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We prove that every finitely generated soluble group that is not virtually abelian has a subgroup of one of a small number of types.

1 Introduction

The aim of this paper is to describe a small number of types of groups, which have the property that every finitely generated soluble group that is not virtually abelian has a subgroup of one of these types.

In Section 2, we shall define a class of 2-generator abelian-by-cyclic groups, each of which can be embedded into a semidirect product $\mathbb{Q}^s \rtimes \mathbb{Z}$ for some $s > 0$. We shall call these *Gc-groups* and prove in Theorem 3.3 in Section 3 that every finitely generated metabelian group that is not virtually abelian has a subgroup that is either a *proper* (that is, non virtually abelian) Gc-group or is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ or to $C_p \wr \mathbb{Z}$ for some prime p , where C_p denotes a cyclic group of order p . Our second main result, Theorem 4.6 in Section 4, is that every finitely generated torsion-free soluble group that is not virtually abelian has a subgroup that is either a proper Gc-group or is isomorphic to \mathbb{Z}^∞ , which in this paper will denote a free abelian group of countably infinite rank. Our main result, Theorem 5.2, proved in Section 5, is a little less satisfactory. We show that every finitely generated soluble group that is not virtually abelian has a subgroup H that is either a proper Gc-group or is isomorphic to \mathbb{Z}^∞ , or is finitely generated with an infinite normal torsion subgroup. We would prefer to be able to further restrict the structure of H in the third case.

These results were motivated by the work of the first author on the class of finitely generated groups whose word problem is an intersection of finitely many context-free languages - known as *poly- \mathcal{CF}* groups. Finitely generated virtually abelian groups are known to all be poly- \mathcal{CF} , and it is conjectured that these are the only soluble poly- \mathcal{CF} groups. Since the class of poly- \mathcal{CF} groups is closed under taking finitely generated subgroups, it would suffice to find a list of finitely generated non-poly- \mathcal{CF} groups such that every finitely generated soluble group either has a subgroup isomorphic to one on the list

or is virtually abelian. The first author has shown that $C_p \wr \mathbb{Z}$ for any prime p , proper Gc-groups and all groups containing \mathbb{Z}^∞ are not poly- \mathcal{CF} . Thus Theorems 3.3 and 4.6 lead to a proof of the conjecture in the metabelian and torsion-free cases respectively. It remains to deal with the groups in the last case of Theorem 5.2. One approach might be to show that a poly- \mathcal{CF} group cannot have an infinite torsion subgroup.

In general not much can be said about the structure of finitely generated soluble groups and their subgroups. For example, B.H. Neumann and H. Neumann proved in 1959 [2, Section 4.1.1] that every countable soluble group of derived length d may be embedded in a 2-generator soluble group of derived length at most $d+2$, and P. Hall proved in 1954 [2, Section 4.1.3] that, for any non-trivial countable abelian group A , there are 2^{\aleph_0} non-isomorphic 2-generator centre-by-metabelian groups G such that $A \cong Z(G) = G''$.

We conclude this introductory section by stating some results that we shall need from the literature.

Proposition 1.1. [5, Theorem 9.64] *Let $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_k)$ be the non-singular diagonal block in the Smith normal form of a matrix M with entries in a euclidean ring R . Define $\gamma_i(M)$ by $\gamma_0(M) = 1$ and for $i > 0$, let $\gamma_i(M)$ be the gcd of all $i \times i$ minors of M . Then $\sigma_i = \gamma_i(M)/\gamma_{i-1}(M)$ for all $1 \leq i \leq k$.*

Proposition 1.2. (Philip Hall: for a proof, see [3, 14.1.3].) *Let G be a finitely generated group, let $N \triangleleft G$ and suppose that G/N is finitely presented. Then N is the normal closure in G of a finite subset of G .*

Proposition 1.3. (Schur: for a proof, see [3, 10.1.4].) *Let G be a group with $|G : Z(G)|$ finite. Then G' is finite.*

Proposition 1.4. (Minkowski: for a proof, see [4].) *There exists a function $L : \mathbb{N} \rightarrow \mathbb{N}$ such that the order of any finite subgroup of $\text{GL}(n, \mathbb{Z})$ divides $L(n)$.*

2 Two-generator metabelian groups

The proof our main result on metabelian groups will rely on some facts about a certain type of two-generator metabelian group. These are groups H satisfying the following hypothesis.

Hypothesis (*): $H = \langle a, b \rangle$, where $|a|$ is infinite, the subgroup $B = \langle b^{a^i} \mid i \in \mathbb{Z} \rangle$ is abelian, and H is not virtually abelian.

We observe that (*) implies that $H = B \rtimes \langle a \rangle$.

Throughout our discussion of metabelian groups, H and B will have these definitions, and for convenience we will use the notation b_i for b^{a^i} . We will also sometimes use additive notation within B .

We will first consider two types of groups satisfying Hypothesis (*): (i) H is torsion-free; (ii) b is a torsion element; before going on to the general case.

2.1 Case 1: H is torsion-free

Suppose that $H = \langle a, b \rangle$ is torsion-free and satisfies Hypothesis (*). If $H \not\cong \mathbb{Z} \wr \mathbb{Z}$, then B is not free abelian on $\{b_i \mid i \in \mathbb{Z}\}$, so there exist integers r, s, c_i with $r \leq s$ and $c_r, c_s \neq 0$ such that $\sum_{i=r}^s c_i b_i = 0$. Since we are assuming that H is torsion-free and not virtually abelian, we must have $r < s$, and we may assume that $\gcd(c_0, \dots, c_s) = 1$. Furthermore, by conjugating by a^{-r} , we may take $r = 0$.

We will eventually show that this relation $\sum_{i=0}^s c_i b_i = 0$ (or in multiplicative notation $b_0^{c_0} b_1^{c_1} \cdots b_s^{c_s} = 1$), together with the fact that B is abelian, fully determines H . To do this, we first consider a certain type of group presentation, and then show that H has a presentation of this type.

2.1.1 The groups $G(\mathbf{c})$

For $\mathbf{c} = (c_0, \dots, c_s) \in \mathbb{Z}^{s+1}$ with $s \geq 1$, $c_0, c_s \neq 0$ and $\gcd(c_0, \dots, c_s) = 1$, let $G(\mathbf{c})$ be the group defined by the presentation $\langle \alpha, \beta \mid \mathcal{R}_{\mathbf{c}} \rangle$, where

$$\mathcal{R}_{\mathbf{c}} = \left\{ [\beta, \beta^{\alpha^i}] \ (i \in \mathbb{Z}), \ \beta^{c_0} (\beta^{\alpha})^{c_1} \cdots (\beta^{\alpha^s})^{c_s} \right\}.$$

We shall call such groups *Gc-groups*, and when we refer to the Gc-group $G(\mathbf{c}) = \langle x, y \rangle$, we will assume that $\mathbf{c} \in \mathbb{Z}^{s+1}$ satisfies the above conditions, and that x replaces α and y replaces β in the above definition of $G(\mathbf{c})$.

Note that the Gc-groups include some polycyclic groups (when $|c_0| = |c_s| = 1$, for example), some of which are even virtually abelian. But they also

include other types of groups, such as the soluble Baumslag-Solitar groups $\langle x, y \mid y^{-1}xy = x^k \rangle$ with $|k| > 1$.

If $\langle X \mid R \rangle$ is a group presentation, denote its abelianisation by $\text{Ab} \langle X \mid R \rangle$. We call such a presentation an *abelian presentation*, and we often use additive notation for its relators.

Lemma 2.1. *Let $G = G(\mathbf{c}) = \langle \alpha, \beta \rangle$ be a Gc-group, and let $J = \langle \beta^{\alpha^i} \mid i \in \mathbb{Z} \rangle$. Then $G = J \rtimes \langle \alpha \rangle$,*

$$J \cong \mathcal{J} := \text{Ab} \left\langle \beta_i \ (i \in \mathbb{Z}) \mid \sum_{i=0}^s c_i \beta_{k+i} \ (k \in \mathbb{Z}) \right\rangle,$$

and G is torsion-free.

Proof. We see from the presentation that $G = J \rtimes \langle \alpha \rangle$. The automorphism ψ of \mathcal{J} given by $\psi(\beta_i) = \beta_{i+1}$ corresponds to the automorphism of J given by conjugation by α . We thus have

$$\mathcal{J} \rtimes \langle \alpha \rangle \cong \langle \beta_i \ (i \in \mathbb{Z}), \alpha \mid [\beta_0, \beta_1], \beta_i^\alpha = \beta_{i+1} \ (i \in \mathbb{Z}), \beta_0^{c_0} \cdots \beta_s^{c_s} \rangle,$$

since all the other relators of \mathcal{J} are consequences of those listed. By eliminating the generators β_i for $i \neq 0$ using $\beta_i = \beta_0^{\alpha^i}$ and replacing β_0 by β , we arrive at a presentation for G , and so $G \cong \mathcal{J} \rtimes \langle \alpha \rangle$, with $J \cong \mathcal{J}$.

It remains to show that G is torsion-free, for which it suffices to show that J is torsion-free. For this, it is enough to show that any subgroup of J of the form $J_I := \langle \beta_i \mid i \in I \rangle$ is torsion-free, where $I = \{k, k+1, \dots, k'\}$.

To show this, we first derive a presentation for J_I . Set $r_k = \sum_{i=0}^s c_i \beta_{k+i}$ for $k \in \mathbb{Z}$. Suppose that $r := \sum_{i=k}^{k'} \delta_i \beta_i$ with $\delta_i \in \mathbb{Z}$ is a relator of J_I . Since r is also a relator of J , it must be equal in the free abelian group on the β_i to an element of the form $\sum_{i=l}^{l'} \gamma_i r_i = \sum_{i=l}^{l'} \gamma_i \sum_{j=0}^s c_j \beta_{j+i}$, where $\gamma_l, \gamma_{l'} \neq 0$. So we must have $k \leq l$ and $l' + s \leq k'$. Conversely, any r_j with $k \leq j \leq k' - s$ is a relator of J_I . Hence $J_I \cong \text{Ab} \langle \beta_i \ (i \in I) \mid r_j \ (k \leq j \leq k' - s) \rangle$.

Let $m = k' - s - k$. Since J_I is a finitely generated abelian group, we can find its isomorphism type by computing the Smith normal form of its presentation matrix, which is a matrix with m rows and $m + s$ columns of

Proposition 2.3. *Let $H = \langle a, b \rangle$ be torsion-free and satisfy Hypothesis (*). Then H is isomorphic to either $\mathbb{Z} \wr \mathbb{Z}$ or a Gc-group.*

Proof. Suppose H is not isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. Then, as we observed earlier, there exist $s \geq 1$ and $\mathbf{c} = (c_0, \dots, c_s) \in \mathbb{Z}^{s+1}$ with $c_0, c_s \neq 0$ and $\gcd(c_0, \dots, c_s) = 1$ such that $\sum_{i=0}^s c_i b_i = 0$ (or in multiplicative notation $b_0^{c_0} \cdots b_s^{c_s} = 1$). We choose \mathbf{c} with s minimal.

Let $G = G(\mathbf{c}) = \langle \alpha, \beta \mid \mathcal{R}_{\mathbf{c}} \rangle$. Then since the relations of G are satisfied by $a, b \in H$, there is an epimorphism $\phi : G \rightarrow H$ with $\phi(\alpha) = a$ and $\phi(\beta) = b$. We shall show that ϕ is an isomorphism. For convenience we shall write β_i for β^{α^i} and b_i for b^{α^i} . Thus $\mathcal{R}_{\mathbf{c}} = \{[\beta_0, \beta_i] \ (i \in \mathbb{Z}), \beta_0^{c_0} \cdots \beta_s^{c_s}\}$.

Let J and K be the subgroups of G defined by $J = \langle \beta_i \mid i \in \mathbb{Z} \rangle$ and $K = \langle \beta_0, \beta_1, \dots, \beta_{s-1} \rangle$. We saw in Lemma 2.1 that J (and hence also K) is abelian, $G = J \rtimes \langle \alpha \rangle$ and G is torsion-free. If $\beta_0, \beta_1, \dots, \beta_{s-1}$ were not linearly independent over \mathbb{Z} , then their images under ϕ , namely b_0, b_1, \dots, b_{s-1} , would not be linearly independent either, contradicting the minimality of s . Thus K is free abelian of rank s .

We now show that J/K is a torsion group. We claim that for any $j \in \mathbb{Z}$, there exists some non-zero $\lambda_j \in \mathbb{Z}$ such that $\lambda_j \beta_j \in K$. This is clearly true for $0 \leq j \leq s-1$, and also for $j = s$ with $\lambda_s = c_s$, since $c_s \beta_s = -\sum_{i=0}^{s-1} c_i \beta_i$. We can then prove the claim by induction for all $j \geq s$ by repeatedly conjugating by α , and for all $j < 0$ by conjugating by α^{-1} . So the generators of J are torsion elements modulo K , and hence J/K is a torsion group. So, for each $g \in J$ there exists $t_g \in \mathbb{N}$ such that $t_g g \in K$.

Let $g \in \ker \phi$. Since $G = J \rtimes \langle \alpha \rangle$, we can write $g = g' \alpha^n$, where $g' \in J$, $n \in \mathbb{Z}$. So $1 = \phi(g) = \phi(g') a^n$, which implies $\phi(g') = a^{-n} = 1$, so $n = 0$ and $g \in J$. Note that ϕ is injective on K , since $\phi(\beta_0), \dots, \phi(\beta_{s-1})$ are linearly independent. So $\phi(g) = 0$ implies $\phi(t_g g) = 0$ with $t_g g \in K$, and hence $t_g g = 0$. Since G is torsion-free, this implies $g = 0$. Thus $\ker \phi$ is trivial and H is isomorphic to G . \square

2.1.3 More about Gc-groups

We finish this section with an embedding result for Gc-groups and some useful lemmas that follow from it.

Proposition 2.4. *Let $G = G(\mathbf{c})$ be a Gc-group. Let $\{x_1, \dots, x_s\}$ be a basis for \mathbb{Q}^s over \mathbb{Q} (the rationals under addition), and let $\mathbb{Z} = \langle y \rangle$. Let $Q = \mathbb{Q}^s \rtimes \mathbb{Z}$, with the action of y on \mathbb{Q}^s define by $x_i^y = x_{i+1}$ ($1 \leq i < s$), $x_s^y = \sum_{i=1}^s -(c_{i-1}/c_s)x_i$. Then G can be embedded in Q .*

Proof. It is routine to show that $y, x_1 \in Q$ satisfy the defining relations of G , so there is a homomorphism $\theta : G \rightarrow Q$ with $\theta(\alpha) = y$, $\theta(\beta) = x_1$. We can show that θ is injective using a similar argument to the proof that ϕ is injective in Proposition 2.3. \square

Lemma 2.5. *Let $G = G(\mathbf{c}) = \langle \alpha, \beta \rangle$ be a Gc-group. Then, for any $t \in \mathbb{N}$, $H = \langle \alpha, \beta^t \rangle$ is a subgroup of finite index in G with $H \cong G$.*

Proof. By Proposition 2.4, we can identify G with a subgroup of $\mathbb{Q}^s \rtimes \mathbb{Z}$ for some s , with $\beta \in \mathbb{Q}^s$. Multiplication by t in \mathbb{Q}^s is a \mathbb{Z} -module isomorphism of \mathbb{Q}^s , and induces an isomorphism $G \rightarrow H$. Let $N = G \cap \mathbb{Q}^s$ and $M = H \cap \mathbb{Q}^s$. Then $|G : H| = |N : M|$. Now a finitely generated subgroup of \mathbb{Q}^s , and hence of N can be generated by at most s elements. So N/M has the property that all of its finitely generated subgroups are finite of order at most t^s . This implies that N/M is finite of order at most t^s , so $|G : H|$ is finite. \square

Lemma 2.6. *Let G be a finitely generated group with finite normal subgroup T such that G/T is isomorphic either to the Gc-group $G(\mathbf{c})$ or to $\mathbb{Z} \wr \mathbb{Z}$. Then G has a finite index subgroup isomorphic to $G(\mathbf{c})$ or $\mathbb{Z} \wr \mathbb{Z}$.*

Proof. Suppose that G/T is isomorphic to the Gc-group $G(\mathbf{c}) = \langle \alpha T, \beta T \rangle$. By the previous lemma, for any $t \in \mathbb{N}$, $\langle \alpha T, \beta^t T \rangle$ is isomorphic to $G(\mathbf{c})$ and has finite index in G/T , so by replacing β by a suitable power we may assume that $\beta \in C_G(T)$. Define $\beta_i = \beta^{\alpha^i}$ for $i \in \mathbb{Z}$, and denote the exponent of T by m . Then the elements β_i^m all commute so by passing to a finite index subgroup again, we may assume that the β_i all commute. Then

$$(\beta_0^m)^{c_0} (\beta_1^m)^{c_1} \dots (\beta_s^m)^{c_s} = (\beta_0^{c_0} \beta_1^{c_1} \dots \beta_s^{c_s})^m = 1,$$

where $\mathbf{c} = (c_0, \dots, c_s)$, so by passing to a finite index subgroup a third time we may assume $\langle \alpha, \beta \rangle \cong G(\mathbf{c})$ is a complement of T in G and hence has finite index in G . The proof for $G/T \cong \mathbb{Z} \wr \mathbb{Z}$ is left to the reader. \square

Henceforth we shall only be interested in Gc-groups which are not virtually abelian. We shall call a non virtually abelian Gc-group a *proper Gc-group*.

2.2 Case 2: b is a torsion element

Proposition 2.7. *Let $H = \langle a, b \rangle$ with b a torsion element and assume H satisfies Hypothesis (*). Then H has a subgroup isomorphic to $C_p \wr \mathbb{Z}$ for some prime p .*

Proof. For any $k \in \mathbb{N}$, $\langle b_{ik} \mid i \in \mathbb{Z} \rangle$ cannot be finitely generated, since if it could, then it would be finite, and so some power of a^k would centralise b and hence B , implying that H is virtually abelian.

The proof is by induction on the order $|b|$ of b . If $|b| = p$ is prime, then $H \cong C_p \wr \mathbb{Z}$. For if not, then some relation $\sum_{i=r}^s \gamma_i b_i = 0$ with $p \nmid \gamma_r, \gamma_s$ is true in H , and by conjugating by powers of a , we see that $B = \langle b_r, \dots, b_{s-1} \rangle$ and H is virtually abelian.

Suppose $|b| = np$ with p prime and $n > 1$, and let $H_1 = \langle a, b^n \rangle$. Since $|b^n| = p$, either $H_1 \cong C_p \wr \mathbb{Z}$ or H_1 is virtually abelian. If H_1 is virtually abelian, then $\langle b_i^n \mid i \in \mathbb{Z} \rangle$ is finitely generated and so some power of a , say a^k , centralises b^n . If $\langle a^k, b \rangle$ is virtually abelian, then $\langle b_{ik} \mid i \in \mathbb{Z} \rangle$ is finitely generated, which we know cannot happen. So we can replace a by a^k and thereby assume that a itself centralises b^n , and hence $b_i^n = b_j^n$ for all $i, j \in \mathbb{Z}$. Put $b' = b_0 b_1^{-1}$, $H_2 = \langle a, b' \rangle$, $b'_i = (b')^{a^i}$ and $\widehat{B} = \langle b'_i \mid i \in \mathbb{Z} \rangle$. Then $|B : \widehat{B}|$ and hence $|H : H_2|$ is finite, so H_2 cannot be virtually abelian. Also, $(b')^n = b_0^n b_1^{-n} = 1$, so $|b'|$ strictly divides n , and the result follows by the inductive hypothesis applied to H_2 . \square

2.3 The general case

Proposition 2.8. *Let $H = \langle a, b \rangle$ and assume H satisfies Hypothesis (*). Then one of the following holds:*

- (i) H has a subgroup isomorphic to $C_p \wr \mathbb{Z}$ for some prime p ;
- (ii) H has a subgroup that is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ or to a proper Gc-group.

Proof. If $|b|$ is finite, then we are done by Proposition 2.7, so assume $|b|$ is infinite. Any torsion elements of H are contained in B . Since B is abelian, its torsion elements form a subgroup T , which is normal in G .

Since B is finitely generated (by b) as a $\mathbb{Z}[a]$ -module, Hilbert's Basis Theorem tells us that all $\mathbb{Z}[a]$ -submodules of B are finitely generated. In particular, T is finitely generated as a $\mathbb{Z}[a]$ module. So there is a finite subset $\{t_1, \dots, t_n\}$ of T such that

$$T = \langle t_i^{p(a)} \mid p(a) \in \mathbb{Z}[a], 1 \leq i \leq n \rangle = \langle t_i^{a^j} \mid j \in \mathbb{Z}, 1 \leq i \leq n \rangle.$$

By Proposition 2.7, we can assume that $\langle a, t \rangle$ is virtually abelian for all $t \in T$, and hence that T is finite. So $H_1 := H/T$ cannot be virtually abelian. Now H_1 is a torsion-free group satisfying Hypothesis (*) and the result follows from Proposition 2.3 and Lemma 2.6. \square

3 Metabelian groups

We require two further results in order to deal with metabelian groups that have no subgroups satisfying Hypothesis (*).

Proposition 3.1. *Let G be a finitely generated group with a normal abelian subgroup N such that G/N is free abelian. If for all $a \in G$, $b \in N$, the subgroup $\langle a, b \rangle$ is virtually abelian, then G is polycyclic.*

Proof. Let G and N be as in the statement of the proposition, with $\langle a, b \rangle$ virtually abelian for all $a \in G$, $b \in N$. Then $\langle b^{a^i} \mid i \in \mathbb{Z} \rangle$ is finitely generated for any $a \in G$, $b \in N$. We will show that N is finitely generated and thus G is polycyclic.

If $G/N = \langle Na_1, \dots, Na_k \rangle$, then any $g \in G$ can be written in the form $g = ba_1^{r_1} \dots a_k^{r_k}$, where $b \in N$, $r_i \in \mathbb{Z}$. It can be shown by a straightforward induction argument that, for $b \in N$, $H_b := \langle b^g \mid g \in G \rangle$ is finitely generated. Since G/N is finitely presented, by Proposition 1.2, N is the normal closure in G of a finite set of elements and hence N is finitely generated as claimed. \square

Lemma 3.2. *If a group G is polycyclic and not virtually abelian, then G has a subgroup that is isomorphic to a proper Gc-group.*

Proof. By the second and third paragraphs of the proof of Theorem 16 in [1], there exist $a \in G$ and a non-trivial free abelian subgroup $N \triangleleft G$ such that

$\langle a, N \rangle$ is not virtually abelian. Let $N = \langle b_1, \dots, b_k \rangle$. If $\langle a, b_i \rangle$ is virtually abelian for some i , then there exist $s_i, t_i \in \mathbb{N}$ such that $a^{s_i} \in C_G(b_i^{t_i})$. If this were true for all i with $1 \leq i \leq k$, then, putting $s = \max\{s_i\}$ and $t = \max\{t_i\}$, $\langle a^s, N^t \rangle$ would be an abelian subgroup of finite index in $\langle a, N \rangle$, contrary to assumption. So at least one of the subgroups $\langle a, b_i \rangle$ is isomorphic to a proper Gc-group. \square

We are now ready to prove the full result for metabelian groups.

Theorem 3.3. *Let G be a finitely generated metabelian group that is not virtually abelian. Then G has a finitely generated subgroup isomorphic to one of the following:*

- (i) $\mathbb{Z} \wr \mathbb{Z}$;
- (ii) $C_p \wr \mathbb{Z}$ for some prime p ;
- (iii) A proper Gc-group.

Proof. By replacing G by a finite index subgroup if necessary, we may assume that G has a normal abelian subgroup N such that G/N is free abelian. The result follows from Proposition 3.1 and Lemma 3.2 if $\langle a, b \rangle$ is virtually abelian for all $a \in G, b \in N$, and from Proposition 2.8 otherwise. \square

4 Torsion-free soluble groups

Proposition 4.1. *Let G be a torsion-free group and A an abelian subgroup of finite index in G . Then $C_G(A)$ is abelian.*

Proof. Since A is abelian, $A \leq Z(C_G(A))$, and hence $Z(C_G(A))$ has finite index in $C_G(A)$. It follows from Proposition 1.3 that $C_G(A)'$ is finite and thus in fact trivial, since G is torsion-free. \square

Proposition 4.2. *Let G be a countable torsion-free locally virtually abelian group and suppose that G does not contain free abelian subgroups of arbitrarily high finite rank. Then G has a finite index normal abelian subgroup which is characteristic.*

Proof. Let $k \in \mathbb{N}$ be maximal such that G has a free abelian subgroup of rank k . We can choose elements $g_i \in G$ for $i \in \mathbb{N}$ such that $G = \langle g_i \mid i \in \mathbb{N} \rangle$ and $\langle g_1, \dots, g_k \rangle \cong \mathbb{Z}^k$.

For $i \geq k$, let $H_i = \langle g_1, \dots, g_i \rangle$ and let A_i be a subgroup of H_i that is maximal subject to having finite index in H_i and being isomorphic to \mathbb{Z}^k . By Proposition 4.1, $C_{H_i}(A_i)$ is abelian, and so $C_{H_i}(A_i) = A_i$.

Suppose B_i is another rank k abelian subgroup of H_i . Then $A_i \cap B_i$ has finite index in B_i , so has rank k , so has finite index in A_i and hence in H_i . Proposition 4.1 implies that $C_{H_i}(A_i \cap B_i)$ is abelian. But $A_i, B_i \leq C_{H_i}(A_i \cap B_i)$, and so by the maximality of A_i , we have $C_{H_i}(A_i \cap B_i) = A_i$, so $B_i \leq A_i$. This proves that A_i is the *unique* maximal abelian subgroup of rank k in H_i . It follows that $A_i \trianglelefteq H_i$ and $A_i \leq A_{i+1}$ for all $i \geq k$. Since A_i is self-centralising, H_i/A_i is isomorphic to a subgroup of $\text{Aut}(A_i) \cong \text{GL}(k, \mathbb{Z})$.

For $i \geq k$, let ι_i be the natural embedding of H_i in H_{i+1} , ϕ_i the natural homomorphism from H_i to H_i/A_i and let $\psi_i : H_i/A_i \rightarrow H_{i+1}/A_{i+1}$ be given by $xA_i \mapsto xA_{i+1}$. Then $\psi_i \circ \phi_i = \phi_{i+1} \circ \iota_i$ and so $\ker \psi_i \circ \phi_i = \ker \phi_{i+1} \circ \iota_i$. Now

$$\ker \phi_{i+1} \circ \iota_i = H_i \cap \ker \phi_{i+1} = H_i \cap A_{i+1} = A_i,$$

where the last equality follows from the maximality of A_i . Thus $\ker \psi_i \circ \phi_i = A_i$. Since already $\ker \phi_i = A_i$, this implies that ψ_i is injective.

Hence, since each H_i/A_i is isomorphic to a finite subgroup of $\text{GL}(m, \mathbb{Z})$, Proposition 1.4 implies that there exists $m \in \mathbb{N}$ such that H_i/A_i is isomorphic to H_m/A_m for all $i \geq m$.

We have a commutative diagram:

$$\begin{array}{ccccccccc} H_m & \xrightarrow{\iota_m} & H_{m+1} & \xrightarrow{\iota_{m+1}} & \dots & \xrightarrow{\iota_{i-1}} & H_i & \xrightarrow{\iota_i} & H_{i+1} \\ \downarrow \phi_m & & \downarrow \phi_{m+1} & & \downarrow & & \downarrow \phi_i & & \downarrow \phi_{i+1} \\ H_m/A_m & \xrightarrow{\psi_m} & H_{m+1}/A_{m+1} & \xrightarrow{\psi_{m+1}} & \dots & \xrightarrow{\psi_{i-1}} & H_i/A_i & \xrightarrow{\psi_i} & H_{i+1}/A_{i+1} \end{array}$$

For $i \geq m$, define $\theta_i : H_i \rightarrow H_m/A_m$ by $\theta_m = \phi_m$ and for $i > m$

$$\theta_i = \psi_m^{-1} \circ \psi_{m+1}^{-1} \dots \circ \psi_{i-1}^{-1} \circ \phi_i.$$

Since each ϕ_i is an epimorphism and all the ψ_j^{-1} ($j \geq m$) are isomorphisms, each θ_i is an epimorphism. Moreover, we have $\theta_{i+1} \circ \iota_i = \theta_i$; that is, $\theta_{i+1}(x) = \theta_i(x)$ for $x \in H_i$.

Define $\theta : G \rightarrow H_m/A_m$ by $\theta(x) = \theta_i(x)$ where $i = \min\{j \geq m \mid x \in H_j\}$. Then θ is a well defined epimorphism with kernel $A := \bigcup_{i \in \mathbb{N}} A_i$. So A is abelian of finite index in G .

Finally, we show that A is characteristic in G . Let B be an abelian subgroup of G containing \mathbb{Z}^k , and choose h_i such that $B = \langle h_i \mid i \in \mathbb{N} \rangle$, put $B_i = \langle h_1, \dots, h_i \rangle$ for $i \geq k$ and assume that $B_k \cong \mathbb{Z}^k$. Then each B_i is contained in some H_{j_i} , and hence in A_{j_i} . So $B = \bigcup_{i \in \mathbb{N}} B_i \leq \bigcup_{i \in \mathbb{N}} A_{j_i} \leq \bigcup_{i \in \mathbb{N}} A_i = A$. Thus A is the unique maximal abelian subgroup of G containing \mathbb{Z}^k and so is characteristic in G . \square

Lemma 4.3. *Let G be a countable torsion-free locally virtually abelian group and let N be an abelian normal subgroup of G such that $G/N \cong \mathbb{Z}^\infty$. Then G has a subgroup isomorphic to \mathbb{Z}^∞ .*

(We recall that \mathbb{Z}^∞ denotes a free abelian group of countably infinite rank.)

Proof. If N contains \mathbb{Z}^k for all $k \in \mathbb{N}$ then we are done, since N is abelian, so assume t is maximal with $\mathbb{Z}^t \leq N$. Write $G = \bigcup_{i \in \mathbb{N}} G_i$, with each G_i finitely generated (and hence virtually abelian), $G_i < G_{i+1}$ for all $i \in \mathbb{N}$, and each $N_i := G_i \cap N$ isomorphic to \mathbb{Z}^t .

Let $A_i = C_{G_i}(N_i)$ and let B_i be an abelian subgroup of finite index in G_i . Then B_i centralises the finite index subgroup $B_i \cap N_i$ of N_i . It is easily seen that an automorphism of a free abelian group of finite rank that centralises a subgroup of finite index must be trivial, so B_i must centralise N_i . Thus $B_i \leq A_i$ so $|G_i : A_i|$ is finite and, since G_i/A_i is isomorphic to a subgroup of $\text{GL}(t, \mathbb{Z})$, $|G_i : A_i|$ is bounded. Hence $A_i \leq A_{i+1}$ for all sufficiently large i .

Also A_i must be abelian, since otherwise there exist $x, y \in A_i$ with $[x, y] = z \in N_i \setminus \{1\}$, but then $\langle x, y \rangle$ is not virtually abelian, contrary to hypothesis. Thus $\bigcup_{i \in \mathbb{N}} A_i$ is abelian and contains finitely generated subgroups of arbitrarily high rank, so it must contain \mathbb{Z}^∞ . \square

Lemma 4.4. *Let G be a group with a normal torsion subgroup T such that $G/T \cong \mathbb{Z}^\infty$. If G is locally virtually abelian, then G has a subgroup isomorphic to \mathbb{Z}^∞ .*

Proof. Let g_1, g_2, \dots be an irredundant generating set for G modulo T . Then for all $i \in \mathbb{N}$, $G_i := \langle g_1, \dots, g_i \rangle$ is virtually abelian, so $T_i := G_i \cap T$ is finite. Also $G_i/T_i \cong \mathbb{Z}^i$.

We shall construct a chain of subgroups $H_1 < H_2 < \dots < H_i < \dots$ such that $H_i \leq G_i$ with $|G_i : H_i|$ finite and $H_i \cong \mathbb{Z}^i$. Suppose that we have defined H_1, \dots, H_i for some $i \geq 0$. Then $g_{i+1} \in G_{i+1}$ centralises $H_i T_{i+1}/T_{i+1}$, while some power g_{i+1}^k of g_{i+1} centralises T_{i+1} . If $l = k|T_{i+1}|$, then g_{i+1}^l centralises H_i . Thus $H_{i+1} := \langle H_i, g_{i+1}^l \rangle$, has finite index in G_{i+1} and is isomorphic to \mathbb{Z}^{i+1} . So we can construct the chain as claimed, and $\bigcup_{i \in \mathbb{N}} H_i \cong \mathbb{Z}^\infty$. \square

Proposition 4.5. *Let G be a countable soluble locally virtually abelian group, and suppose that G has subgroups isomorphic to \mathbb{Z}^k for all $k \in \mathbb{N}$. Then G has a subgroup isomorphic to \mathbb{Z}^∞ .*

Proof. The proof is by induction on the derived length of G . The statement is true for abelian groups. Now suppose it is true for groups of derived length at most n , and let G have derived length $n + 1$, with all finitely generated subgroups of G virtually abelian and $\mathbb{Z}^k \leq G$ for all $k \in \mathbb{N}$.

Let $N = G^{(n)}$. If $\mathbb{Z}^k \leq N$ for all $k \in \mathbb{N}$ then we are done, since N is abelian. So suppose there exists $t \in \mathbb{N}$ maximal such that $\mathbb{Z}^t \in N$. This means that G/N must contain \mathbb{Z}^k for all $k \in \mathbb{N}$. By the induction hypothesis, G/N has a subgroup isomorphic to \mathbb{Z}^∞ . We can assume without loss of generality that $G/N \cong \mathbb{Z}^\infty$.

Let T be the torsion subgroup of N . Then N/T is torsion-free, and so G/T and N/T satisfy the hypothesis of Lemma 4.3. Thus we can replace G by a subgroup and assume $G/T \cong \mathbb{Z}^\infty$. Then by Lemma 4.4, G has a subgroup isomorphic to \mathbb{Z}^∞ . \square

We can now prove our main result for torsion-free finitely generated soluble groups.

Theorem 4.6. *Let G be a finitely generated torsion-free soluble group. Then one of the following holds:*

- (i) G is virtually abelian;
- (ii) G' has a subgroup isomorphic to \mathbb{Z}^∞ ;
- (iii) G has a subgroup isomorphic to a proper Gc -group.

Proof. Let n be the derived length of G . The proof is by induction on n . The statement is vacuously true for $n = 1$, and true for $n = 2$ (G metabelian) by

Theorem 3.3. So assume that $n \geq 2$ and that the result is true for groups with derived length less than n . By applying the result inductively to G' , we may assume that all finitely generated subgroups of G' are virtually abelian.

We may assume that G' does not contain \mathbb{Z}^k for all $k \in \mathbb{N}$, since otherwise we are done by Proposition 4.5. Since G' is countable, Proposition 4.2 shows that G' has a finite index normal abelian subgroup A which is characteristic in G' and hence $A \triangleleft G$. Now G/A is finitely generated with $(G/A)/(G'/A) \cong G/G'$ abelian, while G'/A is finite, so G/A is virtually abelian. But since A is an abelian normal subgroup of G , this implies that G itself is virtually metabelian, and the result follows from Theorem 3.3. \square

5 The main result

We require just one further lemma before proceeding to prove our result for finitely generated soluble groups in general.

Lemma 5.1. *Let G be a group with normal subgroup N and let H be a subgroup of $(G/N)'$. Then G' has a subgroup K such that $K/(N \cap K) \cong H$.*

Proof. This follows from $(G/N)' = G'N/N \cong G'/(G' \cap N)$. \square

Theorem 5.2. *Let G be a finitely generated soluble group. Then at least one of the following holds:*

- (i) G is virtually abelian;
- (ii) G' has a subgroup isomorphic to \mathbb{Z}^∞ ;
- (iii) G has a subgroup isomorphic to a proper Gc -group;
- (iv) G has a finitely generated subgroup H with an infinite normal torsion subgroup U , such that H/U is either free abelian or a proper Gc -group.

Proof. The proof is by induction on the derived length of G . We have already proved it for groups of derived length at most two and when G is torsion-free (Theorems 3.3 and 4.6). Suppose the result holds for groups of derived length at most $n \geq 2$, and let G have derived length $n+1$. So the result holds for G/N , where $N = G^{(n)}$, and we may assume that all finitely generated

subgroups of G' are virtually abelian. Let T be the torsion subgroup of N . For the remainder of the proof, we shall only use the fact that N is abelian, and not that $N = G^{(n)}$. This enables us to replace G by a subgroup when convenient.

If (i) holds for G/N , then G is virtually of derived length 2, and the result follows from Theorem 3.3.

Suppose that (ii) holds for G/N ; that is, $(G/N)'$ contains a subgroup isomorphic to \mathbb{Z}^∞ . Applying Lemma 5.1, let K be a subgroup of G' such that $K/(N \cap K) \cong \mathbb{Z}^\infty$. Then $K/(T \cap K)$ is torsion-free and so, by Lemma 4.3, there is also a \mathbb{Z}^∞ subgroup in K/T . By Lemma 4.4, this implies that K (and hence G') has a subgroup isomorphic to \mathbb{Z}^∞ and so (ii) holds for G .

Suppose that (iii) holds for G/N . Then, by replacing G by a subgroup, we may assume that G/N is isomorphic to a proper Gc-group. So G/T is torsion-free and not virtually abelian, and hence by Theorem 4.6 one of (ii), (iii) holds for G/T . If (ii) holds for G/T , then (ii) holds for G by Lemma 4.4. So suppose that (iii) holds for G/T . By passing to a subgroup we may assume that G/T is a proper Gc-group $\langle aT, bT \rangle$. If T is finite, then the result follows from Lemma 2.6, whereas if T is infinite, then G satisfies (iv).

Finally suppose that G/N satisfies (iv). We may assume that G/N has an infinite normal torsion subgroup U/N , where G/U is either torsion-free abelian or a proper Gc-group. If $N = T$ then G satisfies (iv), so suppose not. Since N is abelian, if N/T contains subgroups isomorphic to \mathbb{Z}^k for all $k \in \mathbb{N}$, then N contains a \mathbb{Z}^∞ subgroup, so we may assume that there is a largest $k > 0$ such that $\mathbb{Z}^k \leq N/T$.

Suppose that U is generated by elements g_i ($i \in \mathbb{N}$) and, for $i \in \mathbb{N}$, define $U_i = \langle g_1, \dots, g_i \rangle$ and $N_i = U_i \cap N$. A finitely generated soluble torsion group is finite, so the groups $U_i/N_i \cong U_i N/N$ are all finite with $\bigcup_{i \in \mathbb{N}} U_i N/N = U/N$. The groups $N_i T/T$ are all finitely generated free abelian groups with $\bigcup_{i \in \mathbb{N}} N_i T/T = N/T$ and so for all sufficiently large i we have $N_i T/T \cong \mathbb{Z}^k$. For $i \in \mathbb{N}$, define the subgroup C_i by $U_i \cap T \leq C_i \leq U_i$ and $C_i T/T = C_{U_i T/T}(N_i T/T)$. By Proposition 1.4, there is a constant L such that $|U_i : C_i| \leq L$ for all i . Since $N_i \leq N_{i+1}$ for each i , we have $C_{i+1} \cap U_i \leq C_i$, and hence $|U_i : C_i| \leq |U_{i+1} : C_{i+1}|$. So, for sufficiently large i , we have $N_i T/T \cong \mathbb{Z}^k$ and $|U_i : C_i| = L$ for some constant L , and so we must also have $C_{i+1} \cap U_i = C_i$. So $C := \bigcup_{i \in \mathbb{N}} C_i$ satisfies $T < C$ and $C/T = C_{U/T}(N/T)$, and $|U : C| = L$.

Now $C \triangleleft G$. If G/U is torsion-free abelian, then G/C is virtually abelian, whereas if G/U is a proper Gc-group, then by Lemma 2.6 G/C is virtually a proper Gc-group. In either case, we can replace G by a finite index subgroup containing C such that $G \cap U = C$, and thereby assume that $U = C$ and hence $U_i = C_i$ for all i . So $N_i T/T \leq Z(U_i T/T)$ for each i , and by Proposition 1.3 $U'_i T/T$ is finite. Since $U' = \bigcup_{i \in \mathbb{N}} U'_i$, $U' T/T$ and hence also $U' T$ is a torsion group. Let $V/U' T$ be the torsion subgroup of the abelian group $U/U' T$. So U/V and hence also G/V is torsion-free, and hence by Theorem 4.6 one of (i), (ii), (iii) holds for G/V .

If G/V is virtually abelian then, since G is not virtually abelian, V must be infinite, and so G satisfies (iv). If G/V satisfies (ii) then so does G by Lemmas 5.1 and 4.4. If (iii) holds for G/V then, by replacing G by a subgroup, we may assume that G/V is isomorphic to a proper Gc-group. Then the result follows by Lemma 2.6 if V is finite, and G satisfies (iv) if V is infinite. \square

References

- [1] D.F. HOLT, S. REES, C.E. RÖVER and R.M. THOMAS. ‘Groups with context-free co-word problem’. *J. London Math. Soc.* (2) 71 (2005) 643–657
- [2] J.C. LENNOX and D.J.S. ROBINSON. *The Theory of Infinite Soluble Groups*. Clarendon Press, Oxford 2004.
- [3] D.J.S. ROBINSON. *A course in the theory of groups*. (Springer-Verlag, New York 1982).
- [4] D.N. ROCKMORE and K.S. TAN. ‘A note on the order of finite subgroups of $GL(n, \mathbb{Z})$ ’. *Arch. Math.*, Vol. 64 (1995), 283–288.
- [5] J.J. ROTMAN. *Advanced Modern Algebra*. Prentice Hall 2002.