Symplectic connections with parallel Ricci tensor

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We dedicate this work to the memory of S. Zakrewski

Abstract

A variational principle introduced to select some symplectic connections leads to field equations which, in the case of the Levi Civita connection of Kähler manifolds, are equivalent to the condition that the Ricci tensor is parallel. This condition, which is stronger than the field equations, is studied in a purely symplectic framework.

1. A symplectic connection ∇ on a symplectic manifold (M, ω) of dimension $2n$ is a torsion free linear connection such that $\nabla\omega = 0$. It is a standard fact [5] that the space $\mathcal{E}$ of symplectic connections on (M, ω) is isomorphic (in a non-canonical way) to the space of completely symmetric, covariant, 3 tensor fields on (M, ω). We have introduced in [3] a variational principle in order to single out particular symplectic connections, which we called preferred. The Lagrangian density is the "square" of the curvature tensor R of ∇ and the scalar product on the space of curvature tensors is induced by ω; it is not positive definite. The functional on $\mathcal{E}$ has the form

$$J = \int R^2 \frac{\omega^n}{n!} \tag{1}$$

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The corresponding field equations are

$$\oint_{X,Y,Z} (\nabla_X r)(Y,Z) = 0 \tag{2}$$

for all vector fields X, Y, Z on M, where $\oint_{X,Y,Z}$ denotes the sum over cyclic permutations of X, Y and Z. The symmetric tensor r is the Ricci tensor of the connection ∇:

$$r(X,Y) = \mathrm{tr}[Z \to R(X,Z)Y].$$

We showed in [3] how to solve the field equations (2) in dimension 2, and we determined for compact surfaces the moduli space of solutions modulo the action of the symplectomorphism group. The aim of this paper is to describe a few preliminary steps useful for the higher dimensional situation.

2. To the curvature endomorphism R of ∇, one associates a symplectic curvature tensor $\underline{R}$:

$$\underline{R}(X,Y,Z,T) = \omega(R(X,Y)Z,T) \tag{3}$$

This tensor is antisymmetric in its first two arguments, symmetric in its last two arguments and satisfies the first and second Bianchi identities

$$\oint_{X,Y,Z} \underline{R}(X,Y,Z,T) = 0 \tag{4}$$

$$\oint_{X,Y,Z} (\nabla_X \underline{R})(Y,Z,T,U) = 0 \tag{5}$$

We consider, as in [7], the decomposition of $\underline{R}$:

$$\underline{R} = E + W \tag{6}$$

where

$$\begin{aligned} E(X,Y,Z,T) &= \frac{-1}{2(n+1)}[2\omega(X,Y)r(Z,T) + \omega(X,Z)r(Y,T) \\ &+ \omega(X,T)r(Y,Z) - \omega(Y,Z)r(X,T) - \omega(Y,T)r(X,Z)], \end{aligned} \tag{7}$$

and observe that the lagrangian density has the form

$$R^2 = E^2 + W^2. \tag{8}$$

Proposition 1 *Let ∇ be a symplectic connection and assume $W = 0$. Then the connection ∇ is preferred. Furthermore there exists a 1-form u such that*

$$(\nabla_X r)(Y,Z) = \omega(X,Y)u(Z) + \omega(X,Z)u(Y) \tag{9}$$

Conversely, if there exists a 1-form u such that (9) holds, then the connection ∇ is preferred and the tensor field W satisfies the second Bianchi identity

$$\oint_{X,Y,Z} (\nabla_X W)(Y,Z,T,V) = 0 \tag{10}$$

Remark 2 The proof will show that condition (10) is sufficient to ensure that ∇ is preferred and that (9) holds.

Proof. Since $W = 0$ we have

$$\oint_{X,Y,Z} (\nabla_X E)(Y,Z,T,U) = 0.$$

Hence using (7):

$$\oint_{X,Y,Z} 2\omega(Y,Z)(\nabla_X r)(T,U) + \omega(Y,T)(\nabla_X r)(Z,U) + \omega(Y,U)(\nabla_X r)(Z,T) \qquad (11)$$
$$- \omega(Z,T)(\nabla_X r)(Y,U) - \omega(Z,U)(\nabla_X r)(Y,T) = 0$$

Choose a local basis X_i ($i \leq 2n$) of the tangent space and let $\omega_{ij} = \omega(X_i, X_j)$. Denote by ω^{ij} the elements of the inverse matrix $\omega^{ij}\omega_{jk} = \delta^i_k$.

Taking in the above relations $Y = X_i$, $Z = X_j$ and multiplying by ω^{ij} we get

$$-2n(\nabla_X r)(T,U) + (\nabla_T r)(X,U) + (\nabla_U r)(X,T) + \omega(X,T)\rho(U) + \omega(X,U)\rho(T) = 0$$

where

$$\rho(U) = \sum_{i,j} (\nabla_{X_j} r)(X_i, U)\omega^{ij}.$$

Making a cyclic sum on X, T, U we get

$$(-2n+2) \oint_{X,T,O} (\nabla_X r)(T,U) = 0$$

which proves that the connection ∇ is preferred if $n > 1$. Recall from [3] that if M is of dimension 2, W vanishes identically and thus the assumption only makes sense if $\dim M \geq 4$. Taking into account that the connection is preferred in $(*)$ we get relation (9) with

$$u = \frac{1}{(2n+1)}\rho.$$

Conversely the substitution of (9) in $\oint(\nabla_X E)(Y,Z,T,U)$ shows that E satisfies the second Bianchi X, Y, Z identities. ▪

3. Among symplectic manifolds are the Kähler or pseudo-Kähler manifolds. For these symplectic manifolds there is a distinguished, symplectic connection: the Levi Civita connection $\dot{\nabla}$. The following exhibits the necessary and sufficient condition for $\dot{\nabla}$ to be preferred in the sense of §1.

Proposition 3 *Let (M, g, J, ω) be a pseudo-Kähler manifold of dimension $2n$ and let $\dot{\nabla}$ be the Levi Civita connection associated to g. Then $\dot{\nabla}$ is a preferred connection if and only if the Ricci tensor r of $\dot{\nabla}$ is parallel*

$$\dot{\nabla} r = 0.$$

Proof. Let us first recall some known facts about the Ricci form of Kähler manifolds [1]. Let R denote the curvature endomorphism of $\dot{\nabla}$ and let $\underline{\underline{R}}$ be the Riemannian curvature tensor

$$\underline{\underline{R}}(X,Y,Z,T) = g(R(X,Y)Z,T).$$

Since g and ω are related by

$$g(X,Y) = \omega(X,JY),$$

the Riemannian curvature tensor $\underline{\underline{R}}$ is related to the symplectic curvature tensor by

$$\underline{\underline{R}}(X,Y,Z,T) = \underline{R}(X,Y,Z,JT).$$

Using the symmetries of $\underline{\underline{R}}$ we get

$$\begin{aligned} g(R(X,JY)JZ,T) &= g(R(JZ,T)JX,JY) = g(JR(JZ,T)X,JY) \\ &= g(R(JZ,T)X,Y) = g(R(X,Y)JZ,T) \end{aligned}$$

which means that

$$R(JX,JY) = R(X,Y).$$

This implies that

$$\begin{aligned} r(JX,JY) &= tr[Z \to R(JX,Z)JY] = tr[Z \to JR(JX,Z)Y] \\ &= -tr[Z \to JR(X,JZ)Y] \\ &= tr[Z \to R(X,Z)Y] = r(X,Y) \end{aligned}$$

The Ricci form ρ is defined by

$$\rho(X,Y) = r(JX,Y).$$

It is a form of type $(1,1)$; indeed

$$\rho(JX,JY) = -r(X,JY) = r(JX,Y) = \rho(X,Y).$$

It is also closed, hence

$$0 = \oint_{X,Y,Z} X_\rho(Y,Z) - \rho([X,Y],Z) = \oint_{X,Y,Z} (\nabla_X \rho)(Y,Z). \qquad (*)$$

Extend ρ complex linearly to the complexified tangent bundle and denote by X', (resp. X'') the holomorphic (resp. antiholomorphic) component of X. In $(*)$ choose $X = X'$, $Y = Y'$, $Z = Z''$ then

$$\nabla_{X'}\rho(Y',Z'') + \nabla_{Y'}\rho(Z'',X') + \nabla_{Z''}\rho(X',Y') = 0.$$

Since ∇ preserves types, the last term vanishes identically and hence

$$\nabla_{X'}\rho(Y',Z'') - \nabla_{Y'}\rho(X',Z'') = 0. \tag{$**$}$$

The field equations (2) can be extended complex linearly to the complex tangent bundle. Choosing, as above, $X = X'$, $Y = Y'$, $Z = Z''$ we get

$$\nabla_{X'}r(Y',Z'') + \nabla_{Y'}r(Z'',X') + \nabla_{Z''}r(X',Y') = 0.$$

This can be written in terms of ρ

$$\nabla_{X'}\rho(Y',Z'') - \nabla_{Y'}\rho(Z'',X') + \nabla_{Z''}\rho(X',Y') = 0.$$

As above, the last term vanishes and thus

$$\nabla_{X'}\rho(Y',Z'') + \nabla_{Y'}\rho(X',Z'') = 0. \tag{$***$}$$

From $(**)$ and $(***)$ we get

$$(\nabla_{X'}\rho)(Y',Z'') = 0.$$

Since ρ is of type $(1,1)$ this implies $\nabla_{X'}\rho = 0$; since ρ is real we get

$$\nabla_X\rho = 0$$

for any real vector field X and thus $\nabla_X r = 0$. ■

If (M,g,J,ω) is Kähler, simply connected and complete (for the Levi Civita connection ∇) the de Rham theorem states that (M,g,J,ω) is isometric to a direct product:

$$(M,\omega,J,g) = \prod_{i=0}^{p}(M_i,\omega_i,J_i,g_i)$$

of simply connected, Kähler, complete manifolds. The factor

$$(M_0,\omega_0,J_0,g) = (\mathbb{C}^p,\omega_0,J_0,g_0)$$

is isometric to the flat manifold $\mathbb{C}^p$ with its standard Kähler structure. Each of the factors (M_i,ω_i,J_i,g_i) has irreducible holonomy. Furthermore the Ricci tensor has the form

$$r = \oplus_{i=1}^{p} r_i$$

where r_i is the Ricci tensor of (M_i,ω_i,J_i,g_i). If the Ricci tensor is parallel ($\nabla r = 0$), then $\nabla^{(i)} r_i = 0$.

If $\nabla^{(i)} r_{(i)} = 0$, the manifold (M_i,ω_i,J_i,g_i) is Kähler-Einstein, i.e. there exists a real number λ_i such that

$$r_i = \lambda_i g_i.$$

Indeed, since g_i is positive definite, there exists, at each point, a g_i-orthonormal basis with respect to which r_i is diagonal. If $V_\mu(x)$ is a proper subspace of M_x where $r_i|_{V_\mu} = \mu I|_{V_\mu}$, the distributions V_μ and $V_\mu^\perp$ are parallel and thus stable by the holonomy. This contradicts the irreducibility and hence $r_i = \lambda_i g_i$.

Corollary 4 *If (M, ω, J, g) is a simply connected, complete Kähler manifold and if the Levi Civita connection is preferred, (M, ω, J, g) is a direct product of Kähler Einstein simply connected complete manifolds.*

We are thus going to study symplectic manifolds admitting a symplectic connection ∇ with parallel Ricci tensor. But before going in this direction we would like to investigate the $W = 0$ condition in the Kähler framework.

Proposition 5 *Let (M, ω, J, g) be a Kähler manifold; assume it is Kähler Einstein and assume that the W tensor associated to the Levi Civita connection vanishes. Then (M, ω, J, g) has constant holomorphic curvature.*

Proof. Let X be a unit tangent vector at $x(\in M)$. The holomorphic curvature

$$\begin{aligned} H_x(X) &= \underline{\underline{R}}(X, JX, X, JX) = -\underline{\underline{R}}(X, JX, X, X) \\ &= -\frac{1}{2(1+n)} 4r(X, X) = -\frac{1}{(1+n)n}\tau \end{aligned}$$

where τ denotes the scalar curvature of (M, g). ∎

Remark 6 If (M, ω, J, g) has constant holomorphic curvature then the W tensor corresponding to the Levi Civita connection vanishes identically.

Corollary 7 *Let (M, ω, g, J) be a simply connected Kähler manifold which is complete with respect to the Levi Civita connection ∇. If ∇ has vanishing W tensor then (M, ω, J, g) is isometric to a product of flat $\mathbb{C}^n$'s, $\mathbb{C}P^k$'s and $\mathbb{H}^k$'s where $\mathbb{H}^k$ is the k-disk with the standard Bergman metric.*

This is a direct consequence of the above analysis and of [4].

4. We now embark on the study of symplectic manifolds (M, ω) admitting a symplectic connection with parallel Ricci tensor. We concentrate on the 4-dimensional situation although quite a number of results generalise to the $2n$-dimensional case.

To the symmetric Ricci tensor r, we can associate an endomorphism A by

$$r(X, Y) = \omega(X, AY).$$

Clearly

$$\omega(X, AY) + \omega(AX, Y) = 0$$

and thus A is an element of the Lie algebra of the symplectic group $Sp(n,\mathbb{R})$.

If r is parallel, so is A and thus the Jordan Chevalley type of A does not depend on the point $x \in M$. In particular, the eigenvalues are constant and the generalised eigenspaces V_λ of A determine parallel distributions.

We recall the following elementary lemma:

Lemma 8 *Let A be an element of the Lie algebra of $Sp(n,\mathbb{R})$; let λ be an eigenvalue of A and let V_λ be the corresponding generalised eigenspace (i.e. $V_\lambda = \{v \in \mathbb{R}^{2n}(\text{ or } \mathbb{C}^{2n}) \,|\, \exists k \in \mathbb{N} \text{ s.t. } (A-\lambda)^k v = 0\}$). Then $V_\lambda^\perp (= \{v \in \mathbb{R}^{2n}(\text{ or } \mathbb{C}^{2n}) \,|\, \omega(v,w) = 0, \forall w \in V_\lambda\})$ is*

$$V_\lambda^\perp = \bigoplus_{\substack{\mu \in spec(A) \\ \mu \neq -\lambda}} V_\mu$$

Remark 9 If necessary, we have extended ω to $\mathbb{C}^{2n}$ complex bilinearly.

Corollary 10 *If A admits a complex eigenvalue $\lambda = a + ib$, $ab \neq 0$ it admits also the eigenvalues $\bar\lambda, -\lambda, -\bar\lambda$. In particular if $\dim M = 4$ A is semi simple and there exists a complex basis $\{e_\lambda, e_{\bar\lambda}, e_{-\lambda}, e_{-\bar\lambda}\}$ composed of eigenvectors of A.*

Corollary 11 *If A admits a real (resp. pure imaginary) eigenvalue a (resp. ib) with a (resp. b) $\neq 0$ it also admits the eigenvalue $-a$ (resp. $-ib$). If A admits the eigenvalue 0, it has necessarily even multiplicity.*

Proposition 12 *Let (M,ω) be a simply connected symplectic manifold and let ∇ be a complete symplectic connection such that the corresponding Ricci tensor is parallel. Assume the Ricci tensor is non-degenerate. Then*

$$(M,\omega,\nabla) = \prod_{i=1}^{p} (M_i,\omega_i,\nabla_i)$$

where each (M_i,ω_i,∇_i) is a complete symplectic space with a parallel Ricci tensor such that the corresponding endomorphism has eigenvalues $\{\lambda_i, -\lambda_i, \overline{\lambda}_i, -\overline{\lambda}_i\}$ (there are just two elements in this set if λ_i is real or imaginary).

Proof. The Ricci tensor defines a pseudo Riemannian structure r_∇; the Levi Civita connection of r_∇ coincides with ∇ since r_∇ is parallel and ∇ is torsion-free. The endomorphism A associated to r_∇ gives a decomposition of the complexified tangent space in sums of generalised eigenspaces. When those are grouped for $\{\lambda_i, -\lambda_i, \overline{\lambda}_i, -\overline{\lambda}_i\}$ they yield real parallel ω- and r_∇-orthogonal distributions on M. The completeness assumption and the simple connectedness of M imply the result by Wu's pseudo Riemannian analogue of the de Rham decomposition theorem [8]. ∎

Proposition 13 *Let (M, ω) be a simply connected symplectic manifold and let ∇ be a complete symplectic connection such that the corresponding Ricci tensor is parallel. Denote by $A_{(r)}$ the endomorphism associated to r. Assume all eigenvalues of $A_{(r)}$ are of multiplicity 1. Then*

(i) (M, ω, ∇) *is a symplectic symmetric space;*

(ii) $(M, \omega, \nabla) = \prod_{i=1}^{p}(M_i, \omega_i, \nabla_i) \times \prod_{j=1}^{q}(M_j, \omega_j, \nabla_j)$ *where each* $(M_i, \omega_i, \nabla_i)$ *(resp. each* $(M_j, \omega_j, \nabla_j)$*) is a* 4*-dimensional (resp.* 2*-dimensional) symmetric symplectic space;*

(iii) each 4*-dimensional factor* $(M_i, \omega_i, \nabla_i)$ *is symplectomorphic and affinely equivalent to the symmetric space* $SL(2, \mathbb{C})/\mathbb{C}^*$ *endowed with its standard connection and with a symplectic form which is the classical Kostant Souriau form corresponding to the adjoint orbit of an element* λH *of* $sl(2, \mathbb{C})$*;* [1]

(iv) Each 2*-dimensional factor* $(M_j, \omega_j, \nabla_j)$ *is symplectomorphic and affinely equivalent* **either** *to the sphere* S^2 *endowed with its standard connection and with a symplectic form which is a multiple of the standard form,* **or** *to the disk* D^2 *endowed with the Levi Civita connection of a Riemannian metric of constant negative curvature and with a symplectic form which is a multiple of the Riemannian volume form,* **or** *to the universal cover of the one-sheeted hyperboloid of Minkowski* 3*-dimensional space endowed with the Levi Civita connection associated to the Lorentz metric of constant curvature and with a symplectic form which is a multiple of the Lorentz volume form.*

Proof. The blocks of the eigendecomposition of the endomorphism $A_{(r)}$ are 4×4 or 2×2 and have the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \overline{\lambda} & 0 \\ 0 & 0 & 0 & -\overline{\lambda} \end{pmatrix}, \quad \lambda = a + ib, \qquad ab \neq 0;$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}_0; \qquad \begin{pmatrix} is & 0 \\ 0 & -is \end{pmatrix}, \quad s \in \mathbb{R}_0.$$

As in the proposition above, (M, ω, ∇) is a product of 4- or 2-dimensional factors, each complete and simply connected.

[1]We denote, as usual, by $\{H, E, F\}$ the basis of the Lie algebra $sl(2, \mathbb{C})$ such that $[E, F] = H, [H, E] = 2E, [H, F] = -2F$ and $\lambda \in \mathbb{C}$ denotes the complex eigenvalue of $A_{(r)}$.

In the 4-dimensional case we can, at each point, choose a basis e_λ of eigenvectors such that $\omega(e_\lambda, e_{-\lambda}) = 1$ and $e_{\bar\lambda} = \bar e_\lambda$, $e_{-\bar\lambda} = \bar e_{-\lambda}$. There exists a local, complex 1-form a such that

$$\nabla_X e_\lambda = a(X) e_\lambda, \qquad \nabla_X e_{-\lambda} = -a(X) e_{-\lambda}$$

and the curvature endomorphism is

$$R(X,Y)e_\lambda = da(X,Y)e_\lambda, \qquad R(X,Y)e_{-\lambda} = -da(X,Y)e_{-\lambda}.$$

The Bianchi identities

$$\oint R(e_\lambda, e_{-\lambda})e_{\bar\lambda} = \oint R(e_\lambda, e_{-\lambda})e_{-\bar\lambda} = 0$$

imply

$$da(e_\lambda, e_{\bar\lambda}) = da(e_\lambda, e_{-\bar\lambda}) = da(e_{-\lambda}, e_{\bar\lambda}) = da(e_{-\lambda}, e_{-\bar\lambda}) = da(e_{\bar\lambda}, e_{-\bar\lambda}) = 0.$$

The only non-vanishing components of the Ricci tensor are

$$r(e_\lambda, e_{-\lambda}) = -da(e_\lambda, e_{-\lambda}) = \lambda; \qquad r(e_{\bar\lambda}, e_{-\bar\lambda}) = \bar\lambda.$$

From this we get

$$(\nabla_X R)(Y,Z)e_\lambda = (\nabla_X R)(Y,Z)e_{-\lambda} = 0.$$

Hence we have local symmetry. To the local symmetric space one associates a symmetric triple $(\mathbf{g}, \sigma, \Omega)$ which determines its local isomorphism class [2, 6]. The Lie algebra $\mathbf{g}$ is constructed as follows: the subspace $\mathbf{p} = \{X \in \mathbf{g} \mid \sigma(X) = -X\}$ is identified with the tangent space at a point, hence $\mathbf{p}^{\mathbb{C}}$ is spanned by $\{e_\lambda, e_{-\lambda}$ and their conjugates $\}$; the subspace $\mathbf{k} = \{X \in \mathbf{g} \mid \sigma(X) = X\}$ is of the form $\mathbf{k} = [\mathbf{p}, \mathbf{p}]$ and its elements can be identified with the endomorphisms of $\mathbf{p}$ given by the curvature. So

$$\begin{cases} [e_\lambda, e_{-\lambda}] = R(e_\lambda, e_{-\lambda}); \\ [R(e_\lambda, e_{-\lambda}), e_\lambda] = -\lambda e_\lambda; \\ [R(e_\lambda, e_{-\lambda}), e_{-\lambda}] = \lambda e_{-\lambda} \end{cases}$$

and the Lie algebra $\mathbf{g}$ is isomorphic to $sl(2,\mathbb{C})$ viewed as a real 6-dimensional Lie algebra. The standard basis $\{H, E, F\}$ is such that

$$e_\lambda = \alpha E, \qquad e_{-\lambda} = \beta F, \qquad R(e_\lambda, e_{-\lambda}) = -\frac{\lambda}{2} H, \qquad (\alpha\beta = -\frac{\lambda}{2}).$$

If $\{\tilde\theta^\lambda\}$ denotes the dual basis to the basis $\{e_\lambda\}$ and $\{\theta^\lambda\}$ the dual basis to $\{E, F, \bar E, \bar F\}$ one has

$$\omega = -\frac{\lambda}{2}(\theta^\lambda \wedge \theta^{-\lambda}) + c \cdot c.$$

This concludes the 4-dimensional case.

In dimension 2 we have either a real or a pure imaginary eigenvalue

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \lambda \in \mathbb{R}_0 \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} is & 0 \\ 0 & -is \end{pmatrix} \quad s \in \mathbb{R}_0 \qquad \omega = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A calculation completely analogous to the above proves the local symmetry. The case of the real eigenvalue leads to the universal cover of the 1-sheeted hyperboloid in Minkowski 3-space. The sphere corresponds to $s > 0$ and the disk to $s < 0$. ■

Proposition 14 *Let (M, ω) be a simply connected symplectic manifold admitting a complete symplectic connection ∇ such that*

(i) its Ricci tensor r is parallel and non-degenerate;

(ii) the corresponding endomorphism $A_{(r)}$ admits a real or purely imaginary eigenvalue λ with multiplicity 2;

then the endomorphism $A_{(r)}|_{V_\lambda}$, where V_λ denotes the generalised eigenspace corresponding to the eigenvalue λ of multiplicity 2, is diagonal.

Proof. Assume $A_{(r)}|_{V_\lambda}$ has a nilpotent part. The argument used in the proof of Proposition 13 reduces us to a 4-dimensional situation. There exists a local basis $\{e_j; j \leq 4\}$ such that

$$A_{(r)} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and λ is real or pure imaginary. There exist two 1-forms a, b such that

$$\nabla_X e_1 = a(X)e_1, \qquad \nabla_X e_3 = -a(X)e_3,$$
$$\nabla_X e_2 = b(X)e_1 + a(X)e_2, \qquad \nabla_X e_4 = b(X)e_3 - a(X)e_4$$

and the curvature has the form

$$R(X,Y)e_1 = da(X,Y)e_1, \qquad R(X,Y)e_2 = db(X,Y)e_1 + da(X,Y)e_2,$$
$$R(X,Y)e_3 = -da(X,Y)e_3, \qquad R(X,Y)e_4 = db(X,Y)e_3 - da(X,Y)e_4.$$

The Bianchi identity

$$\oint_{4,1,2} R(e_4, e_1)e_2 = 0$$

implies $da(e_4, e_1) = 0$. Now the $(1, 4)$ component of the Ricci tensor is on the one hand

$$r_{14} = -\lambda$$

and on the other hand

$$r_{41} = \operatorname{tr}[Z \to R(e_4, Z)e_1] = da(e_4, e_1) = 0,$$

which is a contradiction. ∎

Lemma 15 *Let (M, ω) be a 4-dimensional symplectic manifold and let ∇ be a symplectic connection whose Ricci tensor is parallel. Assume the endomorphism $A_{(r)}$ admits a non-zero real eigenvalue and the eigenvalue 0 and assume $A_{(r)}$ has a nilpotent part. Then (M, ω) contains two transverse symplectic foliations, whose leaves are parallel. The induced connection on each of these leaves is locally symmetric. The simply connected symmetric symplectic space corresponding to one family of leaves is a coadjoint orbit of the group $M(2)$ (the motion group of Minkowski 3-space). The simply connected symmetric space corresponding to the other family of leaves is the universal cover of the one-sheeted hyperboloid in Minkowski 3-space.*

Proof. There exists a local basis such that

$$A_{(r)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Furthermore there exist two 1-forms b, k such that

$$\nabla_X e_1 = 0, \qquad \nabla_X e_2 = b(X)e_1,$$
$$\nabla_X e_3 = k(X)e_3, \qquad \nabla_X e_4 = -k(X)e_4$$

and the curvature has the form

$$R(X, Y)e_1 = 0, \qquad R(X, Y)e_2 = db(X, Y)e_1,$$
$$R(X, Y)e_3 = dk(X, Y)e_3, \qquad R(X, Y)e_4 = -dk(X, Y)e_4.$$

The first Bianchi identity implies

$$db(e_1, e_4) = db(e_1, e_3) = db(e_3, e_4) = 0,$$
$$dk(e_1, e_2) = dk(e_1, e_3) = dk(e_1, e_4) = dk(e_2, e_3) = dk(e_2, e_4) = 0$$

and the form of the Ricci tensor

$$db(e_1, e_2) = 1, \qquad dk(e_3, e_4) = \lambda.$$

From this we deduce the local symmetry and the two types of leaves. ∎

Example Let $(M,\omega) = (M_1,\omega_1)\times(M_2,\omega_2)$ where $M_1 = \mathbb{R}^2$, $\omega_1 = dx\wedge dy$ and (M_2,ω_2) is a symplectic manifold which admits a symplectic connection ∇_2 with parallel Ricci tensor (for instance M_2 is the standard 1-sheeted hyperboloid in Minkowski 3-space with ω_2 its Lorentz volume form). Consider the family of connections on M_1 parametrised by M_2:

$$\nabla_X \partial_x = 0,$$
$$\nabla_X \partial_y = (x+\psi) dy(X) \partial_x,$$

where ψ is a function on M such that $\partial_x \psi = 0$. Combine these and ∇_2 to define a symplectic connection on M. Its Ricci tensor (given by $r(\partial_y,\partial_y) = -1$ and r^{∇_2}) is parallel but the connection is in general not locally symmetric. Indeed, for a tangent vector Z which is tangent to M_2

$$(\nabla_{\partial_y} R)(Z,\partial_y)\partial_y = Z(\partial_y \psi) \neq 0.$$

Remark 16 This shows that there exist Ricci parallel symplectic connections which are not locally symmetric. This also shows that the endomorphism $A_{(r)}$ associated to the Ricci tensor can have a nilpotent part.

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