{q,n}$ to be a face pair by the argument of 3.13. Then $\mathrm{cl}(S^{q,n} - B^{q,n})$ is the opposite face pair with a collar on the boundary.

4.10$_{q+1,n+1}$ Corollary (cf. 3.14). *Let* $M^{q+1,n+1} \subset \mathring{Q}^{q+1,n+1}$ *be manifold pairs. Then* $\mathrm{cl}(Q - M)$ *is a manifold pair.*

The induction step now follows from 4.10 by the same proof as 3.11 but using collars for pairs, 4.6.

The other corollaries to 3.11 all have analogues for pairs, which we leave the reader to state; the proofs are directly analogous to the original ones. Isotopies of pairs are defined in the obvious way and the proof of the absolute regular neighbourhood theorem gives:

4.11 Regular neighbourhood theorem for pairs. *Let* $(N_i, N_{i,0})$ *be regular neighbourhoods of* X *in* (P, P_0) *for* $i = 1, 2$. *Then there is an ambient isotopy of* (P, P_0) *fixed on* X *and with compact support carrying* $(N_1, N_{1,0})$ *to* $(N_2, N_{2,0})$. *Moreover if* (P, P_0) *is a manifold pair and* X *is in the interior then we can assume the isotopy is fixed on any smaller neighbourhood and outside any larger one.*

Collapsing and Shelling for Pairs

Let $(X, X_0) \supset (Y, Y_0)$ with $Y_0 = Y \cap X_0$ and suppose $X \searrow\!\!\!\searrow Y$ across B^n from B^{n-1}. Then we say that the collapse *respects* X_0 if *either* $X_0 = Y_0$ (so that X_0 does not meet $\mathring{B}^n$ or $\mathring{B}^{n-1}$) *or* $B^n \subset X_0$ (so that $X_0 \searrow\!\!\!\searrow Y_0$). In other words, the collapse either misses X_0 completely or else takes place in X_0. A sequence of elementary collapses which respect the subpolyhedron is referred to as a *collapse of pairs* written $(X, X_0) \searrow (Y, Y_0)$. Now suppose $M^{q,n} \supset M_0^{q,n}$ and M^q shells elementarily to M_0^q across B^q from B^{q-1}, then the shelling *respects* M^n if either $M_0^n = M^n$ or else $B^q \cap M^n$ and $B^{q-1} \cap \partial M^n$ are unknotted subballs, so that M^n shells to M_0^n and we are removing an unknotted ball pair from $M^{q,n}$ by an unknotted face. A sequence of elementary shellings which respect the submanifold is a *shelling of pairs*.

Exercise. $M^{q,n}$ shells to $M_0^{q,n}$ implies $M^{q,n} \searrow M_0^{q,n}$.

4.12 Lemma (cf. 3.25). *If $M^{q,n}$ shells to $M_0^{q,n}$ then there is a homeomorphism h: $M^{q,n} \to M_0^{q,n}$ fixed outside any neighbourhood of the shelling.*

4.13 Theorem (cf. 3.26). *Suppose $M^{q,n}$ is a manifold pair and (X, X_0), $(Y, Y_0) \subset M^{q,n}$ with $X_0 = X \cap M^n$ and $Y_0 = Y \cap M^n$. Then if $(X, X_0) \searrow (Y, Y_0)$ then a regular neighbourhood of X in $M^{q,n}$ shells to one of Y in $M^{q,n}$.*

4.14 Corollary (cf. 3.27). *If $(X, X_0) \searrow 0$ then a regular neighbourhood of (X, X_0) in $M^{q,n}$ is an unknotted ball pair.*

Proof of 4.13. Examine the proof of 3.26. There are two cases:

(1) $Y_0 = X_0$, in which case the proof of 3.26 gives a shelling of $N^{q,n}$ to $N_1^{q,n}$ without change.

(2) $(X, X_0) \searrow\!\!\!\searrow (Y, Y_0)$ by a collapse in X_0. In this case the proof of 3.26 generalised to pairs shows that $N^{q,n}$ differs from $N_1^{q,n}$ by the addition of ball pairs by face pairs and then by a similar inductive application of 4.14 we see that both pairs are unknotted. Hence $N^{q,n}$ shells to $N_1^{q,n}$, as required.

We leave the reader to formulate and prove analogues of the other corollaries to 3.26.

Application to Cellular Moves

Two locally flat submanifolds of dimension n, $M_1, M_2 \subset Q$, are said to differ by a *cellular move* provided there is an embedded $(n+1)$-disc $D^{n+1} \subset \mathring{Q}$, which meets M_1 and M_2 in complementary faces, and M_1 agrees with M_2 away from D^{n+1}:

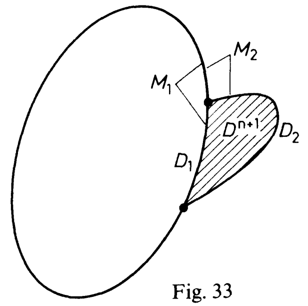


Fig. 33

More precisely,

$$\operatorname{cl}\big((M_1 \cup M_2) - (M_1 \cap M_2)\big) = \partial D^{n+1}$$

and

$$D^{n+1} \cap M_i = \partial D^{n+1} \cap M_i = D_i^n \qquad \text{for } i = 1, 2.$$

Notice that we do not assume that D^{n+1} is locally flat in Q. The usefulness of cellular moves is the following result.

4.15 Proposition. *Let $M_1, M_2 \subset Q$ differ by a cellular move. Then there is an isotopy of Q carrying M_1 to M_2 with support in an arbitrary neighbourhood of D^{n+1}.*

Proof. Triangulate a smaller neighbourhood of D in Q, M_1 and M_2 so that D is a full subcomplex and let N, N_i be the resulting ε-neighbourhoods of D in Q, M_i, $i=1,2$. Then (N, N_i) is a regular neighbourhood pair of (D, D_i) in (Q, M_i) and hence an unknotted ball pair by 4.14 since $(D, D_i) \searrow (D_i, D_i) \searrow 0$. Now $(\dot{N}, \dot{N}_1) = (\dot{N}, \dot{N}_2)$ and by 4.4 there is a homeomorphism h: $(N, N_1) \to (N, N_2)$ extending the identity on boundaries. Then by 3.22(i) h is isotopic to id mod boundaries and the required isotopy of Q is defined by extending this isotopy by the identity.

4.16 Corollary. *Let $S^{q,n}$ be a locally flat sphere pair. Then $S^{q,n}$ is unknotted if and only if S^n bounds an $(n+1)$-ball B^{n+1} in S^q.*

Proof. If $S^{q,n}$ is unknotted then S^n bounds an $(n+1)$-ball since $S^n \subset S^n * S^{q-1}$ bounds $S^n *$ point (cf. 4.3). The result now follows from:

Sublemma. If $S_0^n \subset S^q$, $S_1^n \subset S^q$ both bound $(n+1)$-balls then there is a homeomorphism h: $S^q \to S^q$ such that $h(S_0^n) = S_1^n$.

Proof of sublemma. Triangulate S^q with B^{n+1} (a ball spanning S_0^n) a subcomplex and, after further subdivision if necessary, find an $(n+1)$-simplex $A \in B^{n+1}$ which meets S^n in a top dimensional face. Then $D = \operatorname{cl}(B - A)$ is a ball by 3.25 and S^n, $\dot{A}$ differ by a cellular move across D. So we may assume that S_0^n is the boundary of a simplex which in turn is the face of a q-simplex. This is also true of S_1^n and the result follows since any two q-simplexes are ambient isotopic by the disc theorem (3.34).

Exercise. Define $S^n \subset \mathbb{R}^q$ to be unknotted, if S^n is ambient homeomorphic to $\dot{I}^{n+1} \subset \mathbb{R}^q$. Prove an analogous statement to 4.16.

Disc Theorem for Pairs

Finally we generalise parts of the last section of Chapter 3.

4.17 Proposition (cf. 3.33). *Let $M^{q,n}$ be a manifold pair with M^n connected and $x, y \in \mathring{M}^n$. Then there is an ambient isotopy of $M^{q,n}$ fixed on $\partial M^{q,n}$ of compact support carrying x to y.*

4.18 Proposition (cf. 3.32). *Let $S^{q,n}$ be an unknotted sphere pair and $h\colon S^{q,n} \to S^{q,n}$ a self homeomorphism which preserves orientation of both factors. Then h is ambient isotopic to* id.

Proof. By induction on q. Let $B^{q,n} \subset S^{q,n}$ be an unknotted pair and $C^{q,n}$ the complementary pair and $x \in \mathring{B}^n$. We may assume $h(x)=x$ by 4.17 and that $h(B^{q,n})=B^{q,n}$ by 4.9 and 4.11. Then $h|\partial B^{q,n}$ is isotopic to the identity by induction and we extend this isotopy to $S^{q,n}$ by two applications of 4.19(a) (below); finally use 4.19(b) twice to complete the proof.

4.19 Lemma (cf. 3.22).

(a) *Any isotopy of $\partial M^{q,n}$ extends to $M^{q,n}$.*

(b) *Let $h_i\colon B^{q,n} \to C^{q,n}$, $i=1,2$, be homeomorphisms which agree on $\partial B^{q,n}$. Then h_1 is ambient isotopic to h_2.*

4.20 Theorem (Disc theorem for pairs). *Let $M^{q,n}$ be a connected oriented pair (i.e. both are connected and oriented), and let $h_i\colon I^{q,n} \to \mathring{M}^{q,n}$ be embeddings, $i=1,2$, which preserve orientation on both factors. Then there is an ambient isotopy of $M^{q,n}$ fixed on $\partial M^{q,n}$ and carrying h_1 to h_2.*

Proof. By 4.17, 4.9 and 4.11 we can assume $h_1(I^{q,n})=h_2(I^{q,n})$ (as in proof of 4.18). Then by 4.18 and 4.19(a) we can assume $h_1|\partial I^{q,n}=h_2|\partial I^{q,n}$. Now use 4.19(b).

Remark. Stronger forms of 4.18 and 4.20 are true, in which we assume $h=\mathrm{id}$ on S^n (or $h_1=h_2$ on I^n) and obtain an isotopy fixed on the submanifold. These are proved by using *relative* regular neighbourhoods, which are a more complicated and more general tool than regular neighbourhoods for pairs (see bibliography).

Isotopy Extension

In this final section we study the question mentioned in the last chapter of when a given isotopy $F\colon X \times I \to Y \times I$ is ambient. The spirit of our result is similar to the spirit of the collaring theorem — F is ambient if and only if it is locally ambient (i.e. for each $(x,t)\in X\times I$ we can find a "short" isotopy of a neighbourhood of $F_t(x)$ in Y which covers the restriction of F to a neighbourhood of (x,t)). In fact we will get away with a rather weaker condition (see below). A useful corollary is that an isotopy of a manifold M in a manifold Q is ambient provided $F(M\times J)$ is locally flat in $Q\times J$ for each subinterval $J=[s,t]\subset I$. This is always true in codimension $\geqq 3$ by the unknotting theorem mentioned earlier. The main theorem will follow from existence of collars for pairs and a procedure for making a map level-preserving.

We first prove an extension of 4.5 (the collaring theorem for pairs) to the case where $P_0\subset Q_0$ has a given collar and we wish to extend it to a collar on P in Q.

Definitions

(1) We extend the meaning of a *collar* on P in Q to include an embedding $c\colon P\times J\to Q$ onto a neighbourhood such that c identifies $P\times l$ with P where l is one endpoint of J. Throughout this section J denotes an interval $[s,t]\subset I$.

(2) A collar $c'\colon P\times J'\to Q$ is a *reduction* of c if $c'(x\times J')\subset c(x\times J)$ for each $x\in P$. I.e. near P, c and c' determine the same collar lines, but the parametrisation might well be different.

(3) Let $(P,P_0)\subset(Q,Q_0)$ be a compact locally collarable subpair and $c_0\colon P_0\times I\to Q_0$ a given collar. Then c_0 is *locally extendible* if for each $x\in P_0$ there is a collar pair defined locally whose restriction to P_0 is a reduction of c_0.

4.21 Addendum to 4.5 (Extending collars). *Let $(P,P_0)\subset(Q,Q_0)$ be a compact locally collarable subpair and $c_0\colon P_0\times I\to Q_0$ a locally extendible collar. Then there is an $\varepsilon>0$ and a collar $c\colon P\times[0,\varepsilon]\to Q$ which agrees with c_0 where they are both defined.*

Proof. The proof of 2.25 using the local extensions gives a collar c_1 which restricts to a reduction of c_0. But we can correct c_1 to agree with c_0 on $P_0\times[0,\varepsilon]$, by the following sublemma, where ε is chosen so small that $c_0(P_0\times[0,\varepsilon])\subset c_1(P_0\times[0,1))$.

Sublemma. *Suppose given an embedding $q\colon P_0\times[0,\varepsilon]\to P_0\times[0,1)$, $0<\varepsilon<1$, which is a reduction of the idendity. Then there is a homeomorphism $q_1\colon P\times I\to P\times I$ which extends q and such that $q_1(x\times I)=x\times I$ for each $x\in P$.*

Proof. The construction of q_1 is similar to the construction of g in the proof of 2.26. Use the method of construction of g to define q_1 on

$P_0 \times I$ then extend to $P \times I$ using a cylindrical cell subdivision and inductive conical extension.

4.22 Corollary. *Let $M^{q,n}$ be a manifold pair with compact boundary and c_0: $\partial M^n \times I \to M^n$ a collar, then there is a collar c: $\partial M^q \times [0, \varepsilon] \to M^q$ which agrees with c_0 where they are both defined.*

Proof. Local extendibility follows easily from local flatness.

4.23 Level-preserving lemma. *Suppose X is compact and c: $X \times I \to X \times I$ is a collar on $X \times 0$ in $X \times I$. Then there is an $\varepsilon > 0$ and a collar c_1: $X \times I \to X \times I$ such that $c_1 | X \times [0, \varepsilon]$ is level-preserving. Moreover*

(1) *c and c_1 agree outside an arbitrary neighbourhood of $X \times 0$ and are ambient isotopic fixing $X \times 0$ and the complement of this neighbourhood.*

(2) *If $c | X_0 \times [0, \delta]$ is already level-preserving then we can assume $c_1 | X_0 \times I = c | X_0 \times I$ and the isotopy fixes $X_0 \times I$.*

Proof. Let c: $K \to L$ be a triangulation of c and choose an $\varepsilon > 0$ so that no vertices of K or L lie in $X \times (0, \varepsilon]$. Form deriveds K', L' of K, L near $X \times 0$ be starring each simplex on the ε-level. Then let c_1: $K' \to L'$ be the canonical simplicial embedding. It remains to check the properties listed:

(1) The first half is assured by choosing fine enough triangulations for K and L and the second half follows from 3.23.

(2) Star $c(A)$ at $c(a)$ where $A \in K$ is starred at a.

Now let F: $X \times I \to Y \times I$ be an isotopy of compact polyhedra. We say that F is *locally trivial* if for each subinterval $J \subset I$, the natural collar on $F(X \times \dot{J})$ in $F(X \times J)$ is locally extendible to a collar on $Y \times \dot{J}$ in $Y \times J$.

Remarks

(1) A priori a locally trivial isotopy need not be locally ambient (see the beginning of this section) since a local extension need not

 (a) be level-preserving

 (b) agree precisely with the natural collar on $F(X \times I)$.

However the conditions are in fact equivalent by Theorem 4.26 below.

(2) Local triviality can be reformulated in an intrinsic way without reference to collars, using a notion of "intrinsic dimension" (see bibliography and historical notes).

4.24 Isotopy extension theorem. *Let F: $X \times I \to Y \times I$ be an isotopy of compact polyhedra. Then F is ambient if and only if it is locally trivial.*

4.25 Corollary. *An isotopy F: $M \times I \to Q \times I$ of compact manifolds is ambient provided $F(M \times J) \subset Q \times J$ is locally flat for each subinterval $J \subset I$.*

Proof. Local triviality follows easily from local flatness.

Proof of the theorem. "Only if" is obvious. To prove "if", consider $t \in I$ and $J=[s,t]$ a subinterval. Then by local triviality and 4.21 there is a collar $c\colon Y \times [t-\varepsilon, t] \to Y \times J$ which extends the natural collar on $F(X \times t)$ in $F(X \times J)$ in other words so that $c \circ (F_t \times \mathrm{id}) = F$. By 4.23 we can assume that c is level preserving for perhaps a smaller ε. I.e. we have a "short" isotopy covering F for times "before" t. Similar remarks apply "after" t and we have a short isotopy covering F for all times near t. Therefore, using compactness of I, we can find a finite number of intervals $[t_{i-1}, t_i]$ which cover I, $0=t_0<t_1<\cdots<t_j=1$ and such that for each i there is a short isotopy (i.e. level-preserving homeomorphism):

$$H^{(i)}\colon\ Y \times [t_{i-1}, t_i] \to Y \times [t_{i-1}, t_i]$$

such that

$$H^{(i)} \circ (F_{s_i} \times \mathrm{id}) = F \,|\, X \times [t_{i-1}, t_i]$$

for some (fixed) $s_i \in I$. We form the required isotopy H by piecing together the $H^{(i)}$'s: Define

$$H \,|\, Y \times [0, t_1] = H^{(0)} \circ \big((H_0^{(0)})^{-1} \times \mathrm{id}\big)$$

then

$$\begin{aligned} H \circ (F_0 \times \mathrm{id}) &= H^{(0)} \circ \big((H_0^{(0)})^{-1} \times \mathrm{id}\big) \circ (F_0 \times \mathrm{id}) \\ &= H^{(0)} \circ (F_{s_0} \times \mathrm{id}) \\ &= F, \quad \text{as required.} \end{aligned}$$

In general define $H \,|\, Y \times [t_{i-1}, t_i]$ inductively by

$$H \,|\, Y \times [t_{i-1}, t_i] = H^{(i)} \circ \big((H^{(i)}_{t_{i-1}})^{-1} \times \mathrm{id}\big) \circ (H_{t_{i-1}} \times \mathrm{id})$$

and the covering property is proved similarly.

Exercises

(1) Examine the compactness requirements of 4.24.

(2) By examining the proof of 4.24 show that we can assume H is fixed outside an arbitrary neighbourhood of the *track* of F $\big(= \bigcup_t F_t(X)\big)$.

(3) Prove also that if F is a proper isotopy of manifolds and is fixed on ∂M (i.e. $F_t | \partial M = F_0$) then we can assume H is fixed on ∂Q.

(4) Use 4.23 to prove uniqueness of collars up to isotopy by "shrinking" the time parameter and using the obvious isotopy which matches level-preserving collars. Use 4.24 to deduce a uniqueness theorem up to ambient isotopy in the case when $Q - \mathrm{im}(c)$ is locally collarable at $c(P \times 1)$. Deduce that collars of manifolds are unique up to ambient isotopy.

Chapter 5. General Position and Applications

We give general position theorems for polyhedra in manifolds and applications to unknotting in the stable range, piping and the Whitney lemma. The last two applications will be used in the proof of the h-cobordism theorem in the next chapter.

General Position

We consider two situations

(i) $P, Q \subset M$ are polyhedra. We wish to minimise the dimension of $P \cap Q$ by a small ambient isotopy of P in M.

(ii) $f\colon P \to M$ is a map. We wish to minimise the dimension of the singular set of f by a small homotopy of f.

The program is: first, prove relative theorems for $M = \mathbb{R}^m$ by triangulating and shifting vertices; second, cover P or $f(P)$ by charts in M and inductively apply results for $\mathbb{R}^m$ to each chart in turn.

5.1. *Definitions*

(1) For this section only, *map* means continuous map rather than p.l. map.

(2) Let Y be a metric space. A homotopy $f\colon X \times I \to Y$ is an *ε-homotopy* if for each $(x, t) \in X \times I$, $d\big(f(x, 0), f(x, t)\big) < \varepsilon$. In other words, each point stays in an ε-neighbourhood of its initial position during the homotopy.

(3) An isotopy $F\colon X \times I \to Y \times I$ is an *ε-isotopy* if the composition $\pi_1 \circ F\colon X \times I \to Y \times I \to Y$ is an ε-homotopy.

(4) A map $f\colon X \to Y$ is *closed* if $f(C)$ is closed in Y for each closed set $C \subset X$. Thus an embedding is closed if and only if its image is a closed subset.

(5) The *singular set* of a map $f\colon X \to Y$, denoted $S(f) \subset X$, is defined by

$$S(f) = \mathrm{cl}\{x \mid x \in X,\ f^{-1}f(x) \neq x\}.$$

Thus $f \mid X - S(f)$ is injective.

(6) Let $f: P \to Q$ be p.l. then f is *non-degenerate* if $f^{-1}(y)$ is 0-dimensional for each $y \in f(P)$.

5.2 *Exercises*

(1) Suppose that $f: P \to Q$ is p.l. and P is compact. Then $S(f)$ is a subpolyhedron of P.

(2) Is $S(f)$ a subpolyhedron if P is non-compact? What if P is non-compact and f is closed?

(3) Let P be compact then $f: P \to Q$ is non-degenerate if and only if $f|A$ is injective for each simplex A of P in any triangulation of f.

(4) Let f be non-degenerate and suppose that (P, P_0) is a compact pair. Define $P/f|P_0$ by identifying $x \in P_0$ with $y \in P_0$ if $f(x) = f(y)$. Then $P/f|P_0$ can be given the structure of an abstract polyhedron, so that the quotient map $\pi: P \to P/f|P_0$ is p.l. (see 2.27(3)).

Definition. Suppose $P^p, Q^q \subset M^m$ are subpolyhedra of the unbounded m-manifold M and that $p+q=m$, where $p=\dim(P)$, $q=\dim(Q)$. We say P is *transverse* to Q in M if

(i) $P \cap Q$ consists of a finite set of points,

(ii) for each $p \in P \cap Q$ there are neighbourhoods U_1, U_2, U_3 of p in P, Q, M such that (U_1, U_2, U_3) is p.l. homeomorphic to a neighbourhood of 0 in $(\mathbb{R}^p \times 0, 0 \times \mathbb{R}^q, \mathbb{R}^p \times \mathbb{R}^q)$.

Remark. There are more general definitions of transversality. See bibliography for references.

5.3 General position theorem for embeddings. *Let $Q^q, P_0 \subset P^p$ be closed subpolyhedra of the unbounded manifold M^m with $\mathrm{cl}(P-P_0)$ compact. Let $\varepsilon > 0$ be given. Then there is an ε-isotopy of M with compact support, fixed on P_0 and finishing with $h: M \to M$ such that*

$$\dim\{h(P-P_0) \cap Q\} \leqq p+q-m.$$

Addendum. If $p+q=m$ then we can also arrange that $h(P-P_0)$ meets Q transversely.

We describe the application of 5.3 as "shifting P into general position with respect to Q, keeping P_0 fixed".

5.4 General position theorem for maps. *$P_0 \subset P$ is a closed subpolyhedron with $\mathrm{cl}(P-P_0)$ compact. $f: P^p \to M^m$ is a closed map with $p \leqq m$, such that $f|P_0$ is* p.l. *and non-degenerate. $\varepsilon > 0$ is given. Then there is an ε-homotopy of f* rel P_0 *to f' which is* p.l. *and non-degenerate and such that $\dim(S(f') - P_0) \leqq 2p - m$.*

Addendum. If $m = 2p$ then we can also arrange that the singularities of $f'|P-P_0$ are transverse double points.

We describe 5.4 as "shifting f into general position rel P_0".

Proof of 5.3. Special case $M = \mathbb{R}^m$. Let N be a compact neighbourhood of $\mathrm{cl}(P - P_0)$ in $\mathbb{R}^m$ which meets P, Q in compact polyhedra P_1, Q_1 say. Choose linear triangulations (J, K, L, K_0) of $(N, P_1, Q_1, P_1 \cap P_0)$. Order the vertices of $K - K_0$. Suppose there are t of them. For each vertex in turn define an ε/t-homeomorphism of J by shifting the vertex a distance less than ε/t and extending conewise to the star. This "linear move" is supported by a ball and hence the end of an ε/t-isotopy of compact support by 3.22. Choose the moves in turn to make the set $K^{(0)} \cup L^{(0)}$ maximally affine independent and then the required properties are easily checked.

General case. Let B_i, $i = 1, \ldots, t$, be a cover of $\mathrm{cl}(P - P_0)$ by m-balls in M. Define

$$P_r = P_0 \cup \bigcup_{i=1}^{r} (P \cap B_i)$$

then $P_t = P$.

Induction hypothesis. The theorem is true with P replaced by P_r.

The hypothesis is trivially true for $r = 0$. Suppose it is true for $r - 1$. Let U be an open neighbourhood of B_r homeomorphic with $\mathbb{R}^m$ ($U = B \cup$ open collar) and let $A_1 = P \cap B_r$. Choose, by induction, a δ-isotopy of M carrying P to $h(P)$ with $h(P_{r-1} - P_0) \cap Q$ of minimal dimension and $\delta < \varepsilon/2$ sufficiently small that $h(A_1) \subset U$.

Now define

$$A_0 = h(P_{r-1}) \cap U$$
$$A = h(P_r) \cap U = A_0 \cup h(A_1)$$

then apply the case $M = \mathbb{R}^m$ to A, A_0 in U to get an $\varepsilon/2$-isotopy of U of compact support moving $h(A_1)$ into general position with respect to $Q \cap U$. Extend to M by the identity. Combining the two isotopies establishes the induction step.

Proof of the addendum. The case $M = \mathbb{R}^m$ is easy since by independence the only intersections must lie in the interiors of top dimensional simplexes. The general case follows.

Proof of 5.4. $M = \mathbb{R}^m$. By 5.2(4) we can assume that $f|P_0$ is an embedding (first without loss of generality restrict to a compact neighbourhood of P in P_0). Now triangulate P, P_0 by complexes K, K_0 of sufficiently small mesh that $f(\mathrm{st}(v, K)) \subset \varepsilon/2$-ball for each vertex $v \in K$. Choose images $f'(v)$ for each $v \in K - K_0$ within $\varepsilon/2$ of $f(v)$ so that $f'(K^{(0)})$ is maximally independent. Define f' by extending linearly to simplexes and use the linear homotopy $f \simeq f'$. The required properties are easily checked. The general case and addendum follow as in 5.3.

Exercises. Remove the compactness condition on $\mathrm{cl}(P-P_0)$ by using locally finite covers and a countable induction. Prove theorems for bounded M by first working in $\dot{M}$ and then considering the double of M.

Embedding and Unknotting

5.5 Theorem (Embedding in double dimension). *Let M^m be a compact m-manifold. Then there is an embedding $M^m \to \mathbb{R}^{2m}$, provided $m>2$.*

Proof. It suffices to consider one component, so without loss we may assume M is connected. Let $f\colon M\to\mathbb{R}^{2m}$ be a map in general position. Then f has only double points $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ say, i.e. $f(x_i)=f(y_i)$ and f is an embedding off $X=\{x_i\}\cup\{y_i\}$. By connectivity and general position we can assume that the cone on X, CX is embedded in M extending the inclusion of X, see Fig. 34. Again by general position we have $Cf(CX)$ embedded in $\mathbb{R}^{2m}$ extending the inclusion of $f(CX)$ and meeting $f(M)$ in $f(CX)$. Now choose triangulations so that X, CX, $Cf(CX)$ are subcomplexes and f is simplicial. Take second deriveds so that f is still simplicial and let N_0 be the second derived neighbourhood of CX in M and N the second derived neighbourhood of $Cf(CX)$ in $\mathbb{R}^{2m}$. Then $f\,\partial N_0\subset\partial N$ and $f\,N_0\subset N$. Now N_0, N are balls by 3.27 since CX, $Cf(CX)$ are collapsible, and further $f\,|\,\mathrm{cl}(M-N_0)$ is an embedding. Now redefine f on N_0 as follows. By the cone construction choose an embedding $N_0\to N$ extending f on ∂N_0. We now have the required embedding.

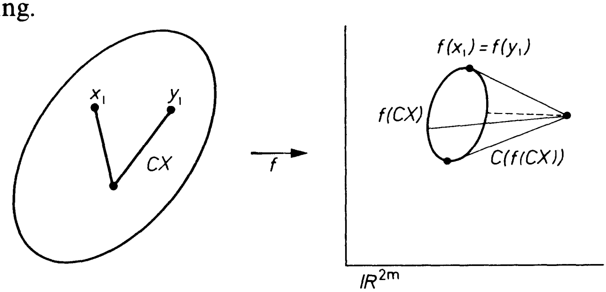


Fig. 34

Remarks

(1) The embedding constructed in 5.5 is locally flat. This follows from 5.7 below.

(2) If $m=2$, the result is still true since a closed 2-manifold is known to be a connected sum of tori and projective planes, each of which embeds in $\mathbb{R}^4$.

(3) In Chapter 7 we will improve 5.5 considerably in the case that M is more highly connected.

5.6 Theorem (Unknotting spheres).

(i) S^1 *unknots in* S^q *for* $q \geqq 4$.

(ii) S^n *unknots in* S^q *for* $q \geqq 2n+1$ *and* $n \geqq 2$.

5.7 Corollary. *A proper manifold pair* $M^{q,n}$ *is locally flat provided* $n=1$, $q \geqq 1$ *or* $n=2$, $q \geqq 5$ *or* $n>2$, $q \geqq 2n$.

Proof. The case $n=1$ is trivial; the other cases follow from 5.6 on looking at link pairs.

Proof of 5.6. (i) By 3.20 we can assume $S^1 \subset \mathbb{R}^q$ and notice that S^1 is locally flat since S^0 unknots in S^{q-1}.

Now choose a point $x \in \mathbb{R}^q$ in "general position" with respect to S^1. More precisely choose $|L| = S^1$ and

$$x \in \mathbb{R}^q - \bigcup \{\langle AB \rangle \mid A, B \in L\}.$$

Recall that $\langle P \rangle$ is the minimal subspace spanned by P. Then xS^1 is a cone, hence a ball, and the result follows from 4.16.

(ii) By (i) any $M^2 \subset M^q$ is locally flat for $q \geqq 5$. This is the start of an induction. Assume inductively that $m \geqq 2$ and any $M^m \subset M^q$ is locally flat for $q \geqq 2m+1$. Let $S^m \subset S^q$ be the given pair. By 3.20 we can assume $S^m \subset \mathbb{R}^q$. We claim that a point $x \in \mathbb{R}^q$ can be chosen in "general position" with respect to S^m so that no line through x meets S^m in more than two points and that each such line is isolated. This is seen as follows:

Choose $|L| = S^m$ and define

$$T = \bigcup \{\langle AB \rangle \mid A, B \in L, \langle AB \rangle \neq \mathbb{R}^q\}.$$

Then $\mathbb{R}^q - T$ is open and dense and if $x \notin T$ and $A, B \in L$ then there is at most one line through x meeting both A and B, for otherwise $x \in \langle AB \rangle$ and $\dim \langle AB \rangle \leqq 2m$ which implies $x \in T$. It follows that only finitely many lines through x meet S^m in more than one point, and further each such line pierces only m-dimensional simplexes in their interiors. Now suppose A, B, C are m-simplexes of L and l is a line which pierces each of A, B, C at an interior point. Call l a transversal and let $T(A, B, C)$ be the union of the transversals of A, B and C. Then $T(A, B, C)$ is part of an algebraic variety of dimension $<q$ and since L is finite we may suppose that $x \notin T(A, B, C)$ for any choice of A, B, C. The required properties of x are now clear.

Now consider the singular cone xS^m. A typical singular ray l_i meets S^m in two points n_i, f_i and we choose the labels so that n_i, the *near* point, is nearer to x than f_i. Define $N = \bigcup \{n_i\}$ the *near set* and $F = \bigcup \{f_i\}$ the *far set*. Since $m \geqq 2$ we can find an arc $\alpha \subset S^m$ with $N \subset \alpha$ and $F \cap \alpha = \emptyset$; then by taking a suitable regular neighbourhood of α we have a ball $B^m \subset S^m$ with $N \subset \mathring{B}^m$, $F \subset S^m - B^m$.

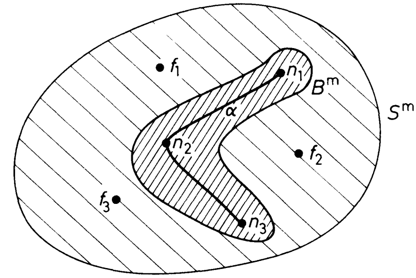


Fig. 35

Define $S_1^m = S^m - B^m \cup x\dot{B}^m$. Then S_1^m differs from S^m by the cellular move across xB^m (which is a cone since B^m contains only near points). But S_1^m bounds the ball $x(S^m - \mathring{B}^m)$ (see Fig. 36).

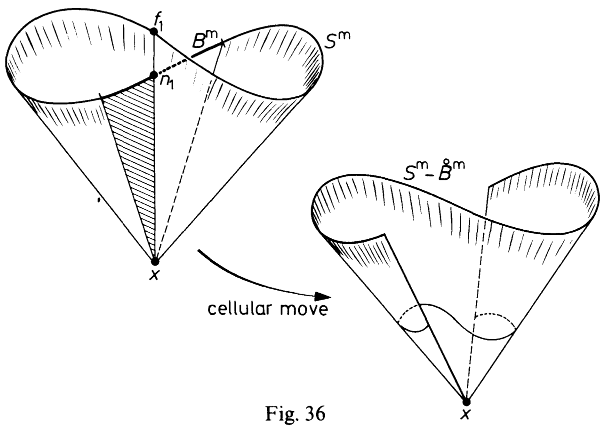


Fig. 36

5.8 Corollary. *Suppose F: $M \times I \to \operatorname{int} Q$ is an embedding and $q \geqq 2m$, then $F_0(M)$ and $F_1(M)$ are ambient isotopic by an isotopy supported by a compact set in* $\operatorname{int} Q$.

Proof. By 3.26 $M \times [-1, 1]$ shells to $M \times [-1, 0]$. Use this shelling to define a series of cellular moves from $F_0(M)$ to $F_1(M)$. The result then follows from 4.15 since any embedding of M in Q is locally flat by 5.7.

Remark. We show later (7.1) that the hypothesis of 5.6 and hence of 5.8 can be weakened to $q - m \geqq 3$.

5.9 Corollary. *Suppose f_0, f_1: $M \to \operatorname{int} Q$ are homotopic embeddings, M is closed and $q \geqq 2m+2$. Then $f_0(M)$ and $f_1(M)$ are ambient isotopic by an isotopy supported by a compact set in* $\operatorname{int} Q$.

Proof. Let $f: M \times I \to \operatorname{Int} Q$ be the homotopy which we can assume to be in general position. Then $S(f) \subset M \times (0,1)$ consists of points $x_1, \ldots, x_n, y_1, \ldots, y_n$ and $f(x_i) = f(y_i)$, $i = 1, \ldots, n$ are n distinct points of $f(M \times I)$. As in the proof of 5.6 choose arcs $\alpha_{(j)}$ in each component of $M \times I$ which contain the x_i but not the y_i and each of which meets $M \times 1$ in one point $x_{(j)}$ and does not meet $M \times 0$, see Fig. 37.

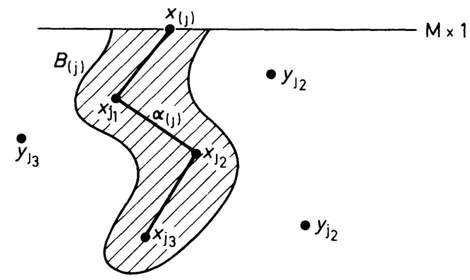


Fig. 37

Take $B_{(j)}$ to be a regular neighbourhood of $\alpha_{(j)}$ which misses the y_i. Then $B_{(j)}$ is a ball and $B_{(j)} \cap M \times 1$ is a face. Then there is a series of cellular moves across the $B_{(j)}$ and $\operatorname{cl}(M \times I - \bigcup B_{(j)}) \cong M \times I$ is embedded by f. The result now follows from 5.8.

Example. $M = S_1^m \cup S_2^m$, $Q = S^{2m+1}$ then there are homotopic embeddings which are not isotopic. They are constructed by winding S_1^m with degree r around S_2^m using the fact that $Q - S_2^m$ has the homotopy type of an m-sphere by 5.6. See final exercises of this chapter for more details.

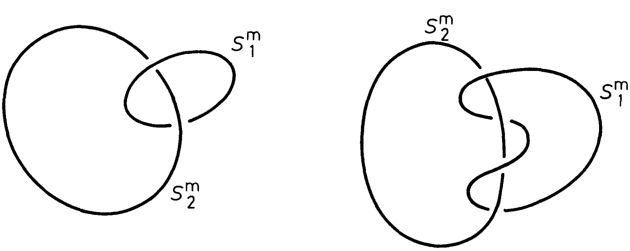


Fig. 38

Exercise. The conclusion of 5.9 still holds if M is not closed provided the homotopy is fixed outside a compact m-manifold $M_0 \subset \operatorname{Int} M$.

Piping

Suppose $M_1^m, M_2^m \subset Q^q$ are two locally flat submanifolds of the connected manifold Q and that $q-m \geqq 2$. We will explain how to form a new submanifold M_3^m by "piping" M_1 and M_2 together. This is done by removing the interiors of small m-discs in each of M_1 and M_2 and running a "tube" between the two "holes" thus formed. The tube is an embedded $S^{m-1} \times I$ hence M_3 is homeomorphic with $M_1 \# M_2$. We can arrange that M_3 is oriented correctly in the case that M_1 and M_2 are both oriented.

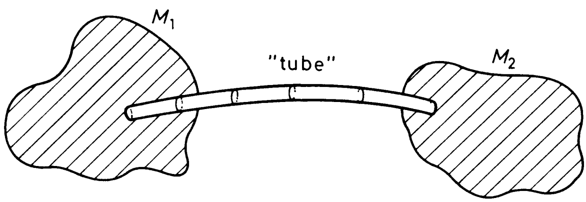


Fig. 39

The tube is found in a neighbourhood of an arc α from $a_1 \in M_1$ to $a_2 \in M_2$; α exists by connectedness and general position.

5.10 Proposition. *Let (N, N_1, N_2) be a regular neighbourhood of α in (Q, M_1, M_2). Then there is a homeomorphism*

$$h\colon (N, N_1, N_2) \to (I^{q-1} \times [-2, 2], I^m \times (-1), I^m \times 1)$$

and h can be chosen, provided $q-m \geqq 2$, to preserve any given orientations.

Using 5.10 we can define the tube to be $h^{-1}(\dot{I}^m \times [-1, 1])$ and the required properties of M_3 are obvious.

Proof of 5.10. Let J triangulate Q so that M_i, $i=1, 2$ appears as a subcomplex P_i and α appears as a full subcomplex K. Let $L \subset K$ be the simplicial complement of a_2. Now let J' be a derived of J near $L \cup a_2$. Without loss we may assume that $N = |N(K, J')|$, $N_i = |N(a_i, P_i')|$. Now by the proof of 3.26 $|N(L, J')| \cap |N(a_2, J')|$ is a $(q-1)$-ball, B^{q-1} say, and by 4.14 $(|N(a_2, J')|, N_2)$ and $(|N(L, J')|, N_1)$ are both unknotted ball pairs. Choose homeomorphisms

$$h_1\colon (|N(L, J')|, N_1) \to (I^{q-1} \times [-2, 0], I^m \times (-1)),$$

$$h_2\colon (N(a_2, J'), N_2) \to (I^{q-1} \times [0, 2], I^m \times (+1)).$$

By composing with suitable reflections we can assume that the h_i preserve orientations. Finally we have only to ensure that $h_i | B^{q-1}$ is a homeomorphism onto $I^{q-1} \times 0$ and that $h_1 | B^{q-1} = h_2 | B^{q-1}$ and then we can define $h=h_1 \cup h_2$. But this follows easily from the disc theorem, applied to give an isotopy of $\partial(I^{q-1}\times[-2,0])-\dot{I}^m\times(-1)$ with compact support carrying $h_1(B^{q-1})$ onto $I^{q-1}\times 0$ (and similarly for h_2). Notice that $q-m\geqq 2$ is used here to conclude that this manifold is connected.

Exercise. The piping tube defined using 5.10 is unique up to ambient isotopy provided M_1, M_2 are both connected, $q-m\geqq 3$ and Q is simply-connected.

Hint. By general position and connectivity α is unique. Now use regular neighbourhoods and induction to match two tubes.

Whitney Lemma and Unlinking Spheres

The Whitney lemma enables us to cancel double points. The situation is this. We are given a pair of connected locally flat submanifolds $P^p, Q^q \subset M^m$ which are transverse, so that $p+q=m$.

If each of P, Q and M are oriented, then we can attach a sign to an intersection point $p\in P\cap Q$ (see below), and the idea is to give conditions under which we can "cancel" a pair of intersections of opposite sign; in other words find an ambient isotopy of P which removes this pair from the set of intersections of P and Q.

Let $p\in P\cap Q$, then by transversality we can find an embedding $h\colon I^m\to M$ such that $h(0)=p$, $h^{-1}(P)=I^p\times 0$ and $h^{-1}(Q)=0\times I^q$.

5.11 Lemma. *The orientation class of h is determined by the orientation classes of $h|I^p\times 0$ and $h|0\times I^q$.*

Proof. Suppose h_1 and h_2 are two such charts and that $h_i|I^p\times 0$ and $0\times I^q$ are in the same class. Then by the S.N.T. (for triples) we can assume $\operatorname{im}(h_1)=\operatorname{im}(h_2)$ and we have $g=h_1^{-1}h_2|\dot{I}^m$ a self-homeomorphism of I^m which preserves $\dot{I}^p\times 0$ and $0\times\dot{I}^q$, and orientation of both of these. Now g is isotopic either to the identity or to r_m. In the first case h_1 is easily seen to be isotopic to h_2.

So assume g is isotopic to r_m. Then $g\colon(\dot{I}^m, 0\times\dot{I}^q)\to(\dot{I}^m, 0\times\dot{I}^q)$ is isotopic to r_m as a homeomorphism of pairs by 4.18 and hence $g|\dot{I}^p\times 0\colon \dot{I}^p\to\dot{I}^m-0\times\dot{I}^q$ is isotopic to r_p. This contradicts the assumption that $g|\dot{I}^p\times 0$ is orientation preserving since $\dot{I}^m-0\times\dot{I}^q$ deformation retracts on $\dot{I}^p\times 0$ and we get a self-homotopy of $\dot{I}^p$ reserving orientation (which is impossible by 3.31).

Using 5.11 we can define the *sign* of p, $\varepsilon(p)=\pm 1$, as follows. Choose h so that $h|I^p\times 0$ and $h|0\times I^q$ are in the given orientation classes for P and Q. Then $\varepsilon(p)=+1$ if h is in the given class for M and -1 if not. We also define the *intersection number* of P and Q, $\varepsilon(P,Q)$, to be $\sum\{\varepsilon(p)\,|\,p\in P\cap Q\}$.

5.12 Whitney lemma (simply-connected version). *Suppose P, Q, M are given as above and that $p, q \in P \cap Q$ satisfy $\varepsilon(p) = -\varepsilon(q)$. Then there is an isotopy of M carrying P to P' with P' transverse to Q in M and with $P' \cap Q = P \cap Q - p - q$; provided either*

(1) $p \geqq 3$, $q \geqq 3$ *and* $\pi_1(M) = 0$ *or*

(2) $p = 2$, $q \geqq 3$ *and* $\pi_1(M - Q) = 0$.

Moreover the isotopy has support in a compact set which does not meet any other intersection points. (See appendix for definition of $\pi_1(\;)$.)

5.13 Corollary. *If $\varepsilon(P, Q) = 0$ and the hypotheses of* 5.12 *are satisfied then we can ambient isotope P off Q, by an isotopy which has compact support.*

Remarks

(1) If $p \geqq 3$ then $\pi_1(M) \cong \pi_1(M - Q)$ by general position; therefore we can restate the lemma with the single hypothesis $\pi_1(M - Q) = 0$.

(2) The Whitney lemma fails for $p = q = 2$, see bibliography.

We will prove 5.12 by induction on $m = \dim M$ together with a theorem on unlinking spheres. By a *link* we mean a triple $S_1^p, S_2^q \subset S^r$ of spheres where (S, S_i) is an unknotted pair for $i = 1, 2$. The *standard* link is $\dot{I}^{p+1} \times (-1)$, $\dot{I}^{q+1} \times 1 \subset \partial(I^r \times [-2, 2])$ and a link is *unlinked* if it is homeomorphic with the standard link.

5.14 *Exercise.* A link is unlinked if and only if there is a ball $B^r \subset S^r$ with $S_1 \subset \mathring{B}$, $S_2 \subset S - B$.

Hint. Use the disc theorem (as in the proof of 5.10).

We are interested in links in the *critical* dimension $r = p + q + 1$. If $r > p + q + 1$ then all links are unlinked by general position (suppose $p \leqq q$ and find a disc spanning S^p in the complement of S^q, then take a suitable regular neighbourhood to be the B^r of 5.14). We say that a link is *homologically trivial* if S_1 is homologous to zero in $S - S_2$ (or more precisely if $i_*\colon \tilde{H}_p(S_1) \to \tilde{H}_p(S - S_2)$ is the zero map). We shall see in the next lemma that this is a symmetric condition.

5.15 Lemma. *The following are equivalent:*

(1) (S, S_1, S_2) *is homologically trivial;*

(2) *for each locally flat disc $D_1 \subset S$ with $\partial D_1 = S_1$ and D_1 transverse to S_2 we have $\varepsilon(D_1, S_2) = 0$;*

(3) *for each disc triple (D, D_1, D_2) with boundary the given link such that (D, D_i) is unknotted $i = 1, 2$ and D_1 transverse to D_2 we have $\varepsilon(D_1, D_2) = 0$.*

Proof. (1) *is equivalent to* (2): Notice that $S-S_2$ deformation retracts on a p-sphere and that a generator of $H_p(S-S_2)$ can be described as the restriction $h|\dot{I}^{p+1}\times 0\to S-S_2$ where h: $(I^{p+1+q}, 0\times I^p)\to(S, S_2)$ is an embedding preserving orientation of both factors (since any two such are ambient isotopic by 4.20). Now let D_1 be the given disc and $p\in D_1\cap S_2$. Then by transversality we can find a p-sphere $S_p=L_p(D_1)$, which represents $\varepsilon(p)$ times the generator of $H_r(S-S_2)$ by definition. Then $D_1-\bigcup\{\mathring{D}_p|p\in D_1\cap S_2\}$ represents a homology between S_1 and $\bigcup S_p$ and hence S_1 represents $\varepsilon(D_1, S_2)$ times the generator which implies the result (for more details on the interpretation of homology used here see Appendix A).

(3) We have $S-S_2$ homotopy equivalent to $D-D_2$ by inclusion. We can then interpret a generator of $H_p(D-D_2)$ in a similar way to part (1) and the argument is now similar.

We now give the unlinking theorem which uses the Whitney lemma and which will be used inductively in the proof of the Whitney lemma:

5.16 Theorem (unlinking spheres). *Let* (S^r, S_1^p, S_2^q) *be a link in the critical dimension and* $r\geqq 4$. *Then* (S, S_1, S_2) *is unlinked if and only if it is homologically trivial.*

Example. The link of 1-spheres in S^3 (Fig. 40) is homologically trivial but not trivial.

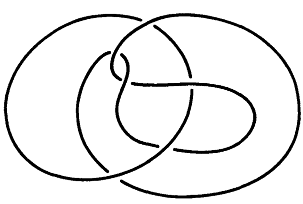


Fig. 40

Proof of 5.16. The "only if" part is obvious. We prove the converse. Without loss of generality we can assume $p\leqq q$. We have two cases:

$p\leqq 1$. The case $p=0$ is easy so assume $p=1$, $q\geqq 2$. Then $S-S_2$ is homotopy equivalent to S^1 and S_1 is homologically trivial, and hence homotopically trivial, in $S-S_2$. The result follows from 5.9.

$p\geqq 2$ *assuming 5.12*. Choose a locally flat disc D_1 with $\partial D_1=S_1$, transverse to S_2 by general position. Then $\varepsilon(D_1, S_2)=0$ by 5.15 and, by Corollary 5.13 applied to $\mathring{D}_1, S_2\subset S-S_1$, we can assume $D_1\cap S_2=\emptyset$. Now let B be a regular neighbourhood of D_1 in $S-S_2$ then $S_1\subset\mathring{B}$, $S_2\subset S-B$ and the link is unlinked by 5.14.

Proof of 5.12 (assuming 5.16 in dimensions $<m$). Join p and q by arcs α, β, in P and Q respectively, which do not run through any other intersections.

Claim. There is a 2-disc $D^2 \subset M$ with $\partial D^2 = \alpha \cup \beta$ and $D^2 \cap (P \cup Q) = \partial D^2$. For if $p \geqq 3$ then by hypothesis there is a map $f\colon D^2 \to M$ with $f(\partial D^2) = \alpha \cup \beta$ and the result follows by general position. If $p=2$ then take a regular neighbourhood of β in M, P, Q say N, N_0, N_1. Then there is a homeomorphism

$$h\colon (N, N_0, N_1) \cong \left(I^p \times I^{q-1} \times [-2,2],\ I^p \times 0 \times (-1 \cup +1),\ 0 \times I^{q-1} \times [-2,2]\right)$$

by a similar argument to the one used in the proof of 5.10. Without loss we may assume that

$$h(\alpha \cap N_0) = [0,1] \times 0 \times (-1 \cup +1).$$

Let $\alpha' = \mathrm{cl}(\alpha - \alpha \cap N_0)$,

$$\beta' = h^{-1}(1 \times 0 \times [-1,1]), \qquad D_1^2 = h^{-1}([0,1] \times 0 \times [-1,1]).$$

The $\alpha' \cup \beta' \subset M - Q$ and by hypothesis there is a map $f\colon D^2 \to M - Q$ which by general position we can take embedded with interior disjoint from D_1^2. Then $f(D^2) \cup D_1^2$ is the required disc.

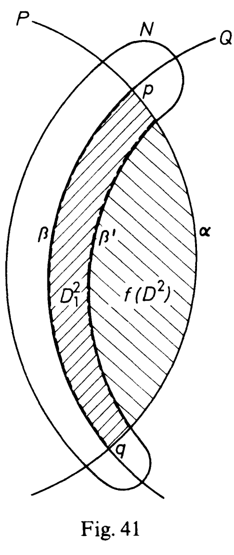


Fig. 41

Now let (N, B_1, B_2) be a regular neighbourhood of D^2 in (M, P, Q). Then (N, B_i) is an unknotted ball pair for $i=1,2$ by 4.14. Consider the link $(S, S_1, S_2) = \partial(N, B_1, B_2)$. Then we have $\varepsilon(B_1, B_2) = 0$ by hypothesis and hence (S, S_1, S_2) is unlinked by 5.15 and 5.16. Therefore there is an unknotted subball B_1' with $\partial B_1' = S_1$ and $B_1' \cap B_2 = \emptyset$ (this follows from the definition of the standard link and 4.4). Then by 4.4 and 3.22 there is an isotopy of N fixing $\dot{N}$ carrying B_1 to B_1' and extending to M by the identity gives the required isotopy of M.

Non-Simply-Connected Whitney Lemma

Now suppose that $\pi_1(M) \neq 0$ and assume P and Q are simply connected and oriented. Choose a basepoint $* \in M$, a local orientation for M at $*$ and basepaths e_P, e_Q from $*$ to basepoints in P and Q respectively. Let $p \in P \cap Q$ then define $\varepsilon(p) = \pm g$, where $g \in \pi_1(M)$ is the element determined by the loop $e_P \rho \tau e_Q$ where ρ is a path in P from the basepoint to p and τ a path in Q from p to the basepoint of Q. The sign of $\varepsilon(p)$ is determined by comparing the local orientation of M at p, which comes from transporting the local orientation at $*$ along $e_P \rho$, with the orientation given by 5.11. Then we can again define $\varepsilon(P, Q) \in \mathbb{Z}(\pi_1(M))$ to be $\sum \{\varepsilon(p) \mid p \in P \cap Q\}$, where $\mathbb{Z}(\pi)$ denotes the integral group ring of π.

The statement of the lemma now makes sense and the hypotheses read either

(1) $p \geqq 3$, $q \geqq 3$, or

(2) $p = 2$, $q \geqq 3$, and $\pi_1(M) \cong \pi_1(M - Q)$.

The proof is then virtually unaltered since $\varepsilon(p) = -\varepsilon(q)$ ensures that $\alpha \cup \beta$ is a trivial loop and hence that $\alpha' \cup \beta'$ is trivial in $M - Q$ if $p = 2$, and that $\varepsilon(B_1, B_2) = 0$ as before.

Final exercises

(1) Define the *homological linking number* of an oriented link as the image of the generator of $\tilde{H}_p(S_1)$ in $\tilde{H}_p(S - S_2)$ and check symmetry using an analogue of 5.15.

(2) Show that $(\dot{I}^{p+1} \times 0, 0 \times \dot{I}^{q+1} \subset \dot{I}^{p+q+1})$ has linking number 1.

(3) Show by piping, using (2), how to construct links with arbitrary linking numbers.

(4) Show that two oriented links (in the critical dimension) are homeomorphic if and only if they have the same linking number, provided $r \geqq 4$. And hence combining (3) and (4) that these links are classified by their linking numbers.

(5) Give an alternative proof of the Whitney lemma, without using links, as follows:

(a) construct a standard picture for a neighbourhood of D in M;

(b) identify the neighbourhood of D in M with the standard picture in three steps:

(i) identify a neighbourhood of β (exactly as in the given proof),

(ii) identify a neighbourhood of $D - D_1$,

(iii) match these two identifications on observing that they meet in a "piping tube" (use uniqueness of piping tubes and model the proof on 5.10);

(c) find, in the standard picture, an unknotted disc B_1' with $\partial B_1' = \partial B_1$ and $B_1' \cap B_2 = \emptyset$. Complete the proof as before.

(6) *Hard.* Notice that there is an element of choice in the arc β' in the case $p=2$ of the Whitney lemma, namely we can alter it by "twisting the I^2 factor on its axis" see Fig. 42. Exploit this element of choice to show that it is not necessary to assume $\pi_1(M) \cong \pi_1(M-Q)$, which implies $\gamma \sim 0$ in $M - Q$ where γ is the loop "once round a transverse disc to Q", but merely that γ is in the centre of $\pi_1(M-Q)$. The idea is to span $\alpha \cup \beta'$ by a disc which meets Q in a finite number of points and deduce that $\alpha \cup \beta' \sim n\gamma$. Then kill this "obstruction" by twisting β' around Q n times.

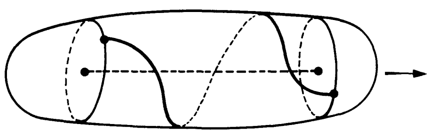


Fig. 42

Chapter 6. Handle Theory

Let W^w be a manifold and H a w-ball such that $W \cap H \subset \partial W$, and suppose that there is a homeomorphism $h\colon I^p \times I^q \to H$, such that $h(\dot{I}^p \times I^q) = H \cap W$. Then we say that (H, h) is a *handle of index p on W*, or simply that "H is a *p-handle*".

Notice that $W' = W \cup H$ is also a w-manifold, since a point has a neighbourhood which is the union of two balls meeting in a common face. If we write $f = h | \dot{I}^p \times I^q$ then we can identify $W \cup H$ with $W \cup_f I^w$ see 2.27(2); thus we say that W' is formed from W by *attaching a handle* by f. Conversely, given any embedding $f\colon \dot{I}^p \times I^q \to \partial W$, then $W' = W \cup_f I^w$ can be regarded as W with an attached p-handle in the obvious way. We write variously $W' = W \cup H = W \cup H^{(p)} = W \cup_f H$.

Terminology. Let (H, h) be a p-handle. Then we call $h(I^p \times 0)$ the *core* of H and $h(0 \times I^q)$ the *cocore*. $h(\dot{I}^p \times 0)$ is the *attaching sphere* (a-sphere) and $h(0 \times \dot{I}^q)$ the *belt sphere* (b-sphere). We also have the *a-tube* $h(\dot{I}^p \times I^q)$ and the *b-tube* $h(I^p \times \dot{I}^q)$. Finally h is the *characteristic map* of H, and $f = h | \dot{I}^p \times I^q$ the *attaching map.*

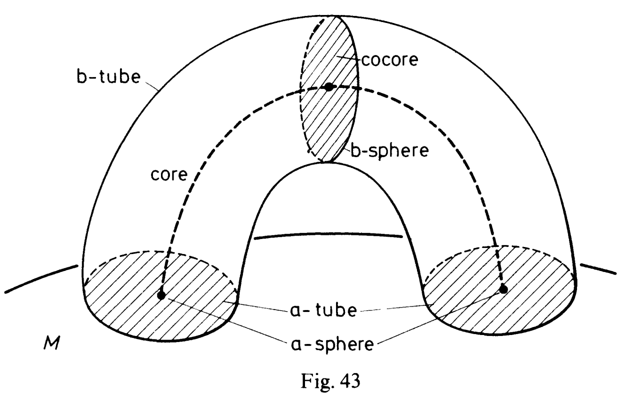


Fig. 43

Fig. 43 shows a 1-handle on a 3-manifold. Note that a 0-handle on W is a w-disc disjoint from W and that, at the other extreme, a w-handle is a disc with its whole boundary equal to a component of ∂W.

The idea of a handle is that it gives an elementary way of enlarging a manifold. We shall see below that any manifold can be regarded as constructed from a ball by attaching handles; such a recipe is called a handle decomposition. However, before examining complete decompositions, we first examine the geometry of two handles added consecutively and we introduce the "handle moves"—reordering, cancelling, adding. The h-cobordism theorem will follow from these moves together with the Whitney lemma and a recipe for computing homology from a handle decomposition. After the proof of the h-cobordism theorem, we will state and prove the two extensions mentioned in Chapter 1.

Handles on a Cobordism

Let (W, M_0, M_1) be a cobordism and H a handle on W then if $H \cap W \subset M_1$ we say H is a *handle on the cobordism.* There is a new cobordism $(W', M_0, \partial W' - M_0)$ which we say is obtained from the original cobordism by attaching a handle. For most of the applications of handles we will be concerned with handles on a cobordism; notice that when $M_0 = \emptyset$ the concept reduces to that of a handle on a manifold.

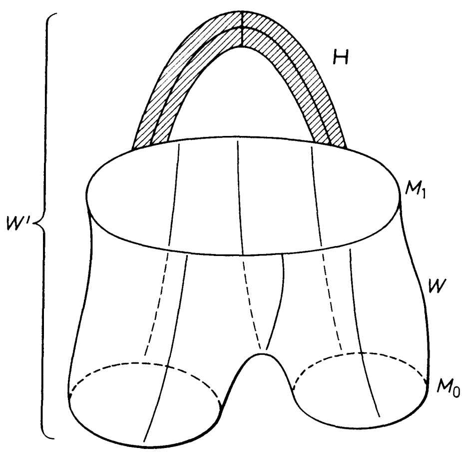


Fig. 44

Our first lemma shows that the result of attaching a handle depends only on the isotopy class of the attaching map:

6.1 Lemma. *Let* $f, g\colon \dot{I}^p \times I^q \to M_1$ *be ambient isotopic embeddings then there is a homeomorphism*

$$h\colon\ W \cup_f H \to W \cup_g H$$

which is the identity outside a collar on M_1 *in* W.

Proof. Let $H_t\colon M_1 \to M_1$ be the covering isotopy and c a collar on M_1. Then H_t extends to W by 3.22 so that it is the identity outside c. Let H_1 be the finishing homeomorphism and define

$$h=\begin{cases} H_1 & \text{on } W \\ \mathrm{id} & \text{on } I^p \times I^q. \end{cases}$$

Reordering Handles

For the next few sections we will be concerned with the result of attaching two handles consecutively. The notation $W \cup H^{(r)} \cup H^{(s)}$ means that $H^{(r)}$ is a r-handle on the cobordism W and $H^{(s)}$ is an s-handle on the cobordism $W \cup H^{(r)}$.

6.2 Reordering lemma. *Let $W' = W \cup H^{(r)} \cup H^{(s)}$ with $s \leqq r$. Then $W' \cong W \cup H^{(s)} \cup H^{(r)}$ with $H^{(r)}$ and $H^{(s)}$ disjoint.*

Proof. Let $f\colon \dot{I}^s \times I^{w-s} \to M_2$ be the attaching map for $H^{(s)}$ where $M_2 = \partial(W \cup H^{(r)}) - M_0$. We will show how to ambient isotope f so as to make its image disjoint from $H^{(r)}$. Then we can clearly attach the handles in reverse order and the result follows from 6.1. Denote by S^{s-1} the a-sphere of $H^{(s)}$ and by S^{w-r-1} the b-sphere of $H^{(r)}$. Then by general position in M_2^{w-1}, we can assume $S^{s-1} \cap S^{w-r-1} = \emptyset$. Now choose regular neighbourhoods N_a of S^{s-1} and N_b of S^{w-r-1} which are disjoint. Then observe that, by the S.N.T., the a-tube N_a' of $H^{(s)}$ is also a regular neighbourhood of S^{s-1} in M_2 so that we can assume $N_a' = N_a$. Similarly, the b-tube N_b' of $H^{(r)}$ is a regular neighbourhood of S^{w-r-1} in M_2 and we have an ambient isotopy of M_2 carrying N_b onto N_b'. This isotopy carries N_a off N_b' and hence carries $\operatorname{im} f$ off $H^{(r)}$ as required. See Fig. 45.

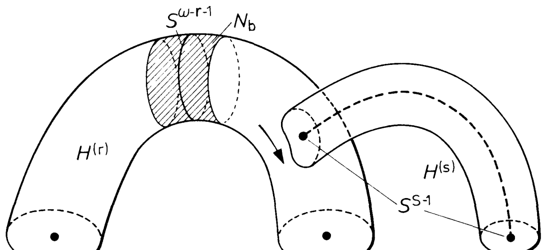


Fig. 45

Handles of Adjacent Index

Suppose $W' = W \cup H^{(r)} \cup H^{(r+1)}$ and let $M_2 = \partial(W \cup H^{(r)}) - M_0$ (as in the last proof). Then the b-sphere S_1 of $H^{(r)}$ and the a-sphere S_2 of $H^{(r+1)}$

are in complementary dimension in M_2. So by a small general position shift of the attaching map of $H^{(r+1)}$ we can assume that S_2 meets S_1 transversally in a finite number of points.

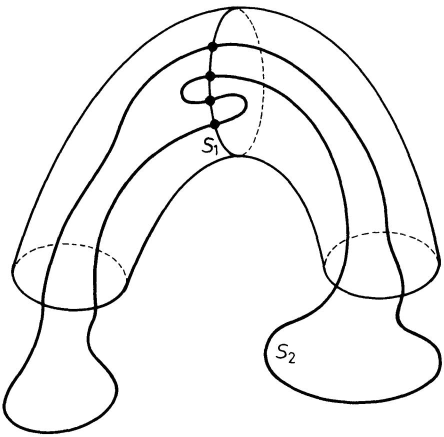


Fig. 46

We can then define the *incidence number* $\varepsilon(H^{(r+1)}, H^{(r)})$ to be the intersection number $\varepsilon(S_1, S_2)$, as defined in the last chapter, since the characteristic maps give standard orientations to S_1, S_2 and the b-tube of $H^{(r)}$. The next result gives an important homology interpretation of this incidence number and shows that it depends only on the homotopy class of the attaching map of $H^{(r+1)}$.

6.3 Lemma. *Let q: $W \cup H^{(r)} \to S^r$ be the (topological) map which sends W to a basepoint $* \in S^r$, collapses $H^{(r)}$ onto its core D^r and identifies $D^r/\partial D^r$ with $S^r/*$. Let g: $S_2 \to S^r$ be the restriction of q, then g has homological degree $\varepsilon(H^{(r+1)}, H^{(r)})$.*

Proof. Let f be the attaching map of $H^{(r+1)}$. The degree of g is unaffected by an isotopy of f. Consider a point $p \in S_1 \cap S_2$. Then the characteristic map h for $H^{(r)}$ defines a standard transverse disc $D_p = h(I^r \times p)$ to S_1 at p. By the definition of transversality and the disc theorem for pairs we can isotope S_2 rel S_1 to make it agree with D_p near p. Do this for each intersection then after a further isotopy which carries a standard neighbourhood of S_1 onto the b-tube we have $S_2 \cap H^{(r)} = \bigcup \{D_p | p \in S_1 \cap S_2\}$. Now $q | D_p$ is the standard identification of $D_p/\partial D_p$ with $S^r/*$ and the result now follows easily from the definition of degree. See Appendix A. (Notice that the orientation of S_2 agrees with D_p if and only if $\varepsilon(p) = +1$.)

Complementary Handles

With the same notation as above, suppose that S_1 and S_2 intersect transversally in just one point p. $H^{(r)}$ and $H^{(r+1)}$ are then said to be *complementary* handles. The importance of such pairs is:

6.4 Cancellation lemma. *Suppose $W' = W \cup H^{(r)} \cup H^{(r+1)}$ with $H^{(r)}$ and $H^{(r+1)}$ complementary. Then there is a homeomorphism $h\colon W' \to W$ which is the identity outside a neighbourhood of $H^{(r)} \cup H^{(r+1)}$.*

Proof. As in the last proof we can assume $S_2 \cap (b\text{-tube of } H^{(r)}) = D_p$ where $S_1 \cap S_2 = p$. Then, by the disc theorem for pairs again, we can assume that $h_1(I^r \times B^r) = h_2(D^r \times I^r)$ where h_1 is the characteristic map for $H^{(r)}$, h_2 that for $H^{(r+1)}$ and B^r, D^r are neighbourhoods of p in S_1, S_2 respectively. Then by expanding a standard neighbourhood of S_1 onto the b-tube of $H^{(r)}$ we can assume that these are the only intersections of $H^{(r)}$ and $H^{(r+1)}$. W' now shells to W in two steps:

(1) shell $H^{(r)}$ from $h_1(I^r \times (S_1 - \mathring{B}))$,

(2) shell $H^{(r+1)}$ onto $h_2((S_2 - \mathring{D}) \times I^r)$.

The result now follows from 3.25.

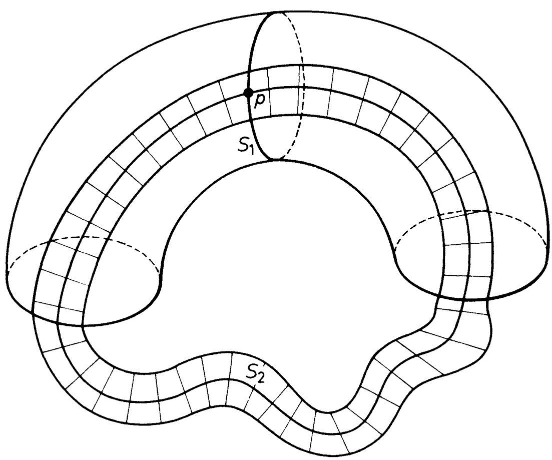


Fig. 47

The next corollary says that handles which are algebraically complementary can be cancelled under extra conditions. This comes from a combination of the cancellation lemma and the Whitney lemma. Here we see why the theory only works well if $w \geqq 6$:

6.5 Corollary. *Suppose $W' = W \cup H^{(r)} \cup H^{(r+1)}$ and M_1 is simply-connected, $w - r \geqq 4$, $r \geqq 2$ and $w \geqq 6$. Then if $\varepsilon(H^{(r+1)}, H^{(r)}) = \pm 1$, $W' \cong W$.*

Proof. Use the terminology of the last proof. Then $\varepsilon(S_1, S_2) = \pm 1$ and we wish to use the Whitney lemma to find an ambient isotopy of S_2 which carries S_2 to S_2' with $S_1 \cap S_2' =$ one point. The result will then follow from 6.1 and 6.4. Now S_1 is in codimension $\geqq 2$, S_2 in codimension $\geqq 3$. Moreover there are deformation retractions of $M_2 - S_1$ and $M_1 -$ (a-sphere of $H^{(r)}$) onto $M_1 -$ (a-tube of $H^{(r)}$) given by using the product structure of $I^r \times I^{w-s}$. It follows from general position that $M_1 -$ (a-sphere of $H^{(r)}$) is simply-connected and hence the Whitney lemma applies.

In the proof of 6.4 we had $H^{(r)} \cap H^{(r+1)}$ a $(w-1)$-ball so that $H^{(r)} \cup H^{(r+1)}$ was a w-ball which, the reader can check, was attached to W by a face. (I.e. we could have done the shelling in one step instead of two.) We now reverse the argument and show how to regard a ball attached to W as a complementary pair of handles of any index:

6.6 Introduction lemma. *Suppose $W' = W \cup B^w$ where $B^w \cap W = B \cap M_1 =$ face B_1 of B. Then we can write $W' = W \cup H^{(r)} \cup H^{(r+1)}$ with $H^{(r)}$ and $H^{(r+1)}$ complementary.*

Moreover if $B^r \subset B_1$ is any locally flat disc then we can assume that the a-sphere of $H^{(r)}$ is ∂B^r and that (a-sphere of $H^{(r+1)}$) $\cap W \subset B^r$.

Proof. Consider the "standard" complementary pair:

$$H_1 = I^r \times ([1,3] \times I^{w-r-1})$$

$$H_2 = I^{r+1} \times I^{w-r-1}$$

with

$$H_1 \cup H_2 = I^r \times [-1, 3] \times I^{w-r-1}$$

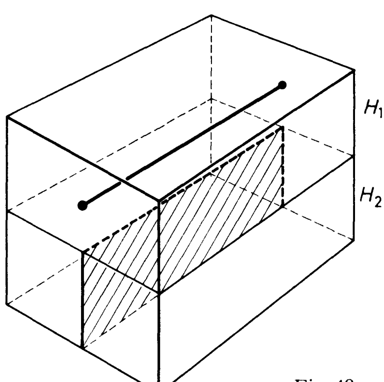


Fig. 48

where the core of H_1 is $I^r \times 2 \times 0$ and the core of H_2 is $I^{r+1} \times 0$. Then $H_1 \cup H_2$ is a ball with face

$$Q = \dot{I}^r \times [-1, 3] \times I^{w-r-1} \cup I^r \times (-1) \times I^{w-r-1}$$

and if we identify the pair $H_1 \cup H_2$, Q with B, B_1 we have the required result. For the last part of the lemma use the following exercise to identify B^r with $\dot{I}^r \times [-1, 2] \cup I^r \times (-1)$.

Exercise. Any two locally flat embeddings $B^r \subset \mathring{B}^m$ are ambient isotopic by an isotopy of compact support.

Hint. Identify $\mathring{B}$ with $\mathbb{R}^m$ and use the proof of 4.16 to show B^r is ambient isotopic to a simplex.

Adding Handles

We now show how to isotope the attaching map of an r-handle by "sliding" it over an adjacent r-handle. This has the result of adding (or subtracting) the incidence numbers of the r-handles with $(r-1)$-handles and is a key step in the algebraic simplification of handle decompositions.

Suppose $W' = W \cup_f H^{(r)}$ and that M_1 is simply-connected and $r \geqq 2$. Then $f | \dot{I}^r$ determines a class in $\pi_r(M_1)$ which we will denote $[f]$.

6.7 Adding lemma. *Suppose* $W' = W \cup_{f_1} H_1 \cup_{f_2} H_2$ *with* $\operatorname{im}(f_1)$ *and* $\operatorname{im}(f_2)$ *disjoint and index* $H_1 =$ *index* $H_2 = r$. *Suppose that* $w - r \geqq 2$, $r \geqq 2$ *and* M_1 *is simply-connected. Then there is an* f_3 *isotopic to* f_2 *such that* $[f_3] = [f_2] + [f_1]$, *and* $\operatorname{im}(f_1) \cap \operatorname{im}(f_3) = \emptyset$.

Alternatively we can find f_3 *so that* $[f_3] = [f_2] - [f_1]$.

Proof. Let h_1 be the characteristic map of H_1 and $c\colon \dot{I}^r \times \dot{I}^{w-r} \times I \to \operatorname{cl}(M_2 - H_1 - H_2)$ a collar on the boundary of the a-tube of H_1, where $M_2 = \partial(W \cup H_1) - M_0$ as usual (see Fig. 49).

Let $S_1 = c(\dot{I}^r \times x \times 1)$ for some $x \in \dot{I}^{w-r}$. Then S_1 bounds the embedded r-disc $D_1 = c(\dot{I}^r \times x \times I) \cup h_1(\dot{I}^r \times x)$. Define $S_2 = a$-sphere of H_2. Form S_3 by piping S_1 and S_2 together in M_2 (see Chapter 4) and define $D = h^{-1}(I^{r-1} \times [-1, 1])$ with the notation of 5.10 (the "solid" piping tube). Then S_2 is ambient isotopic to S_3 by two cellular moves.

Move 1. Across the piping tube D.

Move 2. Across D_1 (see Fig. 50).

Finally by a regular neighbourhood argument we can assume that $f_3 =$ (finishing homeomorphism of this isotopy) $\circ f_2$ is disjoint from f_1. The properties are clear – the sign in the formula comes from the two possible choices of orientation for S_1.

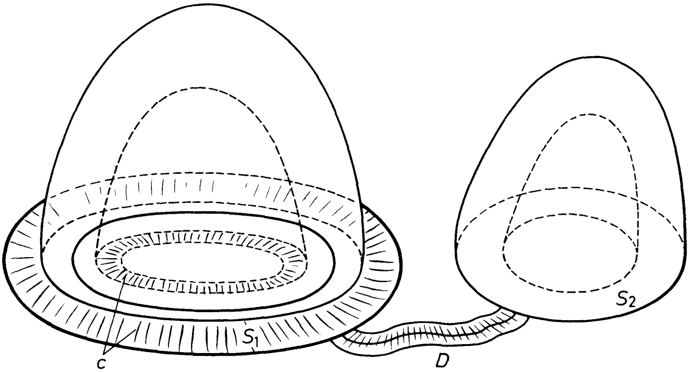


Fig. 49

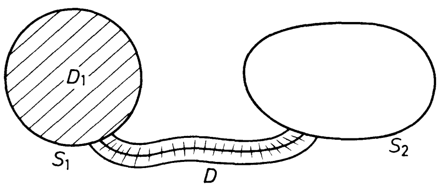


Fig. 50

6.8 *Remark.* Suppose $W = W_1 \cup H^{(r-1)}$ then by 6.3 we have

$$\varepsilon(H_3, H^{(r-1)}) = \varepsilon(H_2, H^{(r-1)}) \pm \varepsilon(H_1, H^{(r-1)}).$$

Handle Decompositions

Let W be a closed manifold. Then a *handle decomposition* of W is a presentation

$$W = H_0 \cup H_1 \cup \cdots \cup H_t$$

where H_0 is a w-ball and H_i is a handle on $W_{i-1} = \bigcup \{H_j \mid j \leqq i-1\}$.

More generally, let (W, M_0, M_1) be a cobordism. Then a *handle decomposition of W on* M_0 is a presentation

$$W = C_0 \cup H_1 \cup \cdots \cup H_t$$

where C_0 is a collar on M_0 in W (which is regarded as a cobordism in the natural way) and H_i is a handle on the cobordism

$$W_{i-1} = C_0 \cup \bigcup \{H_j \mid j \leqq i-1\}.$$

The idea behind a handle decomposition is that it gives an inductive procedure for constructing W from the trivial cobordism.

Now by the collaring theorem we can add a collar C_1 to M_1 without altering W and we have the *symmetrical* decomposition

$$W = C_0 \cup H_1 \cup \cdots \cup H_t \cup C_1.$$

In this case, if we define, $W_{i+1}^c = C_1 \cup \bigcup \{H_j | j \geqq i+1\}$ then we see that H_i can be regarded as a handle H_i^* on W_{i+1}^c with characteristic map $h_i^* = h_i \circ t$. Where t is the automorphism of $I^p \times I^q$ which interchanges the first p coordinates with the last q. So we have the *dual* decomposition

$$W = C_1 \cup H_t^* \cup \cdots \cup H_1^* \cup C_0$$

of W on M_1. Notice that index $(H_i^*) = w -$ index H_i and that the a-tube of H_i is the b-tube of H_i^*.

A decomposition is *nice* if the index of $H_{i+1} \geqq$ index H_i for each i and if handles of the same index are disjoint. It follows from the reordering lemma, applied to successive pairs of handles, that any decomposition gives rise to a nice decomposition which has the same number of handles of each index.

We next prove existence of handle decompositions. Let (K, K_0) be a triangulation of (W, M_0) and let $A_1, \ldots, A_r$ be the simplexes of $K - K_0$ taken in order of increasing dimension. Let K'' be a second derived and define

$$C_0 = |N(K_0, K'')|, \qquad A_i^{**} = |\operatorname{st}(a_i, K'')|.$$

6.9 Proposition.

$$W = C_0 \cup A_1^{**} \cup \cdots \cup A_r^{**}$$

is a handle decomposition of W on M_0 with index $(A_i^{**}) = \dim(A_i)$.

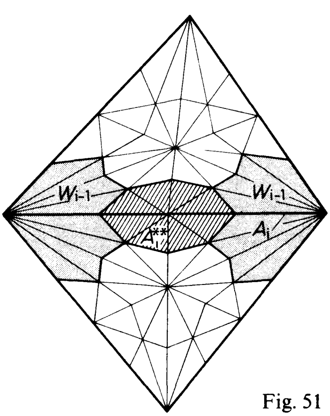


Fig. 51

Proof. C_0 is a collar by 3.9. We have to find a characteristic map h_i for A_i^{**} as a handle on $W_{i-1}=C_0 \cup \bigcup\{A_j^{**} | j \leqq i-1\}$. Now there is a simplicial isomorphism f_i: $A_i^{**} \to \mathrm{st}(a_i, K')'$ defined by pseudo-radial projection from a_i (as in Exercise 7 at the end of Chapter 2) and this carries $A_i^{**} \cap W_{i-1}$ onto $N=|N(\dot{A}_i, \mathrm{lk}(a_i, K')'|$ which is a derived neighbourhood and hence regular. Also $A_i \subset \mathrm{st}(A_i, K)$ is an unknotted ball pair by 4.3 since it is the join of (A_i, A_i) with $(\mathrm{lk}(A_i, K), \emptyset)$.

It follows that we can choose a homeomorphism g_i: $I^p \times I^q \to \mathrm{st}(A_i, K)$ so that $g_i(I^p \times 0)=A_i$ where $\dim A_i = p$ and $p+q=w$; and by the S.N.T. we can assume $g_i(\dot{I}^p \times I^q)=N$. Therefore $h_i=f_i^{-1} \circ g_i$ is a suitable characteristic map for A_i^{**} (see Fig. 51).

The CW Complex Associated with a Decomposition

Notice that, in the last proof, if we shrink all the handles back onto their cores we recover the complex K. More generally given any decomposition of W on M_0 we can construct a CW complex K attached to M_0 of the same homotopy type as W and with one p-cell for each p-handle as follows:

Suppose inductively that we have defined K_{i-1} and a homotopy equivalence

$$l_{i-1}\colon W_{i-1} \to K_{i-1}, \qquad \mathrm{rel}\, M_0.$$

Let r_t: $H_i \to \mathrm{core}(H_i) \cup a\text{-tube}(H_i)$ be the obvious deformation retraction. Then $W_{i-1} \cup_{f_i} H_i$ is homotopy equivalent with $K_{i-1} \cup_{g_i} H_i$, where $g_i = l_{i-1} \circ f_i$, which deformation retracts (by $l_{i-1} \circ r_t$) on $K_{i-1} \cup_{g_i|} I^p$ (index $H_i=p$). Then $K_i=K_{i-1} \cup_{g_i|} I^p$ is a cell complex $K_{i-1} \cup$ attached p-cell, and we have constructed l_i: $W_i \to K_i$.

If the decomposition was nice, the cells will be attached in order of increasing dimension and K will be a CW complex.

Now let $H^{(r)}$, $H^{(r+1)}$ be handles in the decomposition and e^r, e^{r+1} the corresponding cells of K. Then by niceness we can assume $H^{(r)}$, $H^{(r+1)}$ are consecutive and we have the incidence number $\varepsilon(H^{(r+1)}, H^{(r)})$ defined. It follows at once from 6.3 and the definition of incidence numbers in a CW complex (Appendix A) that $\varepsilon(H^{(r+1)}, H^{(r)})=\varepsilon(e^{r+1}, e^r)$. This observation is very important because it means that we can compute $H_*(W, M_0)$ from the list of incidence numbers of a handle decomposition or conversely, as we shall use it in the proof of the h-cobordism theorem, deduce facts about incidence numbers from homological hypotheses.

6.10 *Exercise.* Let $W=C_0 \cup H_1 \cup \cdots \cup H_t$ be a nice decomposition and $W^{(s)}=C_0 \cup \bigcup\{H_i^{(p)} | p \leqq s\}$. Then we have

$$\pi_i(W, W^{(s)})=0 \qquad \text{for } i \leqq s$$

$$\pi_i(W, M^{(s)})=0 \qquad \text{for } i \leqq s,\ n-s-1$$

where $M^{(s)}=\partial W^{(s)}-M_0$.

Hint. Use the CW complexes associated to the decomposition and its dual. Or alternatively use a direct argument and general position.

The Duality Theorems

Let $W=C_0 \cup H_1 \cup \cdots \cup H_t \cup C_1$ be a nice symmetrical decomposition and let K be the associated CW complex. Then the dual decomposition is also nice and we obtain the *dual complex* K^* attached to M_1. Now let $H^{(r)}$, $H^{(r+1)}$ be successive handles and $H^{(w-r)}$, $H^{(w-r-1)}$ their duals and e^r, e^{r+1}, e^{w-r}, e^{w-r-1} the corresponding cells of K and K^*. Then since the a-sphere of $H^{(w-r)}=b$-sphere of $H^{(r)}$ and similarly for $H^{(r+1)}$ we have

$$\varepsilon(H^{(r+1)}, H^{(r)})=\varepsilon(H^{(w-r)}, H^{(w-r-1)}) \mod 2$$

which implies

$$\varepsilon(e^{r+1}, e^r)=\varepsilon(e^{w-r}, e^{w-r-1}) \mod 2.$$

It follows that (cf. Appendix A) there is an isomorphism between the chain complex of K and the cochain complex of K^* with $\mathbb{Z}_2$-coefficients and we have

6.11 Theorem. $H_*(W, M_0; \mathbb{Z}_2) \cong H^{w-*}(W, M_1; \mathbb{Z}_2)$.

Now suppose W is orientable. Then each "level" manifold $M_i=\partial W_i - M_0$ is orientable and we have (with the notation of 6.3) $\varepsilon(S_1, S_2)=\pm\varepsilon(S_2, S_1)$ and hence $\varepsilon(e^{r+1}, e^r)=\pm\varepsilon(e^{w-r}, e^{w-r-1})$. But since $\varepsilon(S_1, S_2)=(-1)^{r(w-r-1)}\varepsilon(S_2, S_1)$ in M_i, and orientation of $H=(-1)^{r(w-r)}$ orientation of H^*, the signs are in fact all positive, and we have

6.12 Theorem. *If W is orientable then*

$$H_*(W, M_0; \mathbb{Z}) \cong H^{w-*}(W, M_1; \mathbb{Z}).$$

The case $M_0=M_1=\emptyset$ of these theorems is usually called "Poincaré duality" and the case $M_0=\emptyset$, "Lefschetz duality".

Simplifying Handle Decompositions

Now we come to the heart of the proof of the h-cobordism theorem, namely using algebraic hypotheses to modify a decomposition.

6.13 Lemma (elimination of 0-handles). *Suppose given a handle decomposition of W on M_0 with i_t t-handles for each t. Suppose that each component of W meets M_0. Then there is another decomposition with no 0-handles, $(i_1 - i_0)$ 1-handles and i_t t-handles for $t > 1$.*

Proof. By the reordering lemma we can assume that indices of handles increase. Now attaching a handle of index 2 does not affect connectivity. It follows that each 0-handle is connected to either another 0-handle or else to C_0 by a 1-handle. But a 0-handle with a 1-handle attached to it by one end only is a complementary pair which can be cancelled. It follows that each 0-handle can be cancelled with a suitable 1-handle.

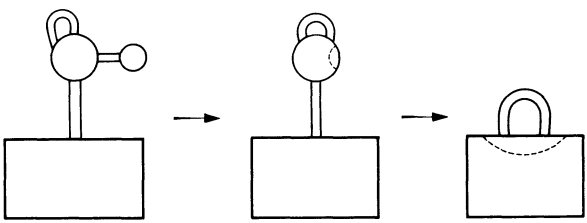


Fig. 52

6.14 Corollary. *Suppose W is connected then W has a handle decomposition on M_0 with*

(i) *no 0- or w-handles if $M_0, M_1 \neq \emptyset$*
(ii) *one 0-handle and no w-handles if $M_0 = \emptyset$, $M_1 \neq \emptyset$*
(iii) *no 0-handles and one w-handle if $M_0 \neq \emptyset$, $M_1 = \emptyset$*
(iv) *one 0-handle and one w-handle if $M_0 = M_1 = \emptyset$.*

Proof. For (i) apply 6.13 to a decomposition and the dual decomposition. For (ii) let $H_0 \cup H_1 \cup \cdots \cup H_t$ be a decomposition. Then H_0 is a 0-handle and we can apply (i) to $C_0 \cup H_1 \cup \cdots \cup H_t$ where C_0 is a collar on ∂H_0 in H_0. Parts (iii) and (iv) follow similarly.

6.15 Lemma (elimination of 1-handles). *Suppose W is connected and we are given a handle decomposition of W on M_0 with no 0-handles and i_t t-handles for $t > 0$. Suppose that $\pi_1(W, M_0) = 0$, and $w \geq 6$. Then there is another decomposition with i_t t-handles for $t \neq 1, 3$, no 1-handles and $(i_1 + i_3)$ 3-handles.*

Proof. We can assume the decomposition is nice. Let (H_1, h_1) be a typical 1-handle. We will show how to "replace" H_1 by a 3-handle and the result then follows by induction. Let $\alpha = h_1(I^1 \times x)$ be an arc in the b-tube of H_1 "parallel" to the core. By general position and regular neighbourhoods (as in 6.2) we can assume α misses the 2-handles and hence lies in $M^{(2)} = \partial W^{(2)} - M_0$. Now by 6.10 $\pi_1(W^{(2)}, C_0) = 0$ and we can find a map $f\colon D^2 \to W^{(2)}$ with $f(\partial D^2) = \alpha \cup \beta$ where β lies in C_0. (The use of 6.10 simply involves pushing f off the cocores of the higher dimensional handles by general position.) Then we can again assume (as in 6.2) that β is embedded in $M^{(2)}$ disjoint from all 1- and 2-handles. Finally homotop f rel ∂D^2 into $M^{(2)}$ by 6.10 (i.e. push off the cores of the 1- and 2-handles by general position) and, by a final application of general position, replace f by a locally flat embedded disc D^2. Now use 6.6 to introduce a complementary 2 and 3 handle pair (H_2, H_3) along a neighbourhood of D^2 so that the a-sphere of H_2 is ∂D^2. Then (H_1, H_2) are complementary and can be cancelled, and we have "replaced" H_1 by H_3, as required, see Fig. 53.

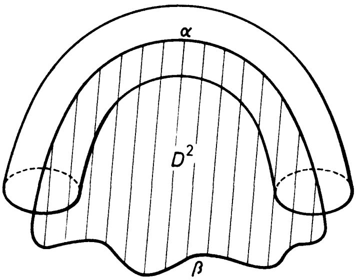


Fig. 53

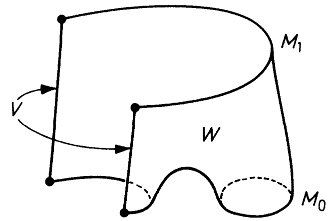


Fig. 54

Remark. Lemma 6.15 works if $w = 5$. The only part of our proof which fails is the final appeal to general position. – We would merely get a locally flat embedding off a finite set which then has to be "piped" over the edge (as in 5.9). The proof of the lemma generalises to "replace" s-handles by $(s+2)$-handles when $\pi_s(W, M_0) = 0$, M_0 is connected and $w \geqq 2s+3$. This result will not be needed.

6.16 Lemma (elimination of s-handles, $2 \leqq s \leqq w-4$). *Suppose given a handle decomposition of W on M_0 with no handles of index $< s$ and i_t handles of index t for $t \geqq s$. Then, if M_0 is simply-connected $2 \leqq s \leqq w-4$, $w \geqq 6$ and $H_s(W, M_0) = 0$, we can find a new decomposition with the same number of t-handles for $t \neq s$, $s+1$, with no s-handles and with $(i_{s+1} - i_s)$ $(s+1)$-handles.*

Proof. We can assume that the decomposition is nice and then we can compute $H_*(W, M_0)$ from the incidence numbers. Let $H^{(s)}$ be a typical s-handle. We show how to eliminate $H^{(s)}$ and the result follows by induction. Let $H_i^{(s+1)}$ be the $(s+1)$-handles and $n_i = \varepsilon(H_i^{(s+1)}, H^{(s)})$. Use 6.7 to add the $(s+1)$-handles so as to reduce $\sum |n_i|$ as far as possible. For example suppose $n_1, n_2 \neq 0$, $|n_1| \geqq |n_2|$, then replace by $n_1, n_1 \pm n_2$ and reduce $\sum |n_i|$. Finally only n_1 say is non-zero and since $H_s(W, M_0) = 0$ we must have $n_1 = \pm 1$. $H^{(s)}$ and $H_1^{(s+1)}$ are then algebraically complementary and the result follows from 6.5.

Proof of the *h*-Cobordism Theorem

6.17 *h*-cobordism theorem. *Let (W, M_0, M_1) be a simply-connected h-cobordism (i.e. $M_0 \subset W$ and $M_1 \subset W$ are both homotopy equivalences). Then, if $w \geqq 6$, $W \cong M_0 \times I$.*

Proof. Choose a decomposition $W = C_0 \cup H_1 \cup \cdots \cup H_t \cup C_1$. We will show how to eliminate all the H_i and then $W \cong C_0 \cup C_1$ and the result is proved. Now by 6.13 and 6.15 we can assume there are no 0- or 1-handles, and, applying these results to the dual decomposition, that there are no w- or $(w-1)$-handles. Now use 6.16 to eliminate all the s-handles for $2 \leqq s \leqq w-4$ and then we have only $(w-3)$- and $(w-2)$-handles. Now apply 6.16 to the dual decomposition to eliminate the $(w-2)$-handles and we then have only $(w-3)$-handles. But $H_{w-3}(W, M_0) = 0$, which implies that there are no $(w-3)$-handles left.

Remark. We actually only used the hypotheses

(1) $\pi_1(W, M_0) = \pi_1(W, M_1) = 0$,

(2) W is simply-connected,

(3) $H_*(W, M_0) = 0$,

(4) $H_*(W, M_1) = 0$.

But (4) follows from (3) and duality (see appendix A.4) so that we have proved the stronger form of the theorem (see end of Chapter 1).

The Relative Case

By a *cobordism with boundary* we mean a compact w-manifold W together with two disjoint $(w-1)$-dimensional submanifolds, $M_0, M_1 \subset \partial W$. Then $V = \mathrm{cl}(\partial W - M_0 - M_1)$ is a cobordism between ∂M_0 and ∂M_1 (see Fig. 54): W is an h-cobordism if $M_0 \subset W$, $M_1 \subset W$, $\partial M_0 \subset V$, $\partial M_1 \subset V$ are all homotopy equivalences.

6.18 Relative *h*-cobordism theorem. *Let (W, M_0, M_1) be a simply-connected h-cobordism with boundary and suppose $V \cong M_0 \times I$ and $w \geqq 6$. Then $(W, V) \cong (M_0, \partial M_0) \times I$.*

Remarks

(1) By assuming that V is a product we avoid having to put conditions on V. Combining 6.18 with the absolute theorem yields a theorem when V is not known already to be a product.

(2) By uniqueness of collars (see end of Chapter 4) we can assume that the product structure on W extends the given structure on V.

Proof. Let (K, K_0) be a triangulation of (W, M_0). Then by 3.17 and hypothesis we can assume

$$V = |N(\partial K_0'', (\partial K - K_0)'')|.$$

Then if we let $A_1, \ldots, A_t$ be the simplexes of K not in K_0 we have

$$W = C_0 \cup A_1^{**} \cup \cdots \cup A_t^{**}$$

where $C_0 = |N(K_0, K'')|$. Then C_0 is a collar which restricts on M_0 to V and we have a "handle decomposition" of W on M_0 *rel* V.

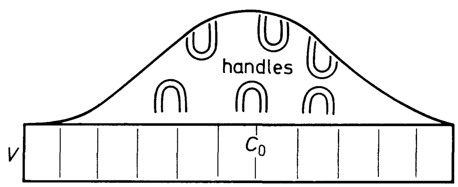


Fig. 55

By the collaring theorem we can assume that this decomposition is symmetrical and it only remains to observe that each of the lemmas used in the proof of the h-cobordism theorem can be applied in this situation and that the resulting homeomorphisms can all be assumed to be fixed on V. Therefore $W \cong C_0 \cup C_1$ rel V which implies the result.

Remark. As with the absolute theorem we have only used the simple connectivity of W, M_0 and M_1, $H_*(W, M_0) = 0$, and duality (which has a similar statement and proof).

The Non-Simply-Connected Case

Let W be a connected h-cobordism; then there is defined a torsion element $\tau(W, M_0) \in \mathrm{Wh}(\pi_1(W))$, see Appendix B.

6.19 s-cobordism theorem. *Let (W, M_0, M_1) be a connected h-cobordism and $w \geqq 6$. Then $W \cong M_0 \times I$ if and only if $\tau(W, M_0) = 0$.*

Remarks

(1) The "only if" part follows from the properties of torsion, so we prove the "if" part.

(2) h-cobordisms may be constructed with any given torsion (see the end of the chapter).

(3) 6.18 and 6.19 can be combined to give a relative s-cobordism theorem which is proved by combining the proofs.

The geometry of the proof of 6.19 is the same as that for the simply-connected case, the main difference being the need to take care with base-points. A "handle" will now mean a *based* handle i.e. we have a specific base-path from the base point of the a-sphere to the base-point of M_0. (This allows us to regard a b-sphere as based by using a standard path between the two spheres.)

The reordering lemma and the notion of complementary handles are unchanged as are the cancellation and introduction lemmas. However we need a non-simply-connected version of Corollary 6.5. Incidence numbers are defined in $\mathbb{Z}\pi$, where $\pi=\pi_1(W)$, as in the last chapter.

6.20 Corollary to 6.4. Suppose $W'=W\cup H^{(r)}\cup H^{(r+1)}$ and $\pi_1(M_1)\cong\pi_1(M_2)\cong\pi_1(W)$ where $M_2=\partial(W\cup H^{(r)})-M_0$ and that $2\leqq r\leqq w-4$. Then if $\varepsilon(H^{(r+1)},H^{(r)})=\pm g$, where $g\in\pi$, $W'\cong W$.

The proof is similar to 6.5 using the non-simply-connected Whitney lemma.

The adding lemma, 6.7, needs to be generalised by allowing $[f_3]=[f_2]\pm[f_1]^g$, where $g\in\pi_1(M_1)$ acts in the usual way. This is proved by choosing the piping tube in a neighbourhood of an arc α which represents g.

Existence of decompositions follows as before and, by a generalisation of 6.3, proved by lifting to the universal cover, we can compute $H_*(\tilde{W},\tilde{M}_0)$ as a $\mathbb{Z}\pi$ module from the incidence numbers when $\pi_1(W)\cong\pi_1(M_0)$. Here $\tilde{X}$ denotes the universal cover of X.

Proof of 6.19. The idea is the same – start with a decomposition and eliminate all the handles, but the method is rather different. We start as before by eliminating 0-, 1-, w-, and $(w-1)$-handles using Lemmas 6.13 and 6.15. Then the idea is to "move" all the handles into two adjacent dimensions. Suppose $H^{(r)}$ is the first handle with $r\geqq 2$ and $w-r\geqq 4$. We show how to replace $H^{(r)}$ by an $(r+2)$-handle $H^{(r+2)}$. Denote the incidence numbers $\varepsilon(H_i^{(r+1)},H^{(r)})$ by t_i, $t_i\in\mathbb{Z}\pi$, where $H_i^{(r+1)}$ are the $(r+1)$-handles. Since $H_r(\tilde{W},\tilde{M})=0$ we must have some linear combination $\sum n_i t_i=1$, $n_i\in\mathbb{Z}\pi$. Introduce a complementary pair $(H^{(r+2)},H^{(r+1)})$. Then by adding suitable combinations of the $H_i^{(r+1)}$ to $H^{(r+1)}$ we can make $\varepsilon(H^{(r+1)},H^{(r)})=\sum n_i t_i=1$. We can then cancel $H^{(r+1)}$ with $H^{(r)}$ by Corollary 6.20. This "replaces" $H^{(r)}$ by $H^{(r+2)}$.

Finally there are handles left of indices $(w-3)$ and $(w-2)$ only, or dually of indices 2 and 3 only. Let A be the matrix over $\mathbb{Z}\pi$ determined by the incidence numbers of $H_i^{(3)}$ with $H_j^{(2)}$. Since $H_*(\tilde{W}, \tilde{M})=0$, A is an invertible $p \times p$ matrix for some p. A determines an element $\tau = \tau(W, M) \in \mathrm{Wh}(\pi)$ (see Appendix B) which measures the obstruction to changing A to the empty matrix by a sequence of the following moves: (1) Replace A by $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ or vice versa. (2) Add a multiple of one row to another. (3) Reorder rows or columns. (4) Multiply a row by an element of π or by -1.

However each of these moves can be realised by a handle operation: (1) Introduce or cancel a pair of (algebraically) complementary handles. (2) Add handles. (3) Renumber handles. (4) Change the base-path or orientation of a handle.

Hence if $\tau = 0$ we can cancel all the 2- and 3-handles and W is a product, as required.

Constructing *h*-Cobordisms

Finally, to end the chapter, we show how to construct h-cobordisms of any given torsion. Let M^n be a given manifold with $n \geqq 3$ and $\tau \in \mathrm{Wh}(\pi)$, $\pi = \pi_1(M)$, a given element. Then τ is determined by an invertible $p \times p$ matrix A with entries in $\mathbb{Z}(\pi)$. Construct a cobordism W with handles of indices 2 and 3 only and matrix (of the last proof) equal to A as follows:

(1) Start with the trivial cobordism $M \times I$ and attach p complementary (2, 3) pairs, $(H_i^{(2)}, H_i^{(3)})$, $i = 1, 2, \ldots, p$.

(2) Attach also p complementary (3, 4) pairs by balls disjoint from the (2, 3) pairs. Forget the 4-handles and call the new 3-handles $\bar{H}_i$, $i = 1, 2, \ldots, p$.

(3) Use the adding lemma to add to each $\bar{H}_i^{(3)}$ a suitable linear combination of the $H_j^{(3)}$ so as to realise the i-th row of A. In other words make $\varepsilon(\bar{H}_i^{(3)}, H_j^{(2)}) = A_{ij}$.

(4) Now forget the $H_j^{(3)}$. Then $W = M \times I \cup \{H_i^{(2)}\} \cup \{\bar{H}_j^{(3)}\}$ is the required cobordism.

Notice that the construction actually embeds W in the trivial cobordism $M \times I \cong M \times I \cup$ all the attached handles.

Exercise. If $n \geqq 4$ then W is an h-cobordism.

Hint. Invertibility of A implies $H_*(\tilde{W}, \tilde{M})=0$. Use 6.10 to check the $\pi_1(\ )$ hypotheses.

6.21 *Exercise* (classification of h-cobordisms). Prove that there is a one-one correspondence between $Wh(\pi)$ and homeomorphism classes (rel M) of h-cobordisms (W, M, M') for $n \geqq 5$, by observing: (1) Any h-cobordism embeds in an s-cobordism and hence in any other h-cobordism. (2) If $W_1 \subset W$ and $\tau(W, M) = \tau(W_1, M)$ then $\mathrm{cl}(W - W_1)$ is an s-cobordism.

Chapter 7. Applications

We give five applications of handle theory:

(1) Unknotting balls and spheres in codimension $\geqq 3$.

(2) A criterion for unknotting in codimension 2.

(3) A weak 5-dimensional h-cobordism and Poincaré theorem.

(4) Engulfing.

(5) Embedding manifolds.

Unknotting Balls and Spheres in Codimension $\geqq 3$

7.1 Theorem. *Any proper (q, n)-ball or sphere pair is unknotted if $q-n \geqq 3$.*

For the following corollaries, see Chapter 4.

7.2 Corollary. *Any proper (q, n)-manifold pair is locally flat provided $q-n \geqq 3$.*

7.3 Corollary. *Any proper isotopy of manifolds is ambient in codimension $\geqq 3$.*

We will deduce the theorem from two lemmas:

7.4 Lemma. *The theorem is true for locally flat (q, n)-ball pairs with $q \geqq 6$.*

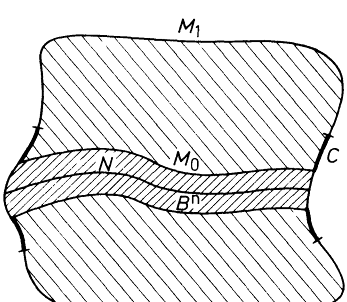


Fig. 56

Proof. Let N be a regular neighbourhood of B^n in B^q, then (N, B^n) is an unknotted ball pair by 4.14. Define $W = \mathrm{cl}(B^q - N)$, $M_0 = W \cap N$; choose a collar C on M_0 in $\mathrm{cl}(\partial W - M_0)$ and define $M_1 = \mathrm{cl}(\partial W - (M_0 \cup C))$. Then (W, M_0, M_1) is a cobordism with boundary C and we claim that it is a simply-connected h-cobordism. First W, M_0, M_1 are all simply-connected by general position, since $W \simeq B^q - B^n$, $M_0 \simeq \partial N - \partial B^n$, $M_1 \simeq \partial B^q - \partial B^n$, and B^n, ∂B^n are in codimension $\geqq 3$. Secondly

$$\begin{aligned} H_*(B^q, N) \cong H_*(B^q, B^n) = 0 \quad & \text{by homotopy} \\ \cong H_*(W, M_0) \quad & \text{by excision.} \end{aligned}$$

It then follows from 6.18 that $W \cong M_0 \times I$ and hence (B^q, B^n) is obtained from the unknotted pair (N, B^n) by gluing a collar on M_0 and is therefore unknotted by 2.25.

7.5 Lemma. *Let $(q-n)$ be fixed then (q, n)-ball pairs unknot for $q \leqq t \Leftrightarrow (q, n)$-sphere pairs unknot for $q \leqq t$.*

Proof. Assume spheres unknot, then given a (q, n)-ball pair, by hypothesis its boundary is unknotted and we can glue on an unknotted ball pair to form a sphere pair, which is unknotted by hypothesis. So the original pair is obtained from an unknotted sphere pair by removing an unknotted ball pair and so the result follows from 4.9.

Conversely, suppose balls unknot and a sphere pair is given; remove a small ball pair, to form a ball pair. Then by hypothesis the original pair is obtained by gluing two unknotted ball pairs along their boundaries and the result is unknotted by 4.3.

Proof of 7.1. By induction on q. By 5.6 sphere pairs unknot for $q \leqq 5$ and hence, by 7.5, ball pairs unknot for $q \leqq 5$. Now suppose the theorem is true for $q \leqq t-1$ and (B^t, B^m) a given pair with $t \geqq 6$, $t - m \geqq 3$. Then looking at links we see that B^m is locally flat in B^t and hence the pair is unknotted by 7.4. It follows from 7.5 that any (t, m)-sphere pair is unknotted and the induction step is established.

A Criterion for Unknotting in Codimension 2

We will need to assume that $\mathrm{Wh}(\mathbb{Z}) = 0$ (see bibliography) so that any h-cobordism between manifolds with fundamental group $\mathbb{Z}$ is simple.

7.6 Theorem. *Let $S^{q,n}$ be a proper locally flat sphere pair with $q - n = 2$, $q \geqq 6$. Then the pair is unknotted if and only if $S^q - S^n$ has the homotopy type of a circle.*

Remarks

(1) The "only if" part is obvious.

(2) The theorem is also known for $q=3$ when it follows from the notorious Dehn lemma and for $q=5$ by "surgery" (see bibliography).

Proof. Let $B^{q,n}$ be the result of removing an unknotted ball pair then by 4.3 it suffices to show $B^{q,n}$ is unknotted. Define N, W, M_0, C, M_1 as in the proof of 7.4. Then $W \simeq S^1$ by hypothesis, $M_0, M_1 \simeq S^1$ since $(\partial N, \partial B^n)$ and $(\partial B^q, \partial B^n)$ are unknotted. Therefore W is an h-cobordism, hence an s-cobordism, hence (by the relative s-cobordism theorem) a product. Therefore $B^{q,n}$ is unknotted as in 7.4.

Weak 5-Dimensional Theorems

7.7 Weak 5-dimensional *h*-cobordism theorem. *Let (W^5, M_0, M_1) be a simply-connected h-cobordism between manifolds without boundary. Then $(W - M_1) \cong M_0 \times [0, 1)$.*

We say W is *invertible* if there is a cobordism $(\overline{W}, M_1, M_2)$ such that $W \cup_{M_1} \overline{W} \cong M_0 \times I$.

7.8 Lemma. *W is invertible with inverse $\overline{W} = W$ with ends reversed.*

Proof. Consider $W \times I$ as a cobordism between the manifolds with boundary $M_0 \times I$ and $W' = (W \times 0 \cup M_1 \times I \cup W \times 1) - \text{collar}$.

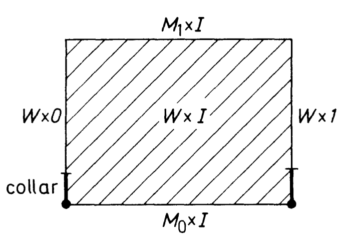


Fig. 57

By the relative 6-dimensional h-cobordism theorem we have $W' \cong M_0 \times I$ so that $W \times 0$ is invertible with inverse $M_1 \times I \cup W \times 1 \cong \overline{W}$, as required.

Proof of 7.7. By collaring $W \cong W \cup M_1 \times I$ and hence

$$W - M_1 \cong W \cup M_1 \times [0, 1).$$

Now consider

$$\overline{\overline{W}} = W \cup_{M_1} \overline{W} \cup_{M_0} W \cup_{M_1} \overline{W} \ldots$$

then this is

$$M_0 \times I \cup M_0 \times I \cup \cdots \quad \text{by 7.8} \quad \cong M_0 \times [0, 1).$$

But by symmetry W is the inverse to $\overline{W}$ hence $\overline{W} \cup_{M_0} W = M_1 \times I$ and so

$$\begin{aligned} \overline{\overline{W}} &\cong W \cup M_1 \times [0,1) && \text{(by pairing the other way)} \\ &\cong W - M_1 && \text{by the first remark.} \end{aligned}$$

Remark. Lemma 7.8 (and hence Theorem 7.7) is also true without the hypothesis of simple connectivity (and for any dimension >5 as well). The proof is similar to our proof but before starting one has to add a cobordism to W to kill torsion (by existence of cobordisms with arbitrary torsion). This of course does not affect invertibility.

7.9 Corollary (weak Poincaré theorem). *Let M^5 be a closed manifold of the homotopy type of S^5. Then M^5 is topologically homeomorphic with S^5.*

Proof. Let $D_1^5 \subset D_2^5 \subset M^5$ be concentric discs and let $D_3^5 \subset M - D_2$ be another. Then $M - (D_1 \cup \mathring{D}_3) \cong \partial D_3 \times [0,1)$ by 7.7, and the argument in Chapter 1. Hence M^5 is covered by two discs D_2 and $D_4 = D_3 \cup \partial D_3 \times [0,t]$ for suitable t. Now ∂D_4 is a locally flat S^4 embedded in $\mathring{D}_2$ and hence bounds a topological ball D_2' in D_2 by 3.39. Therefore $M^5 = D_4 \cup D_2'$ is the union of two topological balls sewn along their boundaries and hence is a topological sphere as required.

Engulfing

Let M be an i-connected manifold (i.e. each map $f\colon K \to M$ can be homotoped to zero if $\dim K \leqq i$) and $X \subset M$ a polyhedron of dimension $\leqq i$. We want to conclude that X is contained in a ball in M and then we say X can be *engulfed* in M.

More generally suppose (W, M_0, M_1) is a cobordism and (W, M_0) is i-connected (i.e. each map $f\colon K \to W$ can be homotoped into M_0 where dimension $K \leqq i$). Then we wish to conclude that $X^i \subset W$ is contained in a collar on M_0, and then say X can be *engulfed from* M_0.

7.10 Engulfing theorem. *Let W be a cobordism (without boundary) and $X^i \subset W$. Then X can be engulfed from M_0 provided (W, M_0) is i-connected and*

(1) $w \geqq 6$, $w - i \leqq 3$ *or*

(2) $w = 4, 5$, $i = 1$ *or*

(3) $w = 5$, $i = 2$ *and M_0 is simply-connected.*

Remarks

(1) The extra hypothesis on M_0 in Part (3) is unnecessary. This is seen by using the non-simply-connected weak h-cobordism theorem (see the proof below).

(2) There are counter-examples to extending the theorem to the case $w-i=2$, $w \geqq 4$. The case $w=3$, $i=1$ is unsolved and equivalent to the Poincaré conjecture in dimension 3 (see bibliography).

We first prove an easy lemma:

7.11 Lemma. *Suppose* (W, M_0) *has a handle decomposition with no handles of index* $\leqq i$ *then* X^i *can be engulfed from* M_0.

Proof. We use induction on the number of handles. So let $W = W_0 \cup H$.

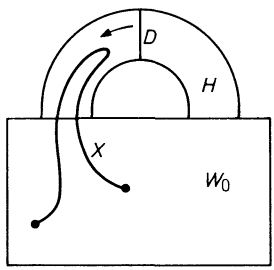


Fig. 58

Let D be the cocore of H. Then by general position we may assume that $X \cap D = \emptyset$. Then by the usual regular neighbourhood argument we can ambient isotope X off H. Then $X \subset W_0$ and the result now follows by induction.

Proof of 7.10

(1) This follows from 7.11 and Lemmas 6.13, 6.15, 6.16; to eliminate $(w-3)$-handles apply 6.5 (or 6.20) to the duals of an algebraically complementary $(w-2, w-3)$-pair.

(2) This follows by changing a homotopy into an ambient isotopy as in 5.9 (we are essentially in the manifold case since in a triangulation of X we can easily engulf the vertices first).

(3) We use the weak h-cobordism theorem. Choose a handle decomposition and eliminate 1-handles by the remark below 6.15. Next ambient isotope X off the 3-, 4- and 5-handles by 7.11. Now use 2-connectivity to find 3-handles $H_i^{(3)}$ which are algebraically complementary to the 2-handles $H_i^{(2)}$ (see proof of 6.19). Then $C_0 \cup \bigcup \{H_i^{(2)}\} \cup \bigcup \{H_i^{(3)}\}$ is a 5-dimensional h-cobordism and hence a "weak" product by 7.7. It is now easy to engulf X.

Exercise. Generalise the engulfing theorem for cobordisms with boundary.

Embedding Manifolds

7.12 Theorem. *Let $f\colon M^n \to Q^q$ be a map of unbounded manifolds with M compact. Then f is homotopic to an embedding provided*

(1) $q-n \geqq 3$

(2) *M is d-connected where $d=2n-q$*

(3) *Q is $(d+1)$-connected.*

7.13 Corollary. *A closed k-connected n-manifold embeds in $\mathbb{R}^{2n-k}$, provided $n-k \geqq 3$.*

Proof of 7.12. We can suppose that f is in general position. We then generalise the method of 5.5 using engulfing. The idea is to find collapsible subsets $C \subset M$, $D \subset Q$ such that $f^{-1}(D)=C$ and $S(f) \subset C$. We can then complete the proof as in 5.5, namely choose regular neighbourhoods N_0, N of C, D in M, Q so that $f^{-1}(N)=N_0$ and $f^{-1}(\dot{N})=\dot{N}_0$ and then replace $f|N_0$ by an embedding into N using the cone construction.

C and D are found by a repeated engulfing argument:

First engulf $S(f)$ in a ball B in M and define $C_1=$(singular) cone on $S(f)$ in B. Then $S(f) \subset C_1$, $\dim(C_1) \leqq d+1$ and $C_1 \searrow 0$.

Next engulf $f(C_1)$ in a ball B' in Q and let $D_1=$singular cone on $f(C_1)$ in B' which we can suppose shifted into general position with respect to $f(M)(\operatorname{rel} f(C_1))$. Then $D_1 \searrow 0$ and $f^{-1}(D_1)=C_1 \cup E_1$ where $\dim(E_1) \leqq d-1$ by general position and codimension $\geqq 3$.

Now engulf E_1 from a regular neighbourhood B of C_1 in M and define

$$C_2 = C_1 \cup (\text{trail of } E_1 \text{ under a collapse } B \searrow C_1).$$

Then $S(f) \subset C_1 \subset C_2 \searrow 0$ and $\dim(C_2 - C_1) \leqq d$.

Next engulf $f(C_2)$ from a regular neighbourhood of D_1 and define

$$D_2 = D_1 \cup \big(\text{trail of } f(C_2) \text{ under a collapse}\big).$$

Then $f(C_2) \subset D_2 \searrow 0$ and $\dim\big(D_2 - f(C_2)\big) \leqq d+1$ so that by general position we can assume $f^{-1}(D_2)=C_2 \cup E_2$ where $\dim E_2 \leqq d-2$.

The process continues with the dimension of the "error term" E_i decreasing at each stage. Eventually $E_n=\emptyset$ and $C=C_n$, $D=D_n$ and the theorem is proved.

7.14 *Exercise.* Prove a version of 7.12 for bounded manifolds where the map is already an embedding on the boundaries by using a collar to replace the problem by an "interior" one. Deduce that the embedding constructed in 7.12 is unique up to *concordance* provided M is $(d+1)$-connected and Q $(d+2)$-connected, where f_0, f_1 are *concordant* if they are restrictions of an embedding $F\colon M \times I \to Q \times I$ which respects the top and bottom levels only. (See also the historical notes.)

Appendix A. Algebraic Results

Here we give definitions and results used in the book. Proofs can be found in [J.1], [J.2] or [J.3] (see bibliography), for a geometrical treatment see below and [J.4].

A.1 Homology

We will assume that abelian homology groups $H_n(X, A)$ are defined for $n=0, 1, \ldots,$ where $A \subset X$ is a pair of topological spaces. These groups have the following properties:

(1) If $f: (X, A) \to (Y, B)$ is a map of pairs then there is an induced natural homomorphism $f_*: H_n(X, A) \to H_n(Y, B)$. *Naturality* means

  (a) $\mathrm{id}_* = \mathrm{id}: H_n(X, A) \to H_n(X, A)$,

  (b) $(f \circ g)_* = f_* \circ g_*$.

(2) There is a natural boundary homomorphism

$$\partial: H_n(X, A) \to H_{n-1}(A) \equiv H_{n-1}(A, \emptyset).$$

Here naturality means that if $f: (X, A) \to (Y, B)$ is a map then the square

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[d, "f|_*"] \\
H_n(Y, B) \arrow[r, "\partial"] & H_{n-1}(B)
\end{tikzcd}$$

commutes.

(3) *Exactness*. The sequence

$$\to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to$$

is exact where the unamed homomorphisms are induced by inclusion.

(4) *Homotopy*. Suppose $f, g: (X, A) \to (Y, B)$ are two maps and that there is a map $h: (X \times I, A \times I) \to (Y, B)$ such that $h|X \times 0 = f$ and $h|X \times 1 = g$. Then $f_* = g_*$.

Remark. h is said to be a homotopy between f and g. Homotopy is an equivalence relation on the set of maps $(X, A) \to (Y, B)$. A map f is a *homotopy equivalence* if there is a map $g: (Y, B) \to X, A)$ so that $f \circ g$ and $g \circ f$ are both homotopic to the relevant identity maps. It follows from (1) and (4) that if f is a homotopy equivalence then f_* is an isomorphism. If the inclusion $A \subset X$ is a homotopy equivalence then from exactness $H_n(X, A) = 0$ for all n. (Write $H_*(X, A) = 0$.) A special sort of homotopy equivalence often used is a (strong) deformation retraction. $A \subset X$ is a *deformation retract* if there is a homotopy of $\mathrm{id}|X$ to a retraction $r: X \to A$ and the homotopy is fixed on A (i.e. $h(a, t) = a$, $a \in A$).

(5) *Excision.* Suppose that $U \subset A$ and $\mathrm{cl}(U) \subset \mathrm{int}(A)$ then the homomorphism $H_n(X - U, A - U) \to H_n(X, A)$ induced by inclusion is an isomorphism.

Remark. If $P, Q_1 \subset Q$ are polyhedra with $Q_1 \supset Q - P$ and we write P_1 for $P \cap Q_1$ then $H_n(Q_1, P_1) \to H_n(Q, P)$ is an isomorphism. This follows from excision and homotopy by a simple argument.

(6) *Dimension.*
$$H_n(\mathrm{pt.}) \cong \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$$

Remark. If X deformation retracts on a point (say X is *contractible*) then $H_n(X) \cong H_n(\mathrm{pt.})$; or equivalently $\tilde{H}_*(X) \equiv \mathrm{Ker}(H_*(X) \to H_*(\mathrm{pt.})) = 0$.

A.2 Geometric Interpretation of Homology

The interpretation given here can be taken as the definition of homology if the reader desires. The properties listed above are easily proved — the excision axiom uses regular neighbourhoods, the dimension axiom follows from the cone construction, for details see [J.4; 3.1].

An *n-cycle* is a polyhedron P which possesses a triangulation K so that each principal simplex of K has dimension n (a simplex is principal if it is the face of no other) and each $(n-1)$-simplex is the face of exactly two n-simplexes. Equivalently (and more intrinsically) there is a polyhedron of dimension $n-2$, $S(P) \subset P$ such that

(1) $P = \mathrm{cl}(P - S(P))$

(2) $P - S(P)$ is an n-manifold without boundary.

In other words P is a "manifold with a codimension 2 singularity" and we call $S(P)$ the *singularity* of P. P is *oriented* if $P - S(P)$ is oriented (use the geometrical definition given in Chapter 3 following the treatment which avoids algebraic topology).

An n-cycle with boundary is a pair $(P, \partial P)$ such that there is an $(n-2)$-dimensional polyhedron $S(P) \subset P$ so that

(1) $P = \mathrm{cl}(P - S(P))$
(2) $P - S(P)$ is an n-manifold with boundary $\partial P - S(P)$
(3) ∂P is an $(n-1)$-cycle with singularity $S(P) \cap \partial P$.

A *singular n-cycle* in X is a pair (P, f) where P is an oriented n-cycle and $f\colon P \to X$ a map. (P_0, f_0) and (P_1, f_1) are *homologous* (or *bordant*) if there is an oriented n-cycle with boundary Q and a map $g\colon Q \to X$ so that $\partial Q \cong P_0 \cup P_1$ and, if we identify Q with $P_0 \cup P_1$ by this isomorphism, then we have $f_0 = g \mid P_0$, $f_1 = g \mid P_1$. (Q, g) is called the *homology* between (P_0, f_0) and (P_1, f_1).

Then $H_n(X)$ is the set of homology classes of singular n-cycles in X. Group structure is given by disjoint union; to see existence of inverses consider $f \circ \pi_1\colon P \times I \to X$.

More generally a singular n-cycle in (X, A) is a pair (P, f) where P is an oriented n-cycle with boundary and f a map of pairs $(P, \partial P) \to (X, A)$. Homology is defined using bordisms with boundary (cf. Chapter 6) and we have the relative homology group $H_n(X, A)$.

A.3 Homology Groups of Spheres

Theorem

(1) $H_i(S^n) = 0$, $i \neq 0, n$
$H_n(S^n) \cong \mathbb{Z} \cong H_0(S^n)$, $n > 0$
$H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(2) id: $S^n \to S^n$ *is a generator of* $H_n(S^n)$ *(under the geometric interpretation).*

(3) (3.31) $r_n\colon S^n \to S^n$ *is not homotopic to* id.

Proof.

(1) By induction on n. For $n = 0$ the result is easy. Suppose the theorem true for $n-1$ and consider $S^n = \dot{I}^{n+1} = D^n_+ \cup D^n_-$ where

$$D^n_+ = \{x \mid x \in S^n, x_n \geqq 0\}$$

and D^n_- has $x_n \leqq 0$. Then $D^n_+ \cap D^n_- = S^{n-1}$ and since balls are contractible we have

$$\begin{aligned} \tilde{H}_*(S^n) &\cong H_*(S^n, D^n_-) && \text{by homotopy and exactness} \\ &\cong H_*(D^n_+, S^{n-1}) && \text{by excision.} \end{aligned}$$

Now use the long exact sequence and induction.

(2) By induction again using the last proof. Consider $\alpha = [\mathrm{id}\colon D^n_+ \to D^n_+]$ then $\partial\alpha = [\mathrm{id}\colon S^{n-1} \to S^{n-1}]$ and so by induction α is a generator of $H_n(D^n_+, S^{n-1})$. But $\beta = [\mathrm{id}\colon S^n \to S^n]$ corresponds under excision to α.

(3) Observe that π_1: $S^n \times I \to S^n$ is a homology between $[\mathrm{id}] \cup [r_n]$ and $[\emptyset]$ which represents zero. So if $\mathrm{id} \simeq r_n$ then $2[\mathrm{id}]=0$ contradicting Part (1).

A.4 Cohomology

There is a dual theory which we mention briefly (for a geometrical treatment see [J.4]). Cohomology groups $H^*(X, A)$ are defined so that the direction of the induced homomorphisms is reversed. I.e. f: $(X, A) \to (Y, B)$ induces f^*: $H^*(Y, B) \to H^*(X, A)$. Cohomology satisfies axioms similar to those for homology. There are cap products

$$\bigcap: H_q(X, A) \otimes H^p(X, A) \to H_{q-p}(X, A)$$

and if M is an oriented manifold then $\bigcap[\mathrm{id}]$: $H^p(M) \to H_{n-p}(M)$ gives the Poincaré duality isomorphism of Chapter 6. The intersection number of two cycles (defined in Chapter 5) is the same as the cap product of one with the Poincaré dual of the other (cf. [J.4; II, 3]). There is an Alexander duality theorem which relates the homology of a polyhedron $P \subset \mathbb{R}^n$ with the cohomology of $\mathbb{R}^n - P$. There is also a universal coefficient theorem which relates cohomology to homology. We used only a weak form of the theorem namely,

Theorem. $H^*(X, A)=0$ *if and only if* $H_*(X, A)=0$.

This weak form is easily deduced from the recipe given in A.7 for computing homology and cohomology from incidence numbers in a CW complex.

A.5 Coefficients

If G is an abelian group then there are defined homology and cohomology groups with coefficients G denoted $H_*(X, A; G)$ and $H^*(X, A; G)$. The ordinary homology groups are the same as those with coefficients $\mathbb{Z}$. Coefficients $\mathbb{Z}_2$ have a simple geometric interpretation as bordism of unoriented cycles. (Co)homology groups with coefficients satisfy the same axioms as those for coefficients $\mathbb{Z}$ except for the dimension axiom which reads $H_n(\text{pt.}; G)=H^n(\text{pt.}; G)=0$, $n \neq 0$, $\cong G$, $n=0$.

A.6 Homotopy Groups

Let X be a space and $* \in X$ a fixed point, the *basepoint*. Then the n-th *homotopy group* $\pi_n(X)$ is the set of homotopy classes of maps $(I^n, \dot{I}^n) \to (X, *)$; group structure is given for $n \geqq 1$ by *track addition*:

$$(f+g)(x_1, x_2, \ldots, x_n) = f(2x_1+1, x_2, \ldots, x_n) \quad x_1 \leqq 0$$
$$= g(2x_1-1, x_2, \ldots, x_n) \quad x_1 \geqq 0.$$

$\pi_1(X)$ is the *fundamental group* of X and $\pi_n(X)$ is abelian for $n \geqq 2$. If $* \in A \subset X$ then the relative groups $\pi_n(X, A)$ are defined to be homotopy classes of maps

$$(I^n, \dot{I}^n, J^{n-1}) \to (X, A, *)$$

where $J^{n-1} = \mathrm{cl}(\dot{I}^n - F^{n-1})$ and F^{n-1} is the face $x_1 = 1$. The boundary map $\partial\colon \pi_n(X, A) \to \pi_{n-1}(A)$ is defined by restricting to F^{n-1} and identifying F^{n-1} with I^{n-1}. The homotopy groups satisfy the axioms for the homology groups (the induced homomorphism is defined by composition) with the exception of the excision axiom and the dimension axiom ($\pi_n(\text{pt.}) = 0$ all n).

The pair (X, A) is *r-connected* if every map $f\colon (P, Q) \to (X, A)$ is homotopic to a map into A, where P is a polyhedron of dimension $\leqq r$. If A is path connected then (X, A) is r-connected if and only if $\pi_i(X, A) = 0$ for $i \leqq r$ (for the if part use a skeletal induction over some triangulation of (P, Q)). 1-connected is usually called *simply-connected*. X is r-connected if $(X, *)$ is r-connected. It is easy to see that X is simply-connected if and only if X is path connected and every loop in X (i.e. map of S^1 in X) extends to a map of D^2 in X.

An action of π_1 on π_n is defined by adding a collar to I^n and mapping the collar lines around the given loop. A similar construction gives a change of basepoint isomorphism and if $\pi_1 = 0$ then π_n is independent of basepoint and is isomorphic with $[S^n, X]$ (notice that $S^n \cong I^n/\dot{I}^n$). Here $[\,,\,]$ denotes the set of homotopy classes of maps.

A.7 *CW* Complexes

If A is a space and $f\colon S^{i-1} \to A$ a map then the identification space $A \cup_f I^i$ is said to be obtained from A by *attaching an i-cell*. The natural map $\phi\colon I^i \to A \cup_f I^i$ is the *characteristic map* for the i-cell and we write $A \cup_f I^i = A \cup e^i$.

A finite *CW* complex X attached to A is obtained by repeatedly attaching cells in order of increasing dimension with cells of the same dimension having disjoint interiors. Observe the similarity with nice handle decompositions. Infinite *CW* complexes are defined by attaching all the i-cells simultaneously.

Let e^i, e^{i+1} be cells in X then the restriction of the characteristic map for e^{i+1} composed with the collapsing map $c\colon A \cup e^i \to (A \cup e^i)/A \cong I^i/\dot{I}^i$ determines a map

$$f\colon S^i \to I^i/\dot{I}^i \cong S^i.$$

The homological degree of f defined by $\deg(f)[\mathrm{id}]=[f]$ (cf. A.3) is called the incidence number of e^{i+1} on e^i denoted $\varepsilon(e^{i+1}, e^i)$.

Now let $C_n(X, A)$ be the free abelian group with basis the n-cells of X and ∂_n: $C_n(X, A) \to C_{n-1}(X, A)$ the homomorphism determined by

$$\partial(e^n) = \sum \{\varepsilon(e^n, e^{n-1})\, e^{n-1} \mid e^{n-1} \in X\}.$$

Theorem

(1) $\partial_{n-1} \circ \partial_n = 0$

(2) $H_n(X, A) \cong \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$.

Sketch of proof. By excision and A.3 we can identify C_n with $H_n(X_n, X_{n-1})$ where $X_i = A \cup \{j\text{-cells} \mid j \leqq i\}$. Moreover $H_i(X_n, X_{n-1}) = 0$ for $i \neq n$. Then ∂_n is the composition

$$H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$$

and part (1) follows from diagram chasing using exactness. Now consider the long exact sequences of the triples $X_{n+1} \supset X_n \supset X_{n-1}$ and $X_{n+1} \supset X_{n-1} \supset X_{n-2}$ (exactness for a triple follows from exactness for a pair and diagram chasing). From the first sequence we deduce that $H_n(X_{n+1}, X_{n-1}) \cong C_n/\mathrm{Im}(\partial_{n+1})$ and from the second that

$$H_n(X_{n+1}, X_{n-2}) \cong \mathrm{Ker}\big(C_n/\mathrm{Im}(\partial_{n+1}) \to C_{n-1}\big) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}).$$

Finally an easy induction argument shows that $H_n(X, A) = H_n(X_{n+1}, X_{n-2})$.

Addenda. (a) If we define δ_n: $C_n \to C_{n+1}$ by $\delta_n(e^n) = \sum \varepsilon(e^{n+1}, e^n)\, e^{n+1}$. Then $H^n(X, A) = \mathrm{Ker}(\delta_n)/\mathrm{Im}(\delta_{n-1})$.

(b) Define $C_n(X, A; G) = C_n(X, A) \otimes G$ then $H_*(X, A; G)$ and $H^*(X, A; G)$ are computed in a similar way.

Whitehead's theorem (quoted in Chapter 1 but not used in the book) states that a map f: $X \to Y$ of 1-connected CW complexes is a homotopy equivalence if and only if f_*: $H_*(X) \to H_*(Y)$ is an isomorphism.

A.8 The Universal Cover

Let X be a path-connected topological space. The universal cover $\tilde{X}$ of X is defined as a set to be the set of homotopy classes of maps f: $(I, 0) \to (X, *)$ where the homotopies are fixed on 1. There is a natural map p: $\tilde{X} \to X$ by $p[f] = f(1)$, and we give $\tilde{X}$ the weakest topology which makes p continuous (i.e. U open if and only if $p(U)$ open in X). $\pi = \pi_1(X)$ acts on $\tilde{X}$ over X by track addition i.e. $f^g = g + [f]$ and it easy to see that $p^{-1}(y_0) = y^\pi$ where $y \in p^{-1}(y_0)$.

Now suppose that (X, A) is a CW complex on A in which all the cells are based (i.e. each cell e^i has a path from $1 = (1, 0, \dots, 0) \in e^i$ to

$*\in A$) and suppose that $\pi_1(A) \cong \pi_1(X)$ by inclusion. Let e^i be a cell with base point y_0 and base path α. Consider the characteristic map for e^i, ϕ: $(I^i, 1) \to (X, y_0)$, then, since I^i is simply-connected, each point $x \in I^i$ determines a unique point $\tilde{x} \in \tilde{X}$, namely the class of $\alpha + \beta$, where β is a path in I^i from 1 to x. This defines a map $\tilde{\phi}$: $I^i \to \tilde{X}$ which is the characteristic map of a cell $\tilde{e}^i$. $(\tilde{e}^i)^g$ is obtained by operating pointwise (or equivalently by changing the base path by adding g).

Thus $(\tilde{X}, \tilde{A})$ becomes a CW complex with cells $(\tilde{e}^i)^g$ for $e^i \in X$ and $g \in \pi$. Now π acts cellwise on $\tilde{X}$ and hence acts on $C_n(\tilde{X}, \tilde{A})$ which thus becomes a $\mathbb{Z}\pi$-module, where $\mathbb{Z}\pi$ is the integral group ring of π. This action carries over to $H_n(\tilde{X}, \tilde{A})$ which is thus also a $\mathbb{Z}\pi$-module.

Now let e^{i+1}, $e^i \in X$ be cells. Define their $\mathbb{Z}\pi$-incidence number to be $\sum \{\varepsilon((\tilde{e}^{i+1})^g, \tilde{e}^i) g \mid g \in \pi\}$. Then $C_*(\tilde{X}, \tilde{A}) \cong C_*(X, A) \otimes \mathbb{Z}\pi$ with boundary on the right given by the $\mathbb{Z}\pi$-incidence numbers. Hence $H_*(\tilde{X}, \tilde{A})$ can be computed from cells of X and $\mathbb{Z}\pi$-incidence numbers.

Appendix B. Torsion

Here we give definitions and results with sketches of proofs. Details are to be found in Cohen [K.3] and Milnor [K.2] (see bibliography).

B.1 Geometrical Definition of Torsion

Let A be a space and A' obtained from A by attaching two cells e^i and e^{i+1}, and suppose that there are characteristic maps h^i and h^{i+1} for e^i and e^{i+1} such that $h^i = h^{i+1} \circ e$ where $e\colon I^i \to I^{i+1}$ is the inclusion of the face F^i. Thus A' may be regarded as obtained from A by attaching the disc I^{i+1} by the map $h^{i+1}|\colon J^i \to A$ (where $J^i = \mathrm{cl}(\dot{I}^{i+1} - F^i)$). Then A' is said to be a *cellular expansion* of A and we say that A' *collapses cellularly* on A. Notice that there is strong deformation retraction of A' on A given by retracting I^{i+1} on J^i.

Now consider pairs (X, A) where X is a finite CW complex on A and $A \subset X$ is a homotopy equivalence, and write $X' \searrow X$ or $X \nearrow X'$ rel A if X' is obtained from X by a sequence of cellular expansions. Write $X' \nearrow\!\searrow X$ rel A if there is a sequence of complexes on A such that $X' = X_0 \searrow X_1 \nearrow X_2 \searrow \cdots \searrow X_n = X$ rel A. $\nearrow\!\searrow$ is then an equivalence relation on the set of complexes X attached to A such that $A \subset X$ is a homotopy equivalence, and we define the *Whitehead group of* A, $\mathrm{Wh}(A)$, to be the set of equivalence classes. The *torsion* $\tau(X, A)$ of a pair (X, A) is the element of $\mathrm{Wh}(A)$ which it determines.

Remark. $\mathrm{Wh}(A)$ is a abelian semi-group with unit the equivalence class of the pair (A, A) and addition given by union identified over A. The fact that it is a group follows from the equivalent algebraic definition (see B.3).

B.2 Geometrical Properties of Torsion

Suppose that X_1 is a subcomplex of X and that $X_1 \searrow X_1'$. Then there is a complex X' obtained by attaching the cells of $X - X_1$ to X_1' by composing their attaching maps with the natural retraction of X_1 on X_1'.

X' is said to be obtained from X by an *internal collapse* written $X \overset{i}{\searrow} X'$. An internal expansion is the reverse of an internal collapse.

Lemma 1. *The torsion of a pair* (X, A) *is unaffected by internal expansions and collapses.*

Sketch of proof. Consider $W = X \times I$ with $A \times I$ identified to A and $X_1 \times 0$ collapsed to $X_1' \times 0$. Then $W \searrow X' \times 0$ by cylindrical collapsing (cf. remarks above 3.25) and $W \searrow X \times 1$ by collapsing cells in $(X - X_1) \times I$ cylindrically, collapsing from the side for $X_1 \times I - X_1' \times I$ and finishing with a cylindrical collapse. Therefore $X_1' \times 0 \nearrow W \searrow X_1 \times 1$.

Lemma 2. *The torsion of a pair* (X, A) *is unaffected by a homotopy of the attaching maps of cells in* $X - A$.

Sketch of proof. Let X' differ from X by a homotopy of attaching maps. Define W by attaching (cells in $X - A) \times I$ by the homotopy. Then we have cylindrical collapses $X \times 0 \nearrow W \searrow X' \times 1$.

Lemma 3. $\mathrm{Wh}(A) = 0$ *if* A *is* 1-*connected.*

Sketch of proof. The idea is to follow the proof of the h-cobordism theorem given in Chapter 6. The analogues of the handle moves are: Introduction of complementary handles – internal expansion. Cancellation of complementary handles – internal collapse. Adding handles – adding cells by homotoping the attaching map of one cell "over" the other.

Then one proceeds to simplify $C_*(X, A)$ exactly as in 6.17 until there are no cells left.

Now suppose that $X_1 \subset X$ is a subcomplex and that $\mathrm{cl}(X - X_1)$ is homeomorphic to a ball B^n attached to X_1 by a face B^{n-1}. Then we say X *poly-collapses* on X_1.

Lemma 4. *The torsion of a pair* (X, A) *is unaffected by poly-expansions and collapses.*

Proof. $X - X_1$ determines a CW complex L on B^{n-1}. Since B^{n-1} is 1-connected $L \nearrow\!\searrow B^{n-1}$ rel B^{n-1} by Lemma 3 and this induces $X \nearrow\!\searrow X_1$, as required.

A CW complex X' on A is a *subdivision* of X if $|X'| = |X|$ and each cell of X' is contained in a cell of X. We write $X' \lhd X$.

Lemma 5. *If* $X' \lhd X$ *then* $\tau(X', A) = \tau(X, A)$.

Proof. Consider $W = X \times I$ with $A \times I$ identified to A and $X \times 0$ subdivided to $X' \times 0$. Then W poly-collapses on both $X' \times 0$ and $X \times 1$ and the result follows from Lemma 4.

B.3 Algebraic Definition of Torsion

Let π be a group and $\mathbb{Z}\pi$ the integral group ring of π. Consider the set of invertible $p\times p$ matrices with entries in $\mathbb{Z}\pi$, for $p=0,1,\ldots$. An equivalence relation on this set is generated by the following operations:

(1) Replace A by $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ or vice versa.

(2) Add a multiple of one row to another.

(3) Reorder rows or columns.

(4) Multiply a row by an element of π or by -1.

The set of equivalence classes is the *Whitehead group* of π denoted $\mathrm{Wh}(\pi)$.

Remark. The multiplication in $\mathrm{Wh}(\pi)$ is given by block addition i.e. $A+B=\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. To see that $\mathrm{Wh}(\pi)$ is a group observe that this multiplication coincides with matrix multiplication since $B\sim\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$ using operations (1) and (3).

Now let A be a space and $\pi=\pi_1(A)$. Let a $p\times p$ matrix A over $\mathbb{Z}\pi$ and an integer $i>1$ be given. Construct a CW complex X attached to A by first attaching p i-cells to the basepoint to form X^i and then further attaching p $(i+1)$-cells so that $\varepsilon(e_j^{i+1}, e_k^i)=A_{jk}$ for each (j,k). (This is done by attaching the cells in $\tilde{X}^i$ using the fact that $\pi_i(\tilde{X}^i,\tilde{A})$ is a free $\mathbb{Z}\pi$-module on p generators.) Now notice that if A is varied by one of the operations (1) to (4) then $\tau(X,A)$ is unaltered since (1) corresponds to an expansion or collapse, (2) to adding cells, (3) to renumbering cells and (4) to changing a basepath or characteristic map.

Thus we have a function $\phi\colon \mathrm{Wh}(\pi)\to\mathrm{Wh}(A)$.

Theorem. *ϕ is an isomorphism.*

Sketch of proof. That ϕ is a homomorphism is clear using block addition. To see that ϕ is onto use a proof like the proof of the s-cobordism theorem to move cells into two adjacent dimensions. To see that ϕ is 1:1 construct a function $\psi\colon \mathrm{Wh}(A)\to\mathrm{Wh}(\pi)$ so that $\psi\circ\phi=\mathrm{id}$. This is done by associating a matrix to each "boundary map" $C_i/B_i\cong B_{i-1}$ using stable bases for the B_i. One then sums the torsion of these matrices using alternating signs. For details here see Milnor [K.2; § 3].

B.4 Torsion and Polyhedra

Let $P\subset Q$ be a compact polyhedral pair with P a deformation retract of Q. Then by considering any triangulation of (Q,P) we get a definition of $\tau(Q, P)$. Now any two triangulations have a common subdivision and it follows from Lemma 5 that $\tau(Q, P)$ is well-defined and a p.l. invariant of (Q, P). We have the following p.l. interpretation of torsion:

Theorem. $\tau(Q, P)=0$ *if and only if there is a sequence of p.l. expansions and collapses (in the sense of Chapter 3)* $Q \nearrow\searrow P$ rel P.

Sketch of proof. If $Q \nearrow\searrow P$ by p.l. expansions and collapses then $\tau(Q, P)=0$ by Lemma 4. Now suppose that $\tau(Q, P)=0$ then an argument similar to the proof of the s-cobordism theorem shows that $Q \nearrow\searrow P$ p.l., handle moves being replaced by p.l. approximations of the corresponding cell moves.

Now suppose that (W, M_0, M_1) is an h-cobordism and that we have a handle decomposition of W on M_0. We used the following result in the proof of the s-cobordism theorem:

Theorem. $\tau(W, M_0)=\tau(K, M_0)$ *where K is the CW complex associated to the given handle decomposition.*

Sketch of proof. This follows from invariance under subdivision and internal collapse on noticing that K is essentially the result of collapsing each handle onto its core. More precisely let J be a triangulation of (W, M_0) so that the handles and their cores are all subcomplexes. Then internal poly-collapses replace each handle by its core and we obtain a subdivision of K.

B.5 Torsion and Homotopy Equivalences

Let $h: X \to Y$ be a homotopy equivalence of CW complexes, such that $h(X_i) \subset Y_i$ for each i. Form the *mapping cylinder* M_h by attaching $X \times I$ to Y by $h|X \times 1$. M_h is then a CW complex and we define the torsion of h, $\tau(h)$ to be $\tau(M_h, X \times 0)$. If $h: P \to Q$ is a p.l. homotopy equivalence of compact polyhedra then M_h can be given the structure of an abstract polyhedron (see [B.1]) and thus $\tau(h)$ is again defined. We then have the following interpretation of $\tau(h)$ (compare Chapter 3).

Theorem. $\tau(h)=0$ *if and only if h is homotopic to a simple homotopy equivalence. The result is also true for CW complexes where "simple" is interpreted using cellular collapses.*

Sketch of proof. If $\tau(h)=0$ then $M_h \nearrow\searrow P \times 0$ and $P \times 0 \nearrow\searrow M_h \searrow Q$ determines a map homotopic to h. Now if $h \simeq h'$ then $M_h \nearrow\searrow M_{h'}$ by considering the mapping cylinder of the homotopy. Consequently if $h \simeq$ simple homotopy equivalence then following $Q \nearrow\searrow P$ gives $M_h \nearrow\searrow M_{\mathrm{id}} \searrow P \times 0$. A similar argument establishes the CW case.

Historical Notes

(Reference numbers refer to the bibliography)

General notes

Polyhedra and p.l. maps have usually been defined using simplicial complexes and simplicial maps. These definitions appear as Theorems 2.11 and 2.14 in our approach. More suitable names for our definitions would be locally-conical sets and maps. The subject arose as a branch of geometric topology in the 1920's, Newman and Alexander being the principal early authors. Geometric topology itself arose out of Poincaré's work on differential equations in the 1890's. The subject was developed by Whitehead in his work on simplicial neighbourhoods in the 1940's. Zeeman's notes [A.7] have been the most important modern influence on the subject.

P.l. topology is now of central importance in geometric topology since Kirby and Siebenmann [R.4] have shown that (in dimensions $\geqq 5$) p.l. notions essentially coincide with topological ones, except for a curious 3-dimensional obstruction. Also smoothing theory (Section Q of bibliography) which links p.l. topology to differential topology, is a well developed subject in which the main problems are now essentially homotopy-theoretic.

Notes on Chapter 1

p. 2: "The house with two rooms" was constructed by Bing [H.3] as an example of a contractible polyhedron which is not collapsible (see also Chapter 3).

p. 6: The "standard mistake" is so called because it has been made in print so often.

p. 7: Our remarks on the definition of a polyhedron apply also to our definition of a p.l. manifold. Notice that a complex which triangulates a p.l. manifold is usually referred to as a combinatorial manifold (see 2.21).

p. 8: The Poincaré conjecture is named in honour of Poincaré, who investigated the 3-dimensional case. He falsely conjectured that a homology 3-sphere is a genuine sphere and discovered counterexamples.

The h-cobordism theorem was proved by Smale [H.4] in the differentiable case, who introduced the idea of a handle and gave essentially the same proof as our Chapter 6. However, for technical reasons, handles work best in the p.l. case and authors in differentiable topology prefer the equivalent notion of a Morse function. This is the attitude taken by Milnor [H.6]. The extension to the p.l. case was realised by several authors, particularly Stallings and Zeeman.

Smale used his h-cobordism theorem to prove the Poincaré conjecture in dimensions $\geqq 5$. In the p.l. case dimension 5 presents a little more difficulty since Smale used the vanishing of $\pi_4(\mathbb{O})$ to show that a 5-dimensional homotopy sphere bounds a contractible 6-manifold (and hence by the argument given in Chapter 1, is h-cobordant to a 5-sphere). In the p.l. case we need to know that $\pi_4(PL)=0$ which uses in addition Cerf's theorem [Q.8].

A weak form of the Poincaré conjecture for dimension $\geqq 5$ (a homotopy sphere is a topological sphere) was proved by Stallings [H.10] and Zeeman [H.11] independently of Smale's work, using engulfing theory (see our Chapter 7, where an engulfing theorem is deduced from handle theory).

The s-cobordism theorem was proved independently by Barden [H.7], Mazur [H.8], and Stallings [A.8]. See also Kervaire [H.9].

Notes on Chapter 2

p. 15: The foundations of p.l. topology (particularly Alexander's work) originally rested heavily on "stellar moves"—the science of stellar subdivision.

p. 18 and p. 20: The subdivision theorem and the treatment of pseudo-radial projection are taken from Zeeman's notes [A.7].

p. 24: The collaring theorem was first proved by Whitehead [B.1] and extended by Zeeman [B.2]. Our treatment is based on Conelly [B.3].

Notes on Chapter 3

Our treatment of regular neighbourhoods is based on Cohen's ideas [C.4]; the earliest result in the chapter is Newman's theorem which appears as our 3.13. Our proof differs little from Cohen's proof [A.6] and is considerably shorter than previous proofs [A.2], Alexander [A.4] (based on stellar moves), Zeeman [A.7] (using a long induction together with the collapsing approach to regular neighbourhoods). Whitehead's paper [B.1] initiated the theory of regular neighbourhoods which was then intimately linked with "simplicial collapsing" which does not appear at all in our treatment. Hudson and Zeeman [C.1] and Cohen [C.4] have extended the theory to "relative regular neighbourhoods".

p. 39: Collapsing and simple homotopy type were invented by Whitehead [B.1] and [K.1], see also Appendix B.

p. 40: The notion of trail is due to Hirsch.

p. 43: 3.32 is due to Gugenheim [G.1].

p. 47: The 3-dimensional case of the Schönflies theorem is due to Alexander [D.1]. The topological theorem was proved by Brown [D.2], Mazur [D.3] and Morse [D.4]. Our 3.38 also follows from the methods of [D.2]. Cohen and Sullivan [D.5] have shown, independently of the unsolved Schönflies conjecture, that any $M^n \subset Q^{n+1}$ (i.e. not necessarily locally flat) has a regular neighbourhood $\cong M \times I$.

Notes on Chapter 4

p. 52: The unknotting theorem for balls and spheres in codimension $\geqq 3$ is due to Zeeman [B.2].

p. 54: The idea of cellular moves is also due to Zeeman [G.5]. (He invented it for precisely the same purpose as our 4.16.)

p. 56: 4.18 and 4.20 (the strong versions mentioned in the remark on p. 56) are due to Hirsch [L.5].

p. 57: The isotopy extension theorem for manifolds (4.25) is due to Hudson and Zeeman [E.1]. Extensions to polyhedra were given by Rourke [E.3] (a weak theorem), Hudson and Lickorish-Siebenmann [E.4] (codimension $\geqq 3$), and the general theorem by Akin [E.5].

p. 58: Akin's hypotheses are constant ambient intrinsic dimension and a weaker local collaring condition.

Notes on Chapter 5

General position is part of p.l. "folklore"; the first systematic treatment appears in Zeeman [A.7], more general theorems are given in Stallings [A.8].

p. 63: Theorem 5.5 is due to Penrose-Whitehead-Zeeman [G.4].

p. 64: Theorem 5.6 and the proof are taken from Zeeman [G.5].

p. 67: Piping is also part of the folklore.

p. 68: The Whitney lemma is due to Whitney! [G.8] in the smooth case. A proof of the p.l. case is given by Weber [G.9] using Zeeman's classification of links [G.7]. (Notice that the exercises at the end of Chapter 5 provide a proof without using links.)

Notes on Chapter 6

The main reference for this chapter is Smale [H.4] (see the notes on Chapter 1).

p. 84: The duality theorem is due to Lefschetz. See also Appendix A.

p. 90: Construction of h-cobordisms is due to Stallings [A.8].

Notes on Chapter 7

p. 91: Unknotting balls and spheres is due to Zeeman [B.2] (by a direct geometrical argument independent of the h-cobordism theorem).

p. 92: The criterion for unknotting in codimension 2 is due to Levine [G.2] for $q \geq 5$ and Papakiriakopoulos [G.3] for $q=3$. See [K.5] for a proof that $\mathrm{Wh}(\mathbb{Z})=0$.

p. 93: The weak 5-dimensional theorems also follow from engulfing theory (which was invented by Stallings [I.2] and Zeeman [A.7] independently of handle theory).

p. 96: The embedding theorem is taken from Irwin [G.6].

p. 96: There is an unknotting theorem due to Zeeman [G.5] which shows that any two embeddings are ambient isotopic under the conditions of 7.14. However Hudson [O.9] has shown that concordance implies isotopy in codimension ≥ 3 (see Rourke [O.10] for a proof using "embedded handle theory"), so that Zeeman's theorem follows from Irwin's theorem and 7.14. However Hudson [O.6], and Casson-Sullivan [N.7; R.9; O.14] have improved both theorems to replace conditions on M and Q by a single condition on the map.

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