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19 EXPECTED UTILITY AS A TOOL IN NON-COOPERATIVE GAME THEORY

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Abstract: This sequel to previous chapters on objective and subjective expected utility reviews conditions for players in a non-cooperative game to be Bayesian rational — i.e., to choose a strategy maximizing the expectation of each von Neumann–Morgenstern utility function in a unique cardinal equivalence class. In classical Nash equilibrium theory, players' mixed strategies involve objective probabilities. In the more recent rationalizability approach pioneered by Bernheim and Pearce, players' possibly inconsistent beliefs about other players' choices are described by unique subjective probabilities. So are their beliefs about other players' beliefs, etc. Trembles, together with various notions of perfection and properness, are seen as motivated by the need to exclude zero probabilities from players' decision trees. The work summarized here, however, leaves several foundational issues unsatisfactorily resolved.

1 Introduction and Outline

1.1 Background

The theory of equilibrium in general non-cooperative games was initially developed for two-person "zero-sum" games by Borel (1921, 1924), von Neumann (1928) and von Neumann and Morgenstern (1944).[1] It was then extended to general n-person games with finite strategy sets by Nash (1950, 1951). This classical theory allows different players to choose stochastically independent "mixed" strategies in the form of objective probability distributions. In this connection, Aumann (1987b, p. 466) gives a very clear and concise account of the role played by objectively expected utility theory in the classical theory of games with independent mixed strategies. A recent extension of classical non-cooperative game theory due to Aumann (1974, 1987a) allows different players to correlate their mixed strategies through some form of correlation device.[2]

[1]See Fréchet (1953) and von Neumann (1953) for contrasting views of Borel's contribution. For a recent brief discussion of some aspects of the history of game theory, see Arrow (2003).
[2]In fact, Aumann (1987a) discusses correlation of subjective rather than objective probabilities. However, these subjective probabilities arise from beliefs that are assumed to be

Von Neumann and Morgenstern also carried out some pioneering work on the decision theoretic foundations of game theory with objective probabilities. Since then, almost the only other published work on this topic has been by Fishburn and Roberts (1978) and by Fishburn (1980, 1982, chs. 7–8). However, this later work was especially concerned with preferences over the restricted domain of product lotteries that result when different players in one fixed game adopt independent mixed strategies.

1.2 Normative Theory and Consequentialism

This survey revisits the decision- and utility-theoretic foundations of non-cooperative game theory. Indeed, it emphasizes the *normative* theory of how players' in a game *should* choose their strategies. Most purportedly descriptive models in economics and social science that use game theory tend to follow this approach, even though normative theory may describe accurately what happens only when a game involves "expert" players who are either well-versed in game theory themselves, or else heed the recommendations of expert consultants who are advising them how to play. Only briefly in Section 6 is there some discussion of quantal responses, which seem likely to offer a more fruitful approach to empirical descriptive modelling in many more realistic settings.

The main question addressed in this chapter will be how the consequentialist theory presented in Chapters 5 and 6 in Volume I of this *Handbook* applies in the context of non-cooperative games. That theory, it may be worth recalling, requires behaviour in each decision tree to generate a consequence choice set that depends only on the feasible set of all possible consequences which can emerge from decisions in that tree.[3] Partly for this reason, the focus will be on games in normal or strategic form; recent work on extensive form concepts will be largely neglected. This will help to prevent the chapter from becoming excessively long. But more pertinently, the basic issues concerning how to apply utility theory as a tool in non-cooperative games all arise in normal form games. Its application to extensive form games is justified in the same way, and to the same extent, as its application to normal form games.

1.3 Normal Form Invariance and Equilibrium

Following the precedent of von Neumann and Morgenstern (1944, 1953), the standard definition of a game includes a payoff function for each player. Yet the existence of such a function ought really to be derived from more primitive axioms. Though von Neumann and Morgenstern did set out to do precisely

common to all individuals. It is hard to see how such common beliefs could come about except if they were really objective probabilities.

[3] Since these chapters were finished, Peter Wakker has kindly pointed out the significance of work by Burks (1977) — see especially Wakker (1998). In particular, Burks also uses decision trees to offer somewhat similar justifications for both the independence axiom and the sure-thing principle. However, the consequentialist approach seems more integrated, especially as essentially the same hypothesis also justifies the existence of a preference ordering.

this, nevertheless their definition of a game is not properly founded on what has since become orthodox single-person decision theory. The fairly obvious remedy proposed in Section 4 is to replace each player's payoff function by a unique cardinal equivalence class of von Neumann–Morgenstern utility functions (NMUFs), each of which is acceptable as a payoff function.

The first part of this chapter concentrates on equilibrium theory. This explores the implications of assuming that players' beliefs can be described by an objective probability distribution over other players' strategies. In fact, the chapter develops an integrated approach describing each person's best response behaviour in all single-person decision problems and in all n-person non-cooperative games. This approach is based on the consequentialist normal form invariance hypothesis described in Section 2, which adapts a similar hypothesis due to von Neumann and Morgenstern themselves. It also generalizes the consequentialist hypothesis for single-person decision theory mentioned in Section 1.2. As in Chapter 5, consequentialist normal form invariance, when combined with the hypotheses of an unrestricted domain, dynamic consistency in continuation subgames, and continuity w.r.t. objective probabilities, implies that each player's best responses to a given joint probability distribution over the other players' strategies are those which maximize the expected value of each NMUF in a unique cardinal equivalence class.

Thereafter, Section 3 discusses some key properties of best responses. Both Nash and correlated equilibrium are briefly reviewed in Section 4, along with Harsanyi's (1967–8) theory of equilibrium in games of incomplete information. Equilibrium requires in particular that all players behave as if they had assigned mutually consistent probabilities to other players' strategies, treating those probabilities as effectively objective, and then choosing expected utility maximizing responses with probability 1.

1.4 The Zero Probability Problem

Chapter 5 demonstrated how, in single-person decision theory, the consequentialist hypothesis implies that all probabilities must be positive, in order to avoid the trivial implication that there is universal indifference — i.e., all lotteries are indifferent. But equilibrium requires that players' inferior responses must be given zero probability. These two requirements are incompatible except in the trivial case when there are no inferior responses because all strategies are indifferent. Alternatively, the game must be regarded as equivalent to one in which all inferior strategies have been deleted from the relevant player's strategy set. But then there may be equilibria which depend on some players using incredible threats to coerce other players into choosing their equilibrium strategies, even though everybody knows that such threats would never be carried out in practice. More precisely, the criterion of subgame perfection due to Selten (1965, 1973) becomes entirely irrelevant. One way out of this difficulty is to use Selten's (1975) own later notion of trembling-hand perfect equilibrium, which Myerson (1978) refined further to proper equilibrium. These two refinements of Nash equilibria are the subject of Section 5.

The trembles discussed in Section 5 are rather *ad hoc* departures from the standard equilibrium idea that players choose best responses with probability 1. Section 6 considers an alternative approach, in which all choice is stochastic, so every possible strategy is chosen with a specified positive probability. Such random choice is the topic of Fishburn's Chapter 7 on stochastic utililty. McKelvey and Palfrey (1995) in particular have considered stochastic strategy choice in non-cooperative game theory. However, as shown in Section 6, for stochastic choice under risk, the consequentialist hypotheses imply the trivial case in which all possible choices occurring with positive probability are equally likely.

1.5 Subjective Probabilities and Rationalizability

As is briefly discussed in an assessment of equilibrium theory in Section 7, it seems therefore that, except in trivial cases, having "objective" or generally agreed probabilities in a non-cooperative game is incompatible with the consequentialist axioms. More recently, following prominent works such as Bernheim (1984), Pearce (1984), and Tan and Werlang (1988), many game theorists have used subjective probabilities to describe each player's beliefs about the other players' strategies. These allow for the possibility that different players' beliefs may not coincide. As in equilibrium theory, players are assumed to choose strategies in order to maximize their respective subjectively expected utilities.[4] In this game theoretic context, it has often been claimed that the existence of subjective probabilities and the subjective expected utility (SEU) hypothesis are justified by Savage's axioms. One problem with this was discussed in Chapter 6: from the consequentialist perspective, Savage's axiom system may be harder to justify than the alternative system due to Anscombe and Aumann (1963). Much more serious, however, is the concern whether *any* axiom system which was originally intended for single-person decision problems can be suitably adapted so that it applies in the more general context of strategic behaviour in non-cooperative games.

More specifically, there could be an important difference in extensive form games between natural nodes at which nature moves exogenously, as opposed to players' information sets where moves are determined endogenously by maximizing the relevant player's expected utility. Recently, this has led Sugden (1991) and Mariotti (1996) in particular to question whether the SEU model applies at all to n-person games.[5] Nevertheless, Section 8 presents one possible justification, based on a construction due to Battigalli, for using the SEU model in this context. The approach is rather different from that of Börgers (1993), and entirely different from Nau and McCardle (1990, 1991).

Using the SEU framework, Section 9 reviews the concept of rationalizability due to Bernheim (1984) and Pearce (1984), but extended in the usual way when

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[4]See also Harsanyi (1966, 1967–8, 1977b, 1980, 1982b, 1983), and note that his debate with Kadane and Larkey (1982) concerned a different issue.

[5]A somewhat different criticism of the SEU model, especially in games, arises in causal decision theory, which is the topic of Chapter 13 in Volume I by Gibbard and Joyce.

there are more than two players, so that each player is allowed to have correlated beliefs regarding other players' pure strategies. Then Section 10 considers the related construction of infinite hierarchies of beliefs, which automatically include players' beliefs about other players' hierarchies of beliefs.

Section 11 reverts to the zero probability problem discussed in Section 5, this time in the context of rationalizability. It begins with a very brief summary of some recent work concerning infinitesimal probabilities. Then it reviews one concept of perfect rationalizability due to Bernheim (1984, p. 1021), but extended to allow correlated beliefs when there are more than two players, as in Section 9. This is followed by a discussion of a more recent concept which Herings and Vannetelbosch (1999, 2000) call "weakly perfect rationalizability". This concept generates the same strategy sets as the "Dekel–Fudenberg procedure" of first eliminating all weakly dominated strategies, then iteratively deleting all strictly dominated strategies from those that remain. The section concludes by considering the stronger concept of proper rationalizability due to Schuhmacher (1999) — see also Asheim (2002).

1.6 Rationalizable Dominance

Next, following a suggestion due to Farquharson (1969, Appendix II), Section 12 steps back entirely from the expected utility or any other probabilistic framework. Instead, it introduces an apparently novel "rationalizable dominance" relation over pure strategies. This binary relation depends only on players' preference orderings over pure strategy profiles and their consequences, implying that players' "payoffs" or utility functions become ordinal rather than cardinal.[6]

Finally, Section 13 provides a brief concluding assessment. This raises the whole issue of whether non-cooperative game theory can be founded more securely on decision-theoretic concepts, and the extent to which utility theory will continue to play the significant role in non-cooperative game theory which it has up to now.

The chapter presumes some basic familiarity with game theory at the level of a standard microeconomic textbook for graduate students such as Mas-Colell *et al.* (1995). Additional reading concerning particular topics in the specialist texts by Osborne and Rubinstein (1994) and by Fudenberg and Tirole (1991) will be suggested at appropriate points in the chapter.

[6] Farquharson's (1969) notion of iterative deletion of strategies that are weakly dominated by other pure strategies shares this ordinality property, So does the "pure strategy dominance" criterion due to Börgers (1993). Neither is based on binary dominance relations, however.

2 Normal Form Invariance

2.1 Games in Normal Form

A *game in normal form* is ordinarily defined as a collection

$$G = \langle I, S^I, v^I \rangle \tag{1}$$

This list begins with a non-empty finite set I of *players*.[7] Each player $i \in I$ has a non-empty *strategy set* S_i, also assumed to be finite in order to avoid technical complications. The Cartesian product of *strategy profiles* is $S^I := \prod_{i \in I} S_i$. The last item in the collection is the list $v^I = \langle v_i \rangle_{i \in I}$ of players' *payoff functions*, with $v_i : S^I \to \mathbb{R}$ for all $i \in I$.

2.2 Consequentialist Game Forms

As explained in the introduction, directly assuming that a payoff function exists fails to place game theory on secure decision-theoretic foundations. To deal with this issue, instead of (1), define a *consequentialist game form*[8] as a collection

$$\Gamma = \langle I, S^I, Y^I, \psi \rangle \tag{2}$$

Here the set I of players and the set S^I of strategy profiles are exactly the same as in the definition (1) of a normal form game. But now $Y^I = \prod_{i \in I} Y_i$ is the Cartesian product of individual consequence domains Y_i, one for each player $i \in I$. Each member $y^I \in Y^I$ is a *consequence profile*. This formulation with individual consequence domains is chosen to allow independent variations in the consequences faced by any one player in the game, which will be important in the construction used in Section 8 below.

Finally, there is an *outcome function* ψ determining the random consequence profile $\psi(s^I) \in \Delta(Y^I)$, with objective probabilities $\psi(y^I; s^I) \in [0, 1]$ for each $y^I \in Y^I$, as a function of the strategy profile $s^I \in S^I$. Here, following the notation of Chapters 5 and 6, $\Delta(Y^I)$ is used to denote the set of *simple lotteries* having finite support on Y^I. Once again, finiteness of the support is merely a simplifying restriction. Then the mapping $\psi : S^I \to \Delta(Y^I)$ in (2) replaces the profile $v^I = \langle v_i \rangle_{i \in I}$ of different players' payoff functions $v_i : S^I \to \mathbb{R}$ in (1).

Of course, (2) collapses to (1) if one takes the case when each outcome $\psi(s^I)$ is a degenerate lottery on Y^I, then puts $Y_i = \mathbb{R}$ and defines each $v_i : S^I \to \mathbb{R}$ so that $\psi(v^I(s^I); s^I) = 1$ for all $s^I \in S^I$. In this case the consequences become real numbers or *rewards*, with each player's utility equal to their respective reward.

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[7] Extensions to games with an infinite set of players, and especially to games with a continuum of players, have appeared prominently in the literature. They raise technical issues of the kind discussed at length by Khan and Sun (2002). For simplicity, and to retain the main focus on utility theory, this chapter ignores this important topic.

[8] The term "game form" is taken deliberately from the literature on implementation and mechanism design — see especially Gibbard (1973).

2.3 Games in Extensive Form

So far the discussion has been limited to games in normal or strategic form, in which each player is regarded as making just one choice of strategy for the entire game. Yet general multi-person games are in *extensive form*. In effect, they are multi-person versions and extensions of the decision trees that were explored in Chapters 5 and 6.

Apart from terminal nodes $x \in X$ and chance nodes belonging to N^0, the set of all remaining nodes $N \setminus (X \cup N^0)$ in the extensive form game tree is partitioned into pairwise disjoint sets N^i, one for each player $i \in I$. The set N^i is defined as consisting of exactly those nodes at which player i is required to make a move in the game. No player other than i is able to move at any node belonging to an information set $n \in N^i$.[9] For some players, N^i could be empty because there are no circumstances in which player i is required to make a move.

Player i, however, may not be able to distinguish between all the different nodes of N^i. Thus, we assume that each player i's set N^i is partitioned into a family $\mathcal{H}_i$ of pairwise disjoint *information sets* $H \subset N^i$.

For any $H \in \mathcal{H}_i$, player i must face the same set of moves at each node of H. Otherwise player i would be able to distinguish between some of the different nodes of H because the set of moves is different, contradicting the definition of an information set. For a more formal discussion, see Osborne and Rubinstein (1994, ch. 11). Given any $i \in I$ and any $H \in \mathcal{H}_i$, let $M_i(H)$ denote the (non-empty) set of moves available to player i at H. For each $m \in M_i(H)$ and each node $n \in H$, there must be a unique node $n_{+1}(n, m)$ which is reached by the move m.

2.4 Perfect Recall

In the tree describing the extensive form of a game, consider any two paths denoted by $p = (n_0, n_1, n_2, \ldots, n_k)$ and by $p' = (n'_0, n'_1, n'_2, \ldots, n'_{k'})$, where $n_0 = n'_0$ is the common initial node, and $n_k, n'_{k'}$ denote the two terminal nodes. Suppose that the particular node n_q of path p belongs to an information set H of player i, but path p' does not intersect H at all. At node n_q, or indeed any other node of H, player i knows that p is still possible, but that p' is impossible. Thus, player i can distinguish between the paths p and p' at node n_r — and indeed at any other node of H, where the information must be the same as at n_q. For player i to have perfect recall, it is required that whenever p intersects any other information set H^* of player i at some node n_r that succeeds n_q because $r > q$, then p' cannot intersect H^*. The reason is that player i should be able to remember that p' is impossible after reaching information set H. This is one requirement of perfect recall — being able to distinguish at later information

[9]This requirement is without loss of generality in finite player games, in which one can add information sets for different players as necessary to make it true. But it does lose generality in games with a continuum of players — see especially Dubey and Kaneko (1984, 1985).

sets whatever pairs of paths could be distinguished at earlier information sets. An important implication is that no path through the tree can intersect any information set more than once. Otherwise, if a path p were to intersect some player i's information set $H \in \mathcal{H}_i$ at node n_q but then also at a subsequent node n_r, it can only be because player i at node n_r has forgotten about having already encountered this information set.

In addition, all players should be able to recall the moves that they themselves made earlier in the game. Indeed, suppose that $m_1, m_2 \in M_i(H)$ are two different moves available to player i at information set $H \in \mathcal{H}_i$. Let $P(m_1)$ and $P(m_2)$ denote the resulting sets of paths that pass through H and then are still possible after i has made the respective moves m_1 and m_2. Suppose that node n^* of the extensive game tree belongs to a different information set $H^* \in \mathcal{H}_i$ for player i, as well as to some path $p \in P(m_1)$. Suppose too that p, which must intersect H, does so at a node n which precedes n^*. Then perfect recall requires that the information set H^* must be disjoint from $P(m_2)$. This is because, whenever any path $p \in P(m_1)$ intersects one of player i's later information sets such as H^*, player i must realize that the earlier move m_1 at information set H makes any path in $P(m_2)$ impossible.

2.5 The Agent Normal Form

For some purposes, especially in connection with subgame perfection and trembling hand perfection as considered in Section 5, it is important to restrict attention to extensive form games in which each player has only one information set. In fact, this can be made true for any game in extensive form, simply by replacing each player $i \in I$ with a team of new players or "agents" labelled $H \in \mathcal{H}_i$, one for each of i's information sets. So each player $H \in \mathcal{H} := \cup_{i \in I} \mathcal{H}_i$ in the modified game has exactly one information set, by construction. Some players $i \in I$ may not have any agents at all, if they happen not to have any information sets in the original game.

This procedure of including one player for each information set transforms the consequentialist game form Γ of (2) into the *agent normal form*

$$\tilde{\Gamma} = \langle \mathcal{H}, \tilde{S}^{\mathcal{H}}, \tilde{Y}^{\mathcal{H}}, \tilde{\psi} \rangle$$

Here each agent $H \in \mathcal{H}_i$ of player i has strategy set $\tilde{S}_H := M_i(H)$ and consequence domain $\tilde{Y}_H := Y_i$. Then $\tilde{S}^{\mathcal{H}} := \prod_{H \in \mathcal{H}} \tilde{S}_H$ is the new set of strategy profiles. The new consequence mapping $\tilde{\psi} : \tilde{S}^{\mathcal{H}} \to \Delta(\tilde{Y}^{\mathcal{H}})$ is defined by

$$\tilde{\psi}(\tilde{y}^{\mathcal{H}}; \tilde{s}^{\mathcal{H}}) = \begin{cases} \psi(y^I; s^I) & \text{if } \tilde{y}_H = y_i \text{ for all } H \in \mathcal{H}_i \text{ and all } i \in I; \\ 0 & \text{otherwise.} \end{cases}$$

where $s^I \in S^I$ is the unique strategy profile in the original game satisfying $s_i = \langle \tilde{s}_H \rangle_{H \in \mathcal{H}_i}$ for all $i \in I$.

Many games with imperfect recall can be converted into games with perfect recall by means of this powerful device. Provided that each path through the game tree meets each information set at most once, no agent ever has earlier information or an earlier move to remember. Indeed, this condition is clearly necessary and sufficient for there to be perfect recall in the agent normal form.

This conversion from a general game with imperfect recall to an agent normal form with perfect recall seems entirely natural, especially in normative game theory, After all, one could take the view that any player forced to have imperfect recall should really be replaced by a “team” of players with shared objectives, each one of whom has perfect recall, even if they are not always allowed to know what other members of their team knew or how they acted earlier in the game. For example, bridge is a popular and prominent card game between two pairs of players — North and South versus East and West. Ideally all four players should have perfect recall of the bidding and play. Nevertheless, to fit it into their framework of two-person zero-sum games, von Neumann and Morgenstern (p. 53) suggested that the two partnerships North–South and East–West should be viewed as two players. Each member of a partnership is then required to “forget” having seen partner’s hand whenever the rules require that player to make a bid or play a card.

One advantage of considering the agent normal form in bridge is to allow for the possibility that one or more of the four human players may be forgetful.

2.6 The Absent-Minded Driver

Suppose there is at least one path through the game tree which intersects some information set more than once. Then the game does not have perfect recall even in its agent normal form. The “absent-minded driver” example of Piccione and Rubinstein (1997) illustrates this — see Figure 1 for a simplified version. Here the game tree may represent possible routes that the driver can choose. There are two decision nodes n_0 and n_1, both of which belong to the same information set, as indicated by the loop surrounding these two nodes in the figure. Each node presents the driver with a choice between the two strategies labelled a and d. Driving to the desired destination requires choosing a at n_0 followed by d at n_1, thus achieving a payoff of 1. However, being absent-minded, at n_1 the driver cannot remember having already chosen a at n_0. That is why n_0 and n_1 are indistinguishable.

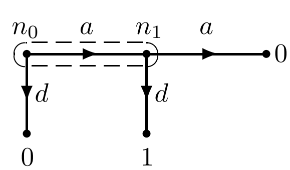

Figure 1 The Absent-Minded Driver

Note that the only pure strategies available are either choosing a at both decision nodes, or d at both decision nodes. Neither pure strategy reaches the desired destination, so each results in a payoff of 0. Yet the highest possible expected payoff is 1/4. This can be achieved by a mixed strategy attaching equal probabilities to a and d at both nodes n_0 and n_1.

Apart from being an example of an agent normal form game with imperfect recall, this example shows how mixed strategies may be optimal for such games. This contrasts with usual decision trees such as in Chapters 5 and 6, in which a mixed strategy is never any better than the best of the pure strategies which occur with positive probability. Indeed, it will turn out that the same is true of best responses in general normal form games with perfect recall.

Agent normal form games with imperfect recall, however, would seem to have limited interest in any normative theory. After all, the absent-minded driver would be well advised to create a simple mnemonic device which enables n_1 to be distinguished from n_0. Even counting with whole hands rather than on one's fingers will work in this simple example!

2.7 Consequentialist Normal Form Invariance

A major assertion by von Neumann and Morgenstern (1953, pp. 79–85) was that no harm is done by normalizing or "simplifying" the extensive form of the game. This normalization involves considering only the corresponding *normal* or *strategic form* in which each player $i \in I$ is represented as making just one choice of strategy specifying what i will do at each possible information set $H \in \mathcal{H}_i$. In other words, each S_i is the Cartesian product set $\prod_{H \in \mathcal{H}_i} M_i(H)$, which is equivalent to the set of mappings m_i with domain $\mathcal{H}_i$ that satisfy the requirement that $m_i(H) \in M_i(H)$ (all $H \in \mathcal{H}_i$).

So, following von Neumann and Morgenstern, the hypothesis of *consequentialist normal form invariance* requires the normal form (2) to be a sufficient description of the game in the following sense: The set of consequences of behaviour by all the players in the game should be invariant to changes in the extensive form of the game which leave this normal form unaffected. Obviously, this is similar to the consequentialist axiom considered in the previous chapters (5 and 6) on expected utility theory.

This concept of invariance is really still too weak, however, because some changes in the consequentialist normal form should leave the set of consequences of behaviour in the game unaffected. After all, in a single-person decision tree or "game against nature" the usual consequentialist axiom requires the set of consequences of behaviour to be determined only by the feasible set of consequences. So decision trees with identical feasible sets are regarded as effectively equivalent. With this analogy in mind, consider the two consequentialist game forms

$$\Gamma = \langle I, S^I, Y^I, \psi \rangle \quad \text{and} \quad \tilde{\Gamma} = \langle I, \tilde{S}^I, Y^I, \tilde{\psi} \rangle$$

with identical sets of players I and identical consequence domains Y^I. These two are said to be *equivalent* iff for each $i \in I$ there exist mappings $\xi_i : S_i \to \tilde{S}_i$ and $\tilde{\xi}_i : \tilde{S}_i \to S_i$ that are both onto and, for all $s^I \in S^I$ and $\tilde{s}^I \in \tilde{S}^I$, the associated products $\xi^I : S^I \to \tilde{S}^I$ and $\tilde{\xi}^I : \tilde{S}^I \to S^I$ satisfy[10]

$$\psi(s^I) = \tilde{\psi}(\xi^I(s^I)) \quad \text{and} \quad \tilde{\psi}(\tilde{s}^I) = \psi(\tilde{\xi}^I(\tilde{s}^I)) \tag{3}$$

That is, every strategy profile $s^I \in S^I$ must have at least one counterpart strategy profile $\tilde{s}^I = \tilde{\xi}^I(\tilde{s}^I) \in \tilde{S}^I$ yielding an identical profile of random consequences, and *vice versa*. Of course, the equivalence is rather obvious when all the mappings $\xi_i : S_i \to \tilde{S}_i$ and $\tilde{\xi}_i : \tilde{S}_i \to S_i$ are one-to-one as well as onto, in which case each $\tilde{\xi}_i$ may as well be the inverse of the corresponding ξ_i. But the definition does not require these mappings to be one-to-one. This is because duplicating one or more players' strategies and their respective consequences does not produce a fundamentally different consequentialist game form, just as it does not in a single-person decision tree.

Indeed, as an illustration, consider the special case when there are two players $I = \{1, 2\}$. Then a consequentialist game form can be represented by a matrix in which the rows correspond to player 1's strategies $s_1 \in S_1$, the columns correspond to player 2's strategies $s_2 \in S_2$, and all the entries are consequence lotteries $\psi(s_1, s_2) \in \Delta(Y_1 \times Y_2)$. In this case, two consequentialist game forms are equivalent if and only if one form can be derived from the other by combining the operations of permuting or duplicating either rows or columns, or of eliminating redundancies among any duplicated rows or columns.

For a general n-person game, given any player $i \in I$, any strategy profile $s^I \in S^I$ can be written in the form (s_i, s_{-i}) where $s_i \in S_i$ and $s_{-i} \in S_{-i} := \prod_{j \in I \setminus \{i\}} S_j$. Then, because each mapping $\xi_i : S_i \to \tilde{S}_i$ $(i \in I)$ must be onto, (3) requires that for all $i \in I$, $s_{-i} \in S_{-i}$, and $\tilde{s}_{-i} \in \tilde{S}_{-i}$ the range sets must satisfy

$$\psi(S_i \times \{s_{-i}\}) = \tilde{\psi}(\tilde{S}_i \times \{\xi_{-i}(s_{-i})\}); \quad \tilde{\psi}(\tilde{S}_i \times \{\tilde{s}_{-i}\}) = \psi(S_i \times \{\tilde{\xi}_{-i}(\tilde{s}_{-i})\})$$

where

$$\xi_{-i}(s_{-i}) = \langle \xi_h(s_h) \rangle_{h \in I \setminus \{i\}} \quad \text{and} \quad \tilde{\xi}_{-i}(\tilde{s}_{-i}) = \langle \tilde{\xi}_h(\tilde{s}_h) \rangle_{h \in I \setminus \{i\}}$$

[10] Defined this way, normal form invariance corresponds to the strategic equivalence concept for extensive form games studied by Elmes and Reny (1994). Unlike earlier concepts due to Thompson (1952) and Dalkey (1953), this limits the domain of allowable extensive form games to those in which all players have perfect recall.

A stronger form of invariance due to Kohlberg and Mertens (1986, p. 1009–10) extends the earlier ideas of Thompson (1952) and Dalkey (1953) in a natural way to allow moves by chance. In addition, their extension would require two games Γ and $\tilde{\Gamma}$ to be equivalent even if (3) is not satisfied for some $s^I \in S^I$ and $\tilde{s}^I \in \tilde{S}^I$, but if instead there exists $i \in I$ such that $\tilde{\psi}(\xi^I(s^I))$ is a probability mixture of the finite collection of lotteries $\psi(s_i', s_{-i})$ $(s_i' \in S_i)$ in $\Delta(Y^S)$. One reason for using the weaker version here is that Kohlberg and Mertens (1986, Ssection 2.8) show how consequentialist reduced normal form invariance, in their stronger sense, is not satisfied by the sequential equilibrium solution set.

For other work on normal form invariance, see Myerson (1986), as well as Mailath, Samuelson and Swinkels (1993).

Hence, corresponding strategy choices by the other players leave each player i facing exactly the same range of possible consequence lotteries as i's strategy s_i varies over S_i.

For most of this chapter, consequentialist normal form invariance will only be invoked in single-person decision trees, as in Chapters 5 and 6. However, the reversal of order axiom (RO) in the latter chapter was given a consequentialist justification based on a three-person game involving both chance and nature, in addition to the decision-maker. This axiom, and the eventual implication that decisions should maximize subjectively expected utility, will be important later on in Section 8.

3 Objective Probabilities and Best Responses

3.1 Expected Utility and Best Responses

For non-cooperative games, it is usual to assume that an equilibrium takes the form of an appropriate commonly known objective probability distribution $\pi \in \Delta(S^I)$ over the space of strategy profiles. In the original case of Nash equilibrium, this distribution takes the form of a product of probability distributions — i.e., $\pi(s^I) = \prod_{i \in I} \mu_i(s_i)$ for each $s^I \in S^I$, as if the players were all playing independent *mixed strategies* $\mu_i \in \Delta(S_i)$. The more general case allows for *correlated equilibrium*, as discussed by Aumann (1974, 1987a) — see also Section 4.4 below.

From the point of view of each player $i \in I$, however, the choice of $s_i \in S_i$ is not determined by an objective probability distribution like μ_i; rather, it is a "free will" choice of whatever pure strategy $s_i \in S_i$ or mixed strategy $\mu_i \in \Delta(S_i)$ seems best for i — see Gilboa (1999). Thus, what player i takes as given in Nash equilibrium is the relevant *marginal distribution* $\pi_i \in \Delta(S_{-i})$ defined by $\pi_i(s_{-i}) = \pi(S_i \times \{s_{-i}\} = \prod_{h \in E \setminus \{i\}} \mu_h(s_h)$ for each $s_{-i} \in S_{-i}$. These represent player i's *expectations* in the form of an appropriate and commonly known objective probability distribution $\pi_i \in \Delta(S_{-i})$ over the set S_{-i} of all possible profiles s_{-i} of the other players' strategies. From player i's point of view, this replaces other players' moves in the game by chance moves. The case of correlated equilibrium is somewhat more complicated — see Section 4.4.

In games with perfect recall, player i's decision problem is accordingly reduced to choosing from the finite feasible set of consequence lotteries generated by each of the pure strategies available to that player. This is a classical single-person decision problem, with uncertainly described by objective probabilities. The usual consequentialist hypotheses of rationality, as discussed in Chapter 5, imply that the player should choose a strategy to maximize objectively expected utility.

There is one important qualification, however. This is the need to exclude zero probability chance moves from decision trees, otherwise Chapter 5 explains how consequentialism would imply universal indifference. So other players' moves which occur with zero probability in equilibrium have to be excluded from the extensive form game. This issue will resurface in Section 5.

For each player $i \in I$, given the expectations $\pi_i \in \Delta(S_{-i})$ and any utility function $v_i : S^I \to \mathbb{R}$ in the unique cardinal equivalence class of NMUFs, there is a unique cardinal equivalence class of (objectively) *expected utility functions*

$$V_i(s_i, \pi_i) := \mathbb{E}_{\pi_i} v_i(s_i, s_{-i}) = \sum_{s_{-i} \in S_{-i}} \pi_i(s_{-i})\, v_i(s_i, s_{-i}) \tag{4}$$

Then rationality requires player i to choose an expected utility maximizing strategy s_i in the *best response* set defined by

$$B_i(\pi_i) := \arg\max_{s_i \in S_i} V_i(s_i, \pi_i) := \{\, s_i^* \in S_i \mid s_i \in S_i \Longrightarrow V_i(s_i^*, \pi_i) \geq V_i(s_i, \pi_i) \,\} \tag{5}$$

Player i's *best response correspondence* $B_i : \Delta(S_{-i}) \twoheadrightarrow S_i$ is the multi-valued mapping defined by $\pi_i \mapsto\!\!\!\rightarrow B_i(\pi_i)$. This correspondence features prominently throughout the rest of the chapter because it underlies many later equilibrium concepts and different forms of rationalizability, etc.

Strategy $s_i \in S_i$ is said to be a *best response* for player i if there exists $\pi_i \in \Delta(S_{-i})$ such that $s_i \in B_i(\pi_i)$. Otherwise s_i is *never a best response.*

Given any finite set Ω of possible states, let $\Delta^0(\Omega)$ denote the subset of $\Delta(\Omega)$ which consists of *interior* probability distributions P satisfying $P(\omega) > 0$ for all $\omega \in \Omega$. Because zero probabilities cause problems, we shall be especially interested in what Pearce (1984) suggests should be called *cautious* best responses. These are defined as strategies such that $s_i \in B_i(\pi_i)$ for some interior probability distribution $\pi_i \in \Delta^0(S_{-i})$. The idea is that it would be incautious to choose a strategy which is a best response only to beliefs that are extreme points of $\Delta(S_{-i})$ satisfying $\pi_i(s_{-i}) = 0$ for some $s_{-i} \in S_{-i}$.

Best responses in games without perfect recall are more complicated. If all players in the agent normal form discussed in Section 2.5 have perfect recall, then every player becomes replaced by a team of agents, each of whom faces a single-person decision problem. In games like the example in Section 2.6 of the absent-minded driver, however, there is imperfect recall even in the agent normal form. Then some mixed strategies may be strictly superior to any pure strategy — a phenomenon that can never arise in one-person decision trees. Nevertheless, one could argue as in Section 2.5 that such absent-mindedness should play no role in a normative theory, because if necessary players should be encouraged to use recording devices in order to facilitate recall.

3.2 Dominance by Pure Strategies

Say that i's strategy $s_i \in S_i$ is *strictly dominated* by the alternative $s_i' \in S_i$ iff

$$v_i(s_i', s_{-i}) > v_i(s_i, s_{-i})$$

for all other players' strategy profiles $s_{-i} \in S_{-i}$.

Similarly, say that i's strategy $s_i \in S_i$ is *weakly dominated* by the alternative $s_i' \in S_i$ iff

$$v_i(s_i', s_{-i}) \geq v_i(s_i, s_{-i})$$

for all other players' strategy profiles $s_{-i} \in S_{-i}$, with strict inequality for at least one such profile.

It is obvious that any strategy that is strictly dominated by another pure strategy is always inferior, so can never be a best response. On the other hand, a strategy $s_i \in S_i$ that is only weakly dominated by an alternative $s_i' \in S_i$ might still be a best response. For s_i to be a best response to some $\pi_i \in \Delta(S_{-i})$, however, it is necessary that $\pi_i(s_{-i}) = 0$ for every other players' strategy profile $s_{-i} \in S_{-i}$ such that $v_i(s_i', s_{-i}) > v_i(s_i, s_{-i})$.

		P_2		
		b_1	b_2	b_3
	a_1	4	1	0
P_1	a_2	0	1	4

Table 1 Example of Dominance Only by Mixed Strategies

(Only Player P_2's Payoffs Are Listed)

A strategy that is not dominated by any alternative may still not be a best response to any $\pi_i \in \Delta(S_{-i})$. Table 1 illustrates an example of this. No pure strategy for player P_2 dominates b_2. Yet b_2 is never a best response for player P_2's because the best response correspondence $B_2(\cdot)$ satisfies

$$b_1 \in B_2(\pi_2) \Longleftrightarrow \pi_2(a_1) \geq \tfrac{1}{2} \quad \text{and} \quad b_3 \in B_2(\pi_2) \Longleftrightarrow \pi_2(a_2) \geq \tfrac{1}{2}$$

3.3 Dominance by Mixed Strategies

The following stronger definition of dominance guarantees that any undominated strategy is a best response for some probability beliefs $\pi_i \in \Delta(S_{-i})$. Since a mixed strategy $\bar{\mu}_i \in \Delta(S_i)$ is a best response only if every pure strategy $s_i \in S_i$ with $\bar{\mu}_i(s_i) > 0$ is a best response, it is enough to consider when pure strategies are undominated.

Say that i's strategy $s_i \in S_i$ is *strictly dominated* if there exists a mixed strategy $\mu_i \in \Delta(S_i)$ such that

$$\sum_{s_i' \in S_i} \mu_i(s_i')\, v_i(s_i', s_{-i}) > v_i(s_i, s_{-i})$$

for all other players' strategy profiles $s_{-i} \in S_{-i}$.

In Table 1, note that player P_2's strategy b_2 is strictly dominated by $\mu_2 \in \Delta(S_2)$ provided that $4\mu_2(b_1) > 1$ and $4\mu_2(b_3) > 1$ — or equivalently, provided that $\min\{\, \mu_2(b_1), \mu_2(b_3)\,\} > 1/4$. Obviously, these inequalities are satisfied in the particular case when $\mu_2(b_1) = \mu_2(b_3) = \frac{1}{2}$.

Similarly, say that i's strategy $s_i \in S_i$ is *weakly dominated* if there exists a mixed strategy $\mu_i \in \Delta(S_i)$ such that

$$\sum_{s'_i \in S_i} \mu_i(s'_i)\, v_i(s'_i, s_{-i}) \geq v_i(s_i, s_{-i})$$

for all other players' strategy profiles $s_{-i} \in S_{-i}$, with strict inequality for at least one such profile.

3.4 *Strategies not Strictly Dominated must be Best Responses*

It is obvious that $s_i \in S_i$ is a best response to some expectations $\pi_i \in \Delta(S_{-i})$ only if s_i is not strictly dominated.

It is also fairly easy to show the converse — see, for example, Osborne and Rubinstein (1994, ch. 4). Alternatively, here is a proof by means of the separating hyperplane theorem.

First, given any $i \in I$ and any $\bar{s}_i \in S_i$, define the two sets

$$\begin{aligned} U_i \quad &:= \quad \{\, \langle u_i(s_{-i}) \rangle_{s_{-i} \in S_{-i}} \in \mathbb{R}^{S_{-i}} \mid \exists \mu_i \in \Delta(S_i) : \\ &\qquad\qquad u_i(s_{-i}) = \sum_{s'_i \in S_i} \mu_i(s'_i)\, v_i(s'_i, s_{-i}) \text{ (all } s_{-i} \in S_{-i}) \,\} \\ W_i \quad &:= \quad \{\, \langle w_i(s_{-i}) \rangle_{s_{-i} \in S_{-i}} \in \mathbb{R}^{S_{-i}} \mid w_i(s_{-i}) > v_i(\bar{s}_i, s_{-i}) \text{ (all } s_{-i} \in S_{-i}) \,\} \end{aligned}$$

These are two non-empty convex sets. If $\bar{s}_i$ is not strictly dominated, then the two sets must be disjoint. Moreover, the vector $\bar{u}_i \in \mathbb{R}^{S_{-i}}$ whose components satisfy $\bar{u}_i(s_{-i}) = v_i(\bar{s}_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ must be a point in the set U_i, as well as a boundary point of W_i. Hence the sets V_i and W_i can be separated by a hyperplane in $\mathbb{R}^{S_{-i}}$ passing through $\bar{u}_i$. That is, there exist real constants $\alpha(s_{-i})$ $(s_{-i} \in S_{-i})$, not all zero, such that

$$\sum_{s_{-i} \in S_{-i}} \alpha(s_{-i})\, u_i(s_{-i}) \leq \sum_{s_{-i} \in S_{-i}} \alpha(s_{-i})\, \bar{u}_i(s_{-i}) \leq \sum_{s_{-i} \in S_{-i}} \alpha(s_{-i})\, w_i(s_{-i}) \quad (6)$$

whenever $\langle u_i(s_{-i}) \rangle_{s_{-i} \in S_{-i}} \in U_i$ and $\langle w_i(s_{-i}) \rangle_{s_{-i} \in S_{-i}} \in W_i$. Then the second inequality in (6) implies that each constant $\alpha(s_{-i})$ $(s_{-i} \in S_{-i})$ must be non-negative. Because not all the constants $\alpha(s_{-i})$ are zero, we can divide by their positive sum in order to normalize and so obtain non-negative probabilities $\pi_i(s_{-i})$ $(s_{-i} \in S_{-i})$ that sum to one. Then the first inequality in (6) implies that

$$\sum_{s_{-i} \in S_{-i}} \pi_i(s_{-i})\, v_i(\bar{s}_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi_i(s_{-i})\, u_i(s_{-i})$$

So the undominated strategy $\bar{s}_i$ is indeed a best response, given the beliefs $\pi_i \in \Delta(S_{-i})$. This confirms that a strategy $\bar{s}_i$ for any player $i \in I$ is not strictly dominated iff it is a best response, given suitable beliefs $\pi_i \in \Delta(S_{-i})$.

3.5 Strategies not Weakly Dominated must be Cautious Best Responses

The results in Section 3.4 for strategies that are not strictly dominated have interesting counterparts for strategies that are not weakly dominated. In the first place, obviously, $s_i \in S_i$ is a (cautious) best response to some interior expectations $\pi_i \in \Delta^0(S_{-i})$ only if s_i is not weakly dominated.

The converse result is really a special case of an important theorem due to Arrow, Barankin and Blackwell (1953) — see also Pearce (1984), van Damme (1987), and Osborne and Rubinstein (1994, p. 64). In the case when the strategy $\bar{s}_i \in S_i$ is not weakly dominated, the set U_i defined in Section 3.4 must be disjoint from the modified set $\tilde{W}_i$ whose members consist of vectors $\langle w_i(s_{-i}) \rangle_{s_{-i} \in S_{-i}}$ in $\mathbb{R}^{S_{-i}}$ satisfying $w_i(s_{-i}) \geq v_i(\bar{s}_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, with strict inequality for at least $s_{-i} \in S_{-i}$. Obviously, U_i and $\tilde{W}_i$ are both convex non-empty subsets of the finite-dimensional space $\mathbb{R}^{S_{-i}}$. The Arrow–Barankin–Blackwell result then guarantees the existence of a separating hyperplane

$$\sum_{s_{-i} \in S_{-i}} \alpha(s_{-i})\, u_i(s_{-i}) = \sum_{s_{-i} \in S_{-i}} \alpha(s_{-i})\, \bar{u}_i(s_{-i})$$

in $\mathbb{R}^{S_{-i}}$ passing through the point $\bar{v}_i$ with the crucial additional property that $\alpha(s_{-i}) > 0$ for all $s_{-i} \in S_{-i}$.

Once again, divide all the constants $\alpha(s_{-i})$ by their sum in order to normalize and so obtain positive probabilities $\pi_i(s_{-i})$ $(s_{-i} \in S_{-i})$ that sum to one. The implication is that $\bar{s}_i$ must be a (cautious) best response to $\pi_i \in \Delta^0(S_{-i})$.

4 Objective Probabilities and Equilibrium

4.1 Two-Person Strictly Competitive Games

Following Borel (1921, 1924), von Neumann (1928) and von Neumann and Morgenstern (1944) focused much of their discussion on the case they called a "zero sum" two-person game, with $I = \{1, 2\}$ and payoff functions satisfying

$$v_1(s_1, s_2) + v_2(s_1, s_2) = 0 \tag{7}$$

for all strategy profiles $(s_1, s_2) \in S_1 \times S_2$. Given any pair $(\pi_1, \pi_2) \in \Delta(S_1) \times \Delta(S_2)$ of mixed strategies, the two players' expected utilities are given by the respective bilinear functions

$$\begin{aligned} W_1(\pi_1, \pi_2) &:= \mathbb{E}_{\pi_1} \mathbb{E}_{\pi_2} v_1(s_1, s_2) \\ W_2(\pi_1, \pi_2) &:= \mathbb{E}_{\pi_1} \mathbb{E}_{\pi_2} v_2(s_1, s_2) = -W_1(\pi_1, \pi_2) \end{aligned} \tag{8}$$

where the last equality holds because of (7). Von Neumann and Morgenstern's *minmax* (or *maxmin*) theorem states that such a game has unique minmax (or maxmin) *values* $(\hat{w}_1, \hat{w}_2)$ for the two players. These are defined to satisfy

$$\hat{w}_1 = -\hat{w}_2 = \max_{\pi_1 \in \Delta(S_1)} \min_{\pi_2 \in \Delta(S_2)} W_1(\pi_1, \pi_2) = \min_{\pi_2 \in \Delta(S_2)} \max_{\pi_1 \in \Delta(S_1)} W_1(\pi_1, \pi_2)$$

These values are generated by any *mixed strategy equilibrium*, which is equivalent to any *saddle point*

$$(\hat{\pi}_1, \hat{\pi}_2) \in \arg \max_{\pi_1 \in \Delta(S_1)} \min_{\pi_2 \in \Delta(S_2)} W_1(\pi_1, \pi_2) = \arg \min_{\pi_2 \in \Delta(S_2)} \max_{\pi_1 \in \Delta(S_1)} W_1(\pi_1, \pi_2)$$

of the function $W_1(\pi_1, \pi_2)$. In particular,

$$W_1(\pi_1, \hat{\pi}_2) \le W_1(\hat{\pi}_1, \hat{\pi}_2) \le W_1(\hat{\pi}_1, \pi_2)$$

for all $(\pi_1, \pi_2) \in \Delta(S_1) \times \Delta(S_2)$. Necessary and sufficient conditions for $(\hat{\pi}_1, \hat{\pi}_2)$ to be such a saddle point are that the pair of inequalities

$$\hat{\pi}_1(s_1) \ge 0, \sum_{s_2 \in S_2} \hat{\pi}_2(s_2)\, v_1(s_1, s_2) \ge \hat{w}_1 \tag{9}$$

should be complementarily slack (i.e., at least one inequality in each pair must hold with equality) for all $s_1 \in S_1$, and also that the pair

$$\hat{\pi}_2(s_2) \ge 0, \sum_{s_1 \in S_1} \hat{\pi}_1(s_1)\, v_1(s_1, s_2) \le -\hat{w}_2 \tag{10}$$

should be complementarily slack for each $s_2 \in S_2$.

This minmax theorem is a special case of the Nash equilibrium existence theorem to be discussed below, with each player $i \in \{1, 2\}$ having a mixed strategy $\hat{\pi}_i$ which matches the other player's belief π_i about i's strategy choice. Alternatively, following Gale (1960, pp. 256–7, Ex. 1), it can be demonstrated that the equilibrium mixed strategies $(\hat{\pi}_1, \hat{\pi}_2)$ must solve the dual pair of linear programs

$$\max_{w_1, \pi_2 \in \Delta(S_2)} \{ w_1 \mid \sum_{s_2 \in S_2} \hat{\pi}_2(s_2)\, v_1(s_1, s_2) \ge w_1 \}$$

$$\min_{w_2, \pi_1 \in \Delta(S_1)} \{ -w_2 \mid \sum_{s_1 \in S_1} \hat{\pi}_1(s_1)\, v_1(s_1, s_2) \le -w_2 \}$$

Because $\Delta(S_1)$ and $\Delta(S_2)$ are compact sets, both these programs have a solution. By the duality theorem of linear programming, the solutions $(\hat{w}_1, \hat{\pi}_2)$ and $(\hat{w}_2, \hat{\pi}_1)$ are such that all the pairs of inequalities (9) and (10) are complementarily slack, for all $s_1 \in S_1$ and all $s_2 \in S_2$ respectively. It follows that $\hat{w}_1 = -\hat{w}_2 = W_1(\pi_1, \pi_2)$. This proves that an equilibrium exists, and also suggests a numerical method for finding one.

Of more concern to this chapter, however, is the fact that (7) holds only for some particular pairs of utility functions chosen from within each player's separate class of cardinally equivalent NMUFs. Rather than a zero-sum game, a more appropriate concept which does not imply any interpersonal comparability of different individuals' utility functions is that of a *strictly competitive* two-person game. Then (7) is replaced by the condition that there exist arbitrary constants C, d_1 and d_2, with d_1 and d_2 both positive, such that

$$d_1\, v_1(s_1, s_2) + d_2\, v_2(s_1, s_2) = C \tag{11}$$

In particular

$$v_1(s_1, s_2) \geq v_1(s_1', s_2') \iff v_2(s_1, s_2) \leq v_2(s_1', s_2')$$

for all $s_1, s_1' \in S_1$ and all $s_2, s_2' \in S_2$. Hence, when W_1 and W_2 are defined by (8), one has

$$W_1(\pi_1, \pi_2) \geq W_1(\pi_1', \pi_2') \iff W_2(\pi_1, \pi_2) \leq W_2(\pi_1', \pi_2')$$

for all $\pi_1, \pi_1' \in \Delta(S_1)$ and all $\pi_2, \pi_2' \in \Delta(S_2)$. This implies that the two players' objectives really are strictly opposed, as the term "strictly competitive" suggests.

It is easy to see that, because the constants C, d_1 and d_2 are arbitrary, (11) remains true after the two players' NMUFs have undergone any independent affine transformations $v_1 \mapsto v_1'$ and $v_2 \mapsto v_2'$ which satisfy

$$v_1'(s_1, s_2) = \alpha_1 + \delta_1 \, v_1(s_1, s_2) \quad \text{and} \quad v_2'(s_1, s_2) = \alpha_2 + \delta_2 \, v_2(s_1, s_2)$$

for arbitrary additive constants α_1, α_2, and arbitrary positive multiplicative constants δ_1, δ_2. Such a game has all the features of a zero sum game, except that the two players' minmax (or maxmin) values (w_1, w_2) obviously satisfy $d_1 \, w_1 + d_2 \, w_2 = C$ instead of $w_1 + w_2 = 0$. Note, however, that a strictly competitive game will usually cease to be strictly competitive if one or both players' attitudes to risk change, resulting in a non-linear transformation of the corresponding NMUF.

4.2 Nash Equilibrium

After this extended detour to discuss the historically important case of two-person games, let us now return to the general n-person framework. A *Nash equilibrium* (Nash, 1950, 1951) of the game G is defined as a profile $\mu^I = \langle \mu_i \rangle_{i \in I}$ of *mixed strategies* $\mu_i \in \Delta(S_i)$, one for each player $i \in I$, with the property that $\mu_i(B_i(\pi_i)) = 1$ for the joint distribution

$$\pi_i = \mu^{I \setminus \{i\}} = \prod_{h \in I \setminus \{i\}} \mu_h \tag{12}$$

induced by the other players' independently chosen mixed strategies. That is, with probability one each player i's mixed strategy μ_i selects a best response $s_i \in B_i(\pi_i)$ to the other players' profile π_i of independent mixed strategies.

Proving that such a Nash equilibrium in mixed strategies always exists is a routine application of Kakutani's fixed point theorem to the correspondence F that maps the Cartesian product space $\prod_{i \in I} \Delta(S_i)$ of mixed strategy profiles μ^I into itself, with $F(\mu^I) := \prod_{i \in I} \Delta(B_i(\pi_i))$ where each π_i satisfies (12). Indeed, the theorem can be applied because the domain $\prod_{i \in I} \Delta(S_i)$ is non-empty, convex and compact, the image sets $F(\mu^I)$ are all non-empty and convex, and the graph of the correspondence F is easily shown to be a closed set.

4.3 Bayesian Nash Equilibrium

A *game with incomplete information* (Harsanyi, 1967–8) is defined as a collection

$$G^* = \langle I, T^I, S^I, v^I, q^I \rangle \tag{13}$$

where I is the set of players and S^I the set of strategy profiles, as in (1). Also, each player $i \in I$ has a finite *type space* T_i. Each possible type $t_i \in T_i$ of player i is assumed to determine probabilistic *prior beliefs* $q_i(\cdot|t_i) \in \Delta(T_{-i})$ about the profile $t_{-i} \in T_{-i} := \prod_{h \in I \setminus \{i\}} T_h$ of all other players' types. Moreover, the profile $t^I \in T^I$ of all players' types is assumed to determine the payoff functions $v_i(s^I; t^I)$ of all the players. Here v_i is allowed to depend on t^I rather than just on t_i to reflect the possibility that there may be a fundamental determinant of player i's payoff which is correlated with other players' types. This is the case in common value auctions, for instance, as discussed by Milgrom and Weber (1982) and by Milgrom (1987).

Equilibrium theory, based on objective probabilities, naturally concentrates on the special case when there is one *common prior* distribution $q \in \Delta(T^I)$ such that each player i's prior distribution $q_i(\cdot|t_i)$ is the conditional distribution derived from q given t_i. That is, for each possible type $t_i \in T_i$ of each player $i \in I$, and for each profile $t_{-i} \in T_{-i}$ of other players' types,

$$q_i(t_{-i}|t_i) = \frac{q(t_i, t_{-i})}{\sum_{t'_{-i} \in T_{-i}} q(t_i, t'_{-i})}$$

In the game G^* each player i's expectations concerning the profile s_{-i} of other players' strategies, conditional on the other players' types t_{-i}, take the form of a distribution $\pi_i(\cdot|t_{-i}) \in \Delta(S_{-i})$. For the special case when, given their profile of types t_{-i}, these other players $h \in I \setminus \{i\}$ choose independent type-dependent mixed strategies $\mu_h(\cdot|t_h) \in \Delta(S_h)$, these probabilities satisfy

$$\pi_i(s_{-i}|t_{-i}) = \prod_{h \in I \setminus \{i\}} \mu_h(s_h|t_h) \tag{14}$$

for all $s_{-i} \in S_{-i}$. Generally, however, player i's conditional beliefs $\pi_i(\cdot|t_{-i})$ about two or more other players' strategies in the profile s_{-i} may be correlated. Even then, when combined with i's own type t_i and prior beliefs $q_i(\cdot|t_i)$ about other players' types t_{-i}, these probabilities still determine player i's expected payoff

$$V_i(s_i, \pi_i; t_i) := \sum_{t_{-i} \in T_{-i}} q_i(t_{-i}|t_i) \sum_{s_{-i} \in S_{-i}} \pi_i(s_{-i}|t_{-i})\, v_i(s_i, s_{-i}; t_i, t_{-i})$$

as a function of i's strategy s_i and type t_i. They also determine player i's best response correspondence, whose values are

$$B_i(\pi_i; t_i) := \arg\max_{s_i \in S_i} V_i(s_i, \pi_i; t_i)$$

There are now two different ways of expressing G^* as an ordinary game in normal form, as in (1). For the first way, let $\tilde{S}_i := S_i^{T_i} := \prod_{t_i \in T_i} S_i(t_i)$, where each $S_i(t_i)$ is a copy of S_i. Thus, $\tilde{S}_i$ is the set of player i's *type-contingent strategies*, each of which is a mapping from T_i to S_i. Also, given the common prior $q \in \Delta(T^I)$ over all players' type profiles, for each type-contingent strategy profile $\tilde{s}^I \in \tilde{S}^I := \prod_{i \in I} \tilde{S}_i$, the expected utility of each player $i \in I$ is obviously given by

$$\tilde{v}_i(\tilde{s}^I) := \sum_{t^I \in T^I} q(t^I)\, v_i(s^I(t^I); t^I)$$

where $s^I(t^I)$ denotes the usual pure strategy profile $\langle s_i(t_i) \rangle_{i \in I}$ in S^I. Then $\tilde{G} := \langle I, \tilde{S}^I, \tilde{v}^I \rangle$ is an ordinary n-person game, except that it has some special separability properties. Given expectations $\tilde{\pi}_i \in \Delta(\tilde{S}_{-i})$ about other players' type-contingent strategies and the common prior $q \in \Delta(T^I)$ over type profiles, player i's expected utility is

$$\tilde{V}_i(\tilde{s}_i, \tilde{\pi}_i) = \sum_{\tilde{s}_{-i} \in \tilde{S}_{-i}} \tilde{\pi}_i(\tilde{s}_{-i})\, \tilde{v}_i(\tilde{s}^I)$$

But $\tilde{\pi}_i(\tilde{s}_{-i}) = \prod_{t_{-i} \in T_{-i}} \pi_i(s_{-i}(t_{-i}) | t_{-i})$ where $s_{-i}(t_{-i}) := \langle s_h(t_h) \rangle_{h \in I \setminus \{i\}}$. It follows that

$$\begin{aligned} \tilde{V}_i(\tilde{s}_i, \tilde{\pi}_i) &= \sum_{\tilde{s}_{-i} \in \tilde{S}_{-i}} \prod_{t_{-i} \in T_{-i}} \pi_i(s_{-i}(t_{-i}) | t_{-i}) \sum_{t^I \in T^I} q(t^I)\, v_i(s^I(t^I); t^I) \\ &= \sum_{t_i \in T_i} p_i(t_i)\, V_i(s_i(t_i), \pi_i; t_i) \end{aligned}$$

where $p_i(t_i) := \sum_{t_{-i} \in T_{-i}} q(t_i, t_{-i})$ and so $q(t^I) = p_i(t_i)\, q_i(t_{-i} | t_i)$. So each player i's set of best responses is given by

$$\begin{aligned} \tilde{B}_i(\tilde{\pi}_i) &:= \arg\max_{\tilde{s}_i \in \tilde{S}_i} \tilde{V}_i(\tilde{s}_i, \tilde{\pi}_i) \\ &= \prod_{t_i \in T_i} \arg\max_{s_i(t_i) \in S_i} V_i(s_i(t_i), \pi_i; t_i) \;=\; \prod_{t_i \in T_i} B_i(\pi_i; t_i) \end{aligned}$$

because the strategies $s_i(t_i)$ can be chosen separately for each $t_i \in T_i$.

The second way to express G^* as an ordinary game in normal form begins by letting $s_{-i,t_{-i}}$ denote the list $\langle s_{j,t_j} \rangle_{j \in I \setminus \{i\}} \rangle$ of strategies chosen by the other players participating in the game when their types are t_{-i}. Then $G^* = \langle I^*, S^{I^*}, v^{I^*} \rangle$ where $I^* := \{ (i, t_i) \mid i \in I, t_i \in T_i \}$ is the set of players, S^{I^*} is the Cartesian product $\prod_{i \in I} \prod_{t_i \in T_i} S_{i,t_i}$ with each strategy set S_{i,t_i} a copy of S_i, independent of i's type, while the payoff functions $v^*_{i,t_i} : S^{I^*} \to \mathbb{R}$ are given by

$$v^*_{i,t_i}(s^{I^*}) = \sum_{t_{-i} \in T_{-i}} q_i(t_{-i} | t_i)\, v_i(s_i, s_{-i,t_{-i}}; t_i, t_{-i})$$

for all $s^{I^*} \in S^{I^*}$ and $(i, t_i) \in I^*$. Thus, G^* has been re-cast as an ordinary game in normal form, but with a special structure. In particular, for each $i \in I$, one player (i, t_i) is selected at random from the set $\{i\} \times T_i$ to make the strategy choice $s_i \in S_i$. This player's best response correspondence is given precisely by $B_i(\pi_i; t_i)$. It follows that this alternative way of expressing G^* as an ordinary game in normal form gives rise to identical best responses, and so to identical Nash equilibria.

A Nash equilibrium of such a game is generally called a *Bayesian–Nash equilibrium*, especially in the case when $q_i(\cdot|t_i)$ is derived from a common prior q by conditioning. Whether there is a common prior or not, such an equilibrium consists of a profile of type-dependent but conditionally independent mixed strategies $\mu_i(\cdot|t_i) \in \Delta(S_i)$ (all $i \in I$ and $t_i \in T_i$) with the property that, when the probabilities $\pi_i(\cdot|t_{-i}) \in \Delta(S_{-i})$ are given by (14), then $\mu_i(B_i(\pi_i; t_i)|t_i) = 1$ for all $i \in I$ and all $t_i \in T_i$.

4.4 Correlated Equilibrium

Finally, a *correlated equilibrium* (Aumann, 1974, 1987a) of the original game (1) of perfect information is a general joint distribution $\mu \in \Delta(S^I)$ on the set S^I of strategy profiles, not necessarily independent, with the property that for all $s^I \in S^I$ with $\mu(s^I) > 0$ and all $i \in I$, if $\pi_i = \mu(\cdot|s_i)$ is the induced conditional distribution on S_{-i} given s_i, then $s_i \in B_i(\pi_i)$. The most plausible interpretation is that a suitable correlation device is used to generate random private signals to each player i that suggest the choice of some particular $s_i \in S_i$. In equilibrium, the distribution μ of the different players' signals must be such that all players are willing to follow these suggestions. For other possible interpretations of correlated equilibria, see Chapter 13 by Joyce and Gibbard in Volume 1 of this *Handbook*.

Of course, any Nash equilibrium is a special case of a correlated equilibrium which arises when μ specifies that different players' signals are independently distributed, implying that individuals' induced mixed strategies happen to be independent. Thus, $\mu(s^I) = \prod_{i \in I} \mu_i(s_i)$. Indeed, in this case π_i is given by (12). Also, because $\mu_i(B_i(\pi_i)) = 1$ and $\prod_{h \in I} \mu_h(s_h) > 0$, it must be true that each $s_i \in B_i(\pi_i)$, as required for Nash equilibrium. Because a Nash equilibrium exists, therefore, so does a correlated equilibrium.[11] But many games have correlated equilibria that are not Nash — e.g., the well-known Battle of the Sexes (Luce and Raiffa, 1957). And in fact, by considering correlation devices that are random mixtures of other correlation devices, it is not difficult to show that the set of correlated equilibria must always be a convex subset of $\Delta(S^I)$ — see, for example, Osborne and Rubinstein (1994). On the other hand, in

[11]See Hart and Schmeidler (1989) for an alternative and elementary direct proof that a correlated equilibrium exists. Their proof uses the duality theory of linear programming rather than a fixed point theorem.

the framework assumed here, Wilson (1971) and Harsanyi (1973) proved that generically there is an odd finite number of mixed strategy Nash equilibria.

5 Perfect and Proper Equilibrium

5.1 Subgame Imperfection of Nash Equilibrium

In an important article that was overlooked for too many years — probably because it was unavailable in English — Selten (1965) noted that some Nash equilibria relied on players being deterred by threats which it would be irrational to carry out if deterrence should happen to fail — see also Selten (1973). Such threats can survive as part of each player's best response in a Nash equilibrium because they are responses to actions which, in equilibrium, are deterred and so occur with probability zero. The difficulty here is very similar to that noticed in Chapter 5, which forced zero probability events to be excluded in order to avoid universal indifference being the only possibility consistent with the consequentialist axioms.

This serious deficiency of the Nash equilibrium concept led Selten to devise the notion of "subgame perfect" equilibria in extensive form games. These are defined as equilibria which rely only on threats that really are credible because the person doing the threatening has an incentive to carry out the threat even if deterrence fails. Kreps and Wilson's (1982) notion of "sequential" equilibria extends the idea to cases where there may not be a properly defined subgame. As in subgame perfect equilibria, they require players to maximize expected utility at each information set, given their probability assessments at that set. In addition, at successive information sets which are reached with positive probability, players must revise their assessments by using Bayes' rule to update the equilibrium probability distribution, based on the knowledge of the information set they have reached.

Exploring subgame perfect and sequential equilibria would require us to consider games in extensive form, which I propose to avoid in this survey. In fact, later Selten (1975) himself came up with the concept of "trembling-hand perfect" equilibria based only on the normal form. These have the attractive property that, provided one considers the agent normal form in which each player has only one information set, only subgame perfect equilibria are chosen in any extensive form game having this agent normal form. Myerson (1978) later refined this concept to "proper" equilibrium. This section will briefly consider each of these two normal form equilibrium concepts in turn. Both replace zero probabilities with vanishing trembles, but in different ways. Many of the results reported here concerning perfect and proper equilibria can be found in van Damme (1987) and in Fudenberg and Tirole (1991, pp. 351–3).

5.2 Trembling-Hand Perfection

Trembling-hand perfection derives its name from the fact that players are prevented from choosing any strategy with perfect certainty. Because the "hand" governing their choice of strategy trembles unpredictably, no strategy occurs with zero probability. Instead, given any strictly positive vector $\eta = \langle\langle \eta_i(s_i)\rangle_{s_i \in S_i}\rangle_{i \in I} \in \prod_{i \in I} \mathbb{R}^{S_i}_{++}$ small enough to satisfy $\sum_{s_i \in S_i} \eta_i(s_i) \leq 1$ for all $i \in I$, consider what happens when each player $i \in I$ is restricted to choosing a mixed strategy μ_i from the set

$$\Delta^\eta(S_i) := \{ \mu_i \in \Delta(S_i) \mid \mu_i(s_i) \geq \eta_i(s_i) \text{ (all } s_i \in S_i) \} \tag{15}$$

of "η-trembles". Given any probability beliefs $\pi_i \in \Delta(S_{-i})$ concerning the other players' strategy profile, player i's η-*constrained* best response set is defined by

$$\begin{aligned} B_i^\eta(\pi_i) \quad &:= \quad \arg\max_{\mu_i} \{ \mathbb{E}_{\mu_i} V_i(s_i, \pi_i) \mid \mu_i \in \Delta^\eta(S_i) \} && (16) \\ &:= \quad \{ \mu_i^\eta \in \Delta^\eta(S_i) \mid \mu_i \in \Delta^\eta(S_i) \Longrightarrow \mathbb{E}_{\mu_i^\eta} V_i(s_i, \pi_i) \geq \mathbb{E}_{\mu_i} V_i(s_i, \pi_i) \} \end{aligned}$$

instead of as the usual unconstrained best response set given by (5). Given that trembles cannot be avoided, this is the best that player i can do. In fact, player i maximizes $\mathbb{E}_{\mu_i} V_i(s_i, \pi_i)$ subject to the constraints $\mu_i(s_i) \geq \eta_i(s_i)$ by choosing $\mu_i(s_i) = \eta_i(s_i)$ unless $s_i \in B_i(\pi_i)$. So if $\mu_i \in B_i^\eta(\pi_i)$, then $\mu_i(s_i) > \eta_i(s_i)$ is only possible when $s_i \in B_i(\pi_i)$.

Next, define an η-*constrained equilibrium* as any profile μ^I of independent mixed strategies which, when $\pi_i = \prod_{h \in I \setminus \{i\}} \mu_h$, satisfies $\mu_i \in B_i^\eta(\pi_i)$ for all $i \in I$. This is a Nash equilibrium in a "perturbed" game where each player $i \in I$ is restricted to completely mixed strategies in $\Delta^\eta(S_i)$, which is a compact convex set. Consider the correspondence F^η from the non-empty convex and compact set $\prod_{i \in I} \Delta^\eta(S_i)$ to itself which is defined by $F^\eta(\mu^I) := \prod_{i \in I} B_i^\eta(\mu^{I \setminus \{i\}})$. Evidently F^η has non-empty convex values. It is easy to verify that its graph is closed. So for each allowable vector $\eta \gg 0$ the correspondence F^η satisfies the conditions needed to apply Kakutani's theorem. It therefore has a fixed point, which must be an η-constrained equilibrium.

Finally, say that $\mu^I \in \prod_{i \in I} \Delta(S_i)$ is a *trembling-hand perfect* (or THP) *equilibrium* if it is the limit as $n \to \infty$ of an infinite sequence μ_n^I $(n = 1, 2, \dots)$ of η_n-constrained equilibria, where $\eta_n \downarrow 0$. Then, for any infinite sequence η_n $(n = 1, 2, \dots)$ satisfying $\eta_n \downarrow 0$, compactness of each set $\Delta(S_i)$ guarantees that any corresponding sequence μ_n^I $(n = 1, 2, \dots)$ of η_n-constrained equilibria has a convergent subsequence, whose limit is by definition a THP equilibrium. Hence, THP equilibrium exists.

On the other hand, suppose that μ^I is a THP equilibrium. Then there exist a sequence $\eta_n \downarrow 0$ and a sequence $\mu_n^I = \langle \mu_{in} \rangle_{i \in I}$ $(n = 1, 2, \dots)$ of η_n-constrained equilibria converging to $\mu^I = \langle \mu_i \rangle_{i \in I}$. In particular, $\mu_{in} \in B_i^{\eta_n}(\pi_{in})$ for all $i \in I$, where $\pi_{in} = \prod_{h \in I \setminus \{h\}} \mu_{hn}$. It follows that $\pi_{in} \to \pi_i = \prod_{h \in I \setminus \{h\}} \mu_h$, and then a routine convergence argument shows that $\mu_i \in B_i(\pi_i)$ for all $i \in I$. This proves that any THP equilibrium is Nash.

Moreover, for games in agent normal form, it is routine to show that any THP equilibrium is also subgame perfect in any subgame of an extensive form game with the given agent normal form. Indeed, any η-constrained equilibrium reaches that subgame with positive probability. Therefore, the agent at the initial information set of that subgame must choose an η-constrained response, which is also an η-constrained response in the subgame. Taking the limit as $\eta \downarrow 0$, it follows that this agent chooses a best response in the subgame, even if the subgame is reached with probability zero in the trembling-hand perfect equilibrium.

5.3 ϵ-Perfect and Perfect Equilibrium

There are two alternative characterizations of THP equilibria which will be useful subsequently. As a preliminary, let $\Delta^0(S_i)$ denote the set of all "completely" mixed strategies μ_i — i.e., those satisfying $\mu_i(s_i) > 0$ for all $s_i \in S_i$.

First, given any $i \in I$, any real $\epsilon \in (0, 1)$, and any $\pi_i \in \Delta(S_{-i})$, the completely mixed strategy $\mu_i \in \Delta^0(S_i)$ is said to be an ϵ-*perfect response* by player i to π_i provided that $\mu_i(s_i) \leq \epsilon$ for all inferior responses $s_i \in S_i \setminus B_i(\pi_i)$. Equivalently, the set of ϵ-perfect responses is given by

$$P_i^\epsilon(\pi_i) := \{\, \mu_i \in \Delta^0(S_i) \mid \mu_i(s_i) > \epsilon \Longrightarrow s_i \in B_i(\pi_i) \,\} \tag{17}$$

From the definitions, it is obvious that $B_i^\eta(\pi_i) \subset P_i^\epsilon(\pi_i)$ whenever $\eta(s_i) \leq \epsilon$ for all $s_i \in S_i$.

Say that the profile μ^I of independent mixed strategies is an ϵ-*perfect equilibrium* if it satisfies $\mu_i \in P_i^\epsilon(\mu_{-i})$ for all $i \in I$.

Obviously, when $\eta(s_i) \leq \epsilon$ for all $i \in I$ and all $s_i \in S_i$, then any η-constrained equilibrium is an ϵ-perfect equilibrium. It follows that any THP equilibrium μ^I is the limit as $n \to \infty$ and $\epsilon_n \downarrow 0$ of a sequence μ_n^I of ϵ_n-perfect equilibria. The converse is also true, but establishing it is helped by introducing a second alternative characterization of THP equilibrium.

Say that μ^I is a *perfect equilibrium* if it is the limit as $n \to \infty$ of a sequence of completely mixed strategy profiles $\mu_n^I \in \prod_{i \in I} \Delta^0(S_i)$ with the property that, for each player i and strategy $s_i \in S_i$, one has

$$\mu_i(s_i) > 0 \Longrightarrow s_i \in B_i(\mu_{-i,n}) \tag{18}$$

Suppose that μ^I is the limit as $\epsilon_n \downarrow 0$ of a sequence μ_n^I of ϵ_n-perfect equilibria. In this case, note that each μ_n^I is a completely mixed strategy profile. Also, if $s_i \in S_i$ is any strategy for player i satisfying $\mu_i(s_i) > 0$, then $\mu_{i,n}(s_i) > \epsilon_n$ for all large n. Then, because μ_n^I is an ϵ_n-perfect equilibrium, it follows that $\mu_{i,n} \in P_i^{\epsilon_n}(\mu_{-i,n})$, so $s_i \in B_i(\mu_{-i,n})$. This is true for all large n, so the limit μ^I must be a perfect equilibrium.

On the other hand, suppose that μ^I is a perfect equilibrium, as the limit of a sequence of completely mixed strategy profiles μ_n^I satisfying (18). For each $i \in I$ and $s_i \in S_i$, define the sequence

$$\eta_{in}(s_i) := \begin{cases} \mu_{i,n}(s_i) & \text{if } \mu_i(s_i) = 0 \\ 1/(n + \#S_i) & \text{if } \mu_i(s_i) > 0 \end{cases}$$

for $n = 1, 2, \ldots$. Then $\sum_{s_i \in S_i} \eta_{in}(s_i) \leq 1$ for all large n. Also, $\mu_{i,n}(s_i) \geq \eta_{in}(s_i)$ for all large n, with strict inequality only if $\mu_i(s_i) > 0$. Because μ_n^I

satisfies (18), for large n it follows that $\mu_{i,n}(s_i) > \eta_{in}(s_i)$ implies $s_i \in B_i(\mu_{-i,n})$. Hence, μ^I_n is an η-constrained equilibrium for large n, and the limit μ^I is a THP equilibrium.

To summarize, μ^I is a THP equilibrium if and only if it is perfect, and also if and only if it is the limit as $n \to \infty$ and $\epsilon_n \downarrow 0$ of a sequence μ^I_n of ϵ_n-perfect equilibria.

5.4 Proper Equilibrium

Consider any subgame which has been reached after one player $i \in I$ in particular has already made some kind of mistake. This leaves player i with the choice between several inferior strategies, of which some are likely to be better than others. Then the argument for considering ϵ-perfect responses suggests that any strategy which is best in the subgame should receive much higher probability than those which are inferior, even within the subgame. Yet the definition of ϵ-perfect responses used above makes no distinction between these different inferior responses. Each of any player i's inferior strategies $s_i \in S_i$ must be given probability no less than ϵ, so it is best to give each of them probability exactly equal to ϵ, without regard to whether some may be better or worse than others. This equal treatment of all inferior strategies allows even a perfect equilibrium to be subgame imperfect, unless one considers perfect equilibria of the agent normal form, with different agents of each player then being required to tremble independently of each other.

To remedy this deficiency, Myerson (1978) refines the definition (18) of $P^\epsilon_i(\pi_i)$ by replacing it with $\hat{P}^\epsilon_i(\pi_i)$, the set of *$\epsilon$-proper responses*, defined as

$$\hat{P}^\epsilon_i(\pi_i) := \{\ \mu_i \in \Delta^0(S_i) \ \mid\ \forall s_i, s'_i \in S_i : \\ V_i(s_i, \pi_i) > V_i(s'_i, \pi_i) \Longrightarrow \mu_i(s'_i) \leq \epsilon\, \mu_i(s_i)\ \} \tag{19}$$

Thus, player i gives all inferior strategies low positive probability, but these probabilities are much lower for worse strategies. Note that $\hat{P}^\epsilon_i(\pi_i)$ really does refine the set $P^\epsilon_i(\pi_i)$ of ϵ-perfect responses, because the latter evidently satisfies

$$P^\epsilon_i(\pi_i) = \{\ \mu_i \in \Delta^0(S_i) \mid \forall s_i, s'_i \in S_i : V_i(s_i, \pi_i) > V_i(s'_i, \pi_i) \Longrightarrow \mu_i(s'_i) \leq \epsilon\ \}$$

Next, define an *ϵ-proper equilibrium* as any profile μ^I of independent totally mixed strategies which, when $\pi_i = \prod_{h \in I \setminus \{i\}} \mu_h$, satisfies $\mu_i \in P^\epsilon_i(\pi_i)$ for all $i \in I$. Finally, say that $\mu^I \in \prod_{i \in I} \Delta(S_i)$ is a *proper equilibrium* if it is the limit as $n \to \infty$ of an infinite sequence μ^I_n $(n = 1, 2, \dots)$ of ϵ_n-proper equilibria, where $\epsilon_n \downarrow 0$ in $\mathbb{R}$.

Because $\hat{P}^\epsilon_i(\pi_i) \subset P^\epsilon_i(\pi_i)$, it is obvious that any ϵ-proper equilibrium is ϵ-perfect. Taking limits as $\epsilon \downarrow 0$, it follows that any proper equilibrium, if one exists, must be trembling-hand perfect.

In fact, existence of proper equilibrium can be proved fairly easily by restricting completely mixed strategies to the closed convex set

$$\tilde{\Delta}^\epsilon(S_i) := \{\ \mu_i \in S_i \mid \mu_i(s_i) \geq \epsilon^m/m \quad (\text{all } s_i \in S_i)\ \} \tag{20}$$

for each $i \in I$ and $\epsilon \in (0,1)$, where $m := \max_{i \in I} \# S_i$ denotes the maximum number of pure strategies available to any one player — see Fudenberg and Tirole (1991, p. 357). Then, instead of $\hat{P}_i^\epsilon(\pi_i)$ or $B_i^\epsilon(\pi_i)$, consider

$$A_i^\epsilon(\pi_i) := \hat{P}_i^\epsilon(\pi_i) \cap \tilde{\Delta}^\epsilon(S_i) \tag{21}$$

For each fixed $\epsilon \in (0,1)$, the set $A_i^\epsilon(\pi_i)$ is obviously non-empty, closed, and convex. Also, because the expected utility $V_i(s_i, \pi_i)$ is a continuous function of player i's probability beliefs $\pi_i \in \Delta(S_{-i})$, it is easy to show that the correspondence $\pi_i \mapsto\!\!\!\rightarrow A_i^\epsilon(\pi_i)$ has a closed graph, for each fixed $\epsilon \in (0,1)$.

Next, define the correspondence F^ϵ from the non-empty compact convex set $\prod_{i \in I} \tilde{\Delta}^\epsilon(S_i)$ into itself by $F^\epsilon(\mu^I) := \prod_{i \in I} A_i^\epsilon(\mu^{I \setminus \{i\}})$. The argument used in Section 5.2 can be repeated to demonstrate that for each $\epsilon \in (0,1)$ there exists a fixed point $\mu^I(\epsilon) \in F^\epsilon(\mu^I(\epsilon))$, which must be an ϵ-proper equilibrium. Existence of a proper equilibrium can then be proved by taking the limit as $\epsilon \downarrow 0$, following the argument used in Section 5.2 to demonstrate existence of a perfect equilibrium.

It is also easy to show that any proper equilibrium is not only trembling-hand perfect, but now also subgame perfect, even outside the agent normal form. Indeed, not only is each subgame reached with positive probability in any ϵ-proper equilibrium; in addition, strategies that are inferior in the subgame must be played with much lower probability, and so, in the limit as $\epsilon \to 0$, with zero probability in proper equilibrium. Furthermore, Kohlberg and Mertens (1986) proved that proper equilibria are also sequential. They also give an example showing that, even in a single-person game, a proper equilibrium need not be trembling-hand perfect in the agent normal form.

5.5 Importance of Best Responses

This very brief survey of the Nash, Bayesian, correlated, perfect, and proper equilibrium concepts illustrates in particular how the best response correspondence lies at the heart of non-cooperative game theory. For each player, this correspondence specifies how behaviour depends on expectations. The equilibrium concepts considered so far treat the probability distribution $\mu \in \Delta(S^I)$ as objective. Then the best response correspondence comes from maximizing objectively expected utility. In equilibrium, μ attaches probability 1 to the set of strategy profiles in which each player chooses a best response.

Section 9 will discuss some significant extensions of equilibrium theory. Nevertheless, these extensions concern the determination of players' expectations; they still assume that, given their expectations, all players choose strategies from their respective sets of best responses. Or later in Section 11, at least from their respective sets of ϵ-perfect or ϵ-proper responses. The next section removes this requirement completely.

6 Quantal Response Equilibrium

6.1 Motivation

The previous section considered how small trembles may help to resolve the zero probability problem created by the requirement that each player's set of best responses be given probability 1. Such trembles, however, play no role in classical single-person decision theory. In fact, ideas from stochastic utility theory (as reviewed by Fishburn in Chapter 7 of Volume 1) are introduced artificially. Instead, it seems worth investigating the implications of applying a fully articulated stochastic decision theory to non-cooperative games. In an attempt to provide a more accurate description of observed behaviour in games, McKelvey and Palfrey (1995) have initiated one important line of research in this area. Some of the most recent analysis appears in Haile, Hortaçsu and Kosenok (2003).

Consider an underlying set Y of consequences, and the set $\Delta(Y)$ of simple consequence lotteries. Ordinary decision theory, as explored in Chapters 5 and 6 of Volume 1, considers a *choice function* C defined on $\mathcal{F}$, the collection of non-empty finite subsets F of $\Delta(Y)$, with $C(F) \in \mathcal{F}$ and $C(F) \subset F$ for all $F \in \mathcal{F}$. *Stochastic decision theory*, on the other hand, as surveyed by Fishburn in Chapter 7 of Volume 1, considers a simple lottery $q(F) \in \Delta(F)$ defined for each $F \in \mathcal{F}$. Thus, we may write $q(\lambda, F)$ for the probability of choosing $\lambda \in F$ when the agent is presented with the feasible set $F \in \mathcal{F}$.

The main goal of our inquiry remains to explore the implications of the consequentialist normal form invariance axiom described in Section 2.7, but adapted to fit the stochastic choice framework considered in this Section. It will be shown that stochastic utility theory is of no help in avoiding the zero probability problem because only trivial extensions of the usual expected utility maximizing decision rule satisfy consequentialist normal form invariance.

6.2 Ordinality

First, for each non-empty finite set $F \subset \Delta(Y)$, define

$$C(F) := \{ \lambda \in F \mid q(\lambda, F) > 0 \} \tag{22}$$

as the set of elements that are chosen with positive probability from F. Next, define the binary *stochastic weak preference relation* $\succsim$ on $\Delta(Y)$ by

$$\lambda \succsim \mu \iff \lambda \in C(\{ \lambda, \mu \}) \iff q(\lambda, \{ \lambda, \mu \}) > 0 \tag{23}$$

It is immediate from the definition that the relation $\succsim$ must be complete. The corresponding *stochastic strict preference* and *stochastic indifference* relations obviously satisfy

$$\lambda \succ \mu \iff q(\lambda, \{ \lambda, \mu \}) = 1 \quad \text{and} \quad \lambda \sim \mu \iff 0 < q(\lambda, \{ \lambda, \mu \}) < 1 \tag{24}$$

Now, arguing exactly as in Section 5.6 of Chapter 5, consequentialism implies that the relation $\succsim$ must be transitive, as well as complete, so a preference ordering. Moreover, one must have $C(F) := \{ \lambda \in F \mid \forall \mu \in F : \lambda \succsim \mu \}$ — in other words,

$$q(\lambda, F) > 0 \iff \forall \mu \in F : q(\lambda, \{ \lambda, \mu \}) > 0$$

Thus, the elements that are chosen with positive probability from F are exactly those which maximize the ordering $\succsim$.

6.3 Luce's Superstrong Transitivity Axiom

Having considered the qualitative issue of what elements of F are chosen with positive probability, we now consider the quantitative issue of what probabilities are assigned to the elements of $C(F)$, defined by (22). Note that all elements of $C(F)$ are stochastically indifferent, according to the definition (24) of the symmetric relation $\sim$. From now on, let E denote any non-empty finite set of stochastically indifferent lotteries in $\Delta(Y)$.

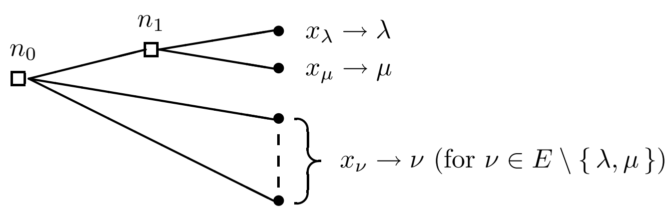

Figure 2 Decision Tree Illustrating Superstrong Transitivity

The next stage of the argument considers essentially the same decision tree as that used in Section 5.6 of Chapter 5 to prove that consequentialism implies ordinality. Here consequentialism requires that the probability $q(\lambda, E)$ of choosing λ from E should be the same whether the agent faces the tree illustrated in Figure 2, or else is forced to make one single decision (in a trivial tree, with only one chance node, and one terminal node for each member of E).

In the decision tree of Figure 2, consequentialism implies that the probability of moving from n_0 to n_1 is $q(\lambda, E) + q(\mu, E)$, equal to the probability of choosing one of the two lotteries λ and μ from the set E. Given this earlier choice, the conditional probability of choosing λ in the subtree emanating from n_1 is $q(\lambda, \{ \lambda, \mu \})$. But consequentialist normal form invariance requires that $q(\lambda, E)$ be equal to the compound probability of choosing λ in the tree as a whole. So

$$q(\lambda, E) = q(\lambda, \{ \lambda, \mu \}) \, [q(\lambda, E) + q(\mu, E)] \tag{25}$$

Of course, this equation must be satisfied for each combination λ, μ, E with $\{ \lambda, \mu \} \subset E \subset \Delta(Y)$ and all elements of E stochastically indifferent. Similarly, replacing λ by μ gives

$$q(\mu, E) = q(\mu, \{ \lambda, \mu \}) \, [q(\lambda, E) + q(\mu, E)] \tag{26}$$

At this point, it is helpful to introduce the notation

$$\ell(\lambda,\mu) := \frac{q(\lambda,\{\,\lambda,\mu\,\})}{q(\mu,\{\,\lambda,\mu\,\})} \in (0,1) \tag{27}$$

for the *choice likelihood ratio* between the choices λ and μ from the pair set $\{\,\lambda,\mu\,\}$, where $\lambda \sim \mu$. Obviously, (25), (26) and (27) together imply that

$$\frac{q(\lambda,E)}{q(\mu,E)} = \ell(\lambda,\mu) \tag{28}$$

for each combination λ,μ,E with $\{\,\lambda,\mu\,\} \subset E \subset \Delta(Y)$ and all elements of E stochastically indifferent. In particular, when E is the three-member set $\{\,\lambda,\mu,\nu\,\}$, (28) implies that

$$\ell(\lambda,\nu) = \frac{q(\lambda,E)}{q(\nu,E)} = \frac{q(\lambda,E)}{q(\mu,E)}\,\frac{q(\mu,E)}{q(\nu,E)} = \ell(\lambda,\mu)\,\ell(\mu,\nu) \tag{29}$$

This important property is a form of transitivity. Provided we define $\ell(\lambda,\lambda) := 1$ for all $\lambda \in \Delta(Y)$, it is also trivially valid when two or more of λ,μ,ν coincide.

Because (29) is so much stronger than most transitivity axioms considered in stochastic choice theory, it will be called *superstrong transitivity.*[12]

6.4 Luce's Model

For each stochastic indifference class $E \subset \Delta(Y)$, fix an arbitrary lottery $\lambda_E \in E$, and then define the positive-valued function f_E on E by $f_E(\lambda) := \ell(\lambda,\lambda_E)$ for all $\lambda \in E$. Let $E^*(F)$ denote the unique stochastic indifference class in $\Delta(Y)$ such that $C(F) \subset E^*(F)$, where $C(F)$ is the non-empty subset of F defined by (22). Then (28) evidently implies that $q(\lambda,F) = \alpha_F f_{E^*(F)}(\lambda)$ for all $\lambda \in C(F)$, where α_F is a suitable positive constant. Because of the definition (22) of $C(F)$, it follows that $\sum_{\lambda\in C(F)} q(\lambda,F) = 1$, and also that $q(\lambda,F) = 0$ for all $\lambda \in F \setminus C(F)$. Hence $\alpha_F = 1/\sum_{\lambda\in C(F)} f_{E^*(F)}(\lambda)$, implying that

$$q(\lambda,F) = f_{E^*(F)}(\lambda)/\sum_{\lambda'\in C(F)} f_{E^*(F)}(\lambda') \tag{30}$$

for all $\lambda \in C(F)$. Then *Luce's model* of stochastic choice is the special case that results by imposing the requirement that $q(\lambda,F) > 0$ for all $\lambda \in F \in \mathcal{F}$.[13] In this special case, one has $E^*(F) = F$ for all $F \in \mathcal{F}$, so (30) simplifies to

$$q(\lambda,F) = f(\lambda)/\sum_{\lambda'\in F} f(\lambda') \tag{31}$$

[12] Luce (1958, 1959) describes it as a "choice axiom".

[13] On p. 285 of Volume I, Fishburn offers other names, and also ascribes the basic idea to Bradley and Terry (1952).

for all $\lambda \in F$. Here one may call $f : \Delta(Y) \to \mathbb{R}$ a *stochastic utility* function. Note that (31) is invariant to transformations of f that take the form $\tilde{f}(\lambda) \equiv \rho f(\lambda)$ for a suitable mutiplicative constant $\rho > 0$. Thus, f is a positive-valued function defined up to a ratio scale.

Much econometric work on discrete choice uses the special *multinomial logit* version of Luce's model, in which $\ln f(\lambda) \equiv \beta U(\lambda)$ for a suitable *logit utility* function U on $\Delta(Y)$ and a suitable constant $\beta > 0$. McFadden (1974) proved how the associated form of $q(\lambda, F)$ corresponds to the maximization over F of a "random utility" function $\beta U(\lambda) + \epsilon(\lambda)$ in which the different errors $\epsilon(\lambda)$ $(\lambda \in F)$ are independent random variables sharing a common cumulative distribution function $\exp(-e^{-\epsilon})$.[14]

6.5 *Equilibrium*

Consider the normal form game $G = \langle I, S^I, v^I \rangle$ as in (1). Recall from (4) the notation $V_i(s_i, \pi_i)$ for the expected payoff of each player $i \in I$ from the strategy $s_i \in S_i$, given the probability beliefs $\pi_i \in \Delta(S_{-i})$ about the profile of other players' strategies. For each player $i \in I$, assume that Luce's model applies directly to the choice of strategy $s_i \in S_i$. Specifically, assume that there is a stochastic utility function of the form $f_i(s_i) = \phi_i(V_i(s_i, \pi_i))$, where the transformation $\phi_i : \mathbb{R} \to \mathbb{R}_+$ is positive-valued, strictly increasing, and continuous. In the special case of the multinomial logit model, this transformation takes the form $\phi_i(V_i) \equiv e^{\beta_i V_i}$ for a suitable constant $\beta_i > 0$. In the general case, each player has a corresponding *stochastic response function* $p_i : \Delta(S_{-i}) \to \Delta(S_i)$ satisfying $p_i(\pi_i)(s_i) := \phi_i(V_i(s_i, \pi_i)) / \sum_{s'_i \in S_i} \phi_i(V_i(s'_i, \pi_i))$ for all $\pi_i \in \Delta(S_{-i})$ and all $s_i \in S_i$.

A *quantal response equilibrium* is a profile $\hat{\mu}^I \in \prod_{i \in I} \Delta(S_i)$ of independent mixed strategies satisfying $\hat{\mu}_i(s_i) = p_i(\hat{\pi}_i)(s_i)$ for each player $i \in I$ and each strategy $s_i \in S_i$, where $\hat{\pi}_i = \hat{\mu}^{I \setminus \{i\}} = \prod_{h \in I \setminus \{i\}} \hat{\mu}_h$ as in (12). In fact, such an equilibrium must be a fixed point of the mapping $p : D \to D$ defined on the domain $D := \prod_{i \in I} \Delta(S_i)$ by $p(\mu^I)(s^I) = \langle p_i(\mu^{I \setminus \{i\}})(s_i) \rangle_{s_i \in S_i}$. When D is given the topology of the Euclidean space $\prod_{i \in I} \mathbb{R}^{S_i}$, it is easy to see that the mapping p is continuous. Because D is non-empty and convex, Brouwer's fixed point theorem can be used to prove that such an equilibrium exists.

6.6 *Strategic Choice versus Consequentialism*

Actually, consequentialism really requires a different formulation, starting with the consequentialist game form $\Gamma = \langle I, S^I, Y^I, \psi \rangle$ as in (2). For each player $i \in I$, define the *strategic outcome function* $\phi_i : S_i \times \Delta(S_{-i}) \to \Delta(Y_i)$ so that $\phi_i(s_i, \pi_i) := \sum_{s_{-i} \in S_{-i}} \pi_i(s_{-i}) \psi_i(s^I)$ is the lottery in $\Delta(Y_i)$ that results, from player i's perspective, when i plays $s_i \in S_i$ and has probabilistic beliefs

[14] See Amemiya (1981, 1985), for example, who provides a much fuller discussion of what he calls "qualitative response models". Note that the standard utility maximizing model emerges in the limit as $\beta \to \infty$.

about the other players' strategies described by $\pi_i \in \Delta(S_{-i})$. Let $\Phi(\pi_i) := \{ \phi_i(s_i, \pi_i) \mid s_i \in S_i \}$ denote the range of possible lotteries available to player i.

Next, for each $i \in I$, let $\mathcal{F}(\Delta(Y_i))$ denote the family of non-empty finite subsets of $\Delta(Y_i)$. Then, for each $F \in \mathcal{F}(\Delta(Y_i))$ and each $\lambda \in F$, let $q_i^C(\lambda, F)$ specify i's "consequentialist" stochastic choice probability for λ when i faces the feasible set F. To achieve these consequentialist stochastic choice probabilities in the game, given the beliefs $\pi_i \in \Delta(S_{-i})$ over other players' strategies, player i can choose any mixed strategy $q_i \in \Delta(S_i)$ belonging to the set $Q_i(\pi_i)$ of all such q_i which satisfy

$$q_i(\{ s_i \in S_i \mid \phi_i(s_i, \pi_i) = \lambda \}) = q_i^C(\lambda, \Phi(\pi_i))$$

for all $\lambda \in \Phi(\pi_i)$. Whenever there happen to be two different strategies $s_i', s_i'' \in S_i$ such that $\phi_i(s_i', \pi_i) = \phi_i(s_i'', \pi_i)$, the relative probabilities of s_i' and s_i'' will be indeterminate. In fact, the set $Q_i(\pi_i)$ will include a non-trivial line segment of different mixed strategies. Hence, this consequentialist approach is inconsistent with the unique stochastic choice $p_i(\pi_i)$ that emerges when Luce's model is applied directly to player i's choice from S_i, as described in Section 6.5.

6.7 *Consequentialist Stochastic Choice is Trivial*

To revert to our discussion of consequentialist normal invariance, consider next the decision tree illustrated in Figure 3, which is the same as that used in Section 6.3 of Chapter 5 to prove that consequentialism implies the independence axiom. Here, consequentialism requires that whenever λ and μ are stochastically indifferent, the choice likelihood ratio $\ell(\lambda, \mu)$ in the subtree following n_1 should satisfy

$$\ell(\lambda, \mu) = \ell(\alpha\,\lambda + (1-\alpha)\,\nu, \alpha\,\mu + (1-\alpha)\,\nu) \tag{32}$$

That is, it should be the same as the corresponding choice likelihood ratio in the trivial decision tree with only one initial decision node and with two terminal nodes leading to the random consequences $\alpha\,\lambda + (1-\alpha)\,\nu$ and $\alpha\,\mu + (1-\alpha)\,\nu$ respectively.

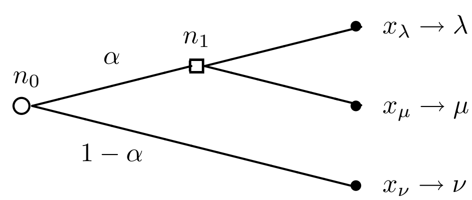

Figure 3 Decision Tree Illustrating Triviality

Now consider any stochastically indifferent pair $\lambda, \mu \in \Delta(Y)$, and take $\nu = \mu$, $\alpha = \frac{1}{2}$ in (32). The result is

$$\ell(\lambda, \mu) = \ell(\tfrac{1}{2}\,\lambda + \tfrac{1}{2}\,\mu, \mu) \tag{33}$$

Fix any $\bar{\nu} \in \Delta(Y)$ belonging to the same stochastic indifference class E as λ and μ. Because of superstrong transitivity (29), equation (33) implies that

$$\ell(\lambda, \bar{\nu}) = \ell(\lambda, \mu)\, \ell(\mu, \bar{\nu}) = \ell(\tfrac{1}{2}\,\lambda + \tfrac{1}{2}\,\mu, \mu)\, \ell(\mu, \bar{\nu}) = \ell(\tfrac{1}{2}\,\lambda + \tfrac{1}{2}\,\mu, \bar{\nu})$$

even when $\lambda = \bar{\nu}$ or $\mu = \bar{\nu}$. But the same argument with λ and μ interchanged shows that

$$\ell(\mu, \bar{\nu}) = \ell(\tfrac{1}{2}\,\lambda + \tfrac{1}{2}\,\mu, \bar{\nu}) = \ell(\lambda, \bar{\nu})$$

even when $\lambda = \bar{\nu}$ or $\mu = \bar{\nu}$. Invoking superstrong transitivity (29) once again implies that

$$\ell(\mu, \bar{\nu}) = \ell(\lambda, \bar{\nu}) = \ell(\lambda, \mu)\, \ell(\mu, \bar{\nu})$$

Because $\lambda, \mu, \bar{\nu} \in E$ and so $\ell(\mu, \bar{\nu}) > 0$, it follows that $\ell(\lambda, \mu) = 1$.

This argument is valid for any pair λ, μ in the same stochastic indifference class E of $\Delta(Y)$. It follows that, given any non-empty finite feasible set $F \subset \Delta(Y)$, if $E = C(F)$ is the top indifference class of all elements in F that meximize the ordering $\succsim$, then $q(\lambda, F) = q(\mu, E) = 1/\#E$ for all $\lambda, \mu \in E$. Thus, all elements in E are chosen with equal probability, as when the principle of insufficient reason is used to specify a probability distribution.

This argument has shown that stochastic choice which satisfies the consequentialist axioms — especially consequentialist normal form invariance — allows only a trivial extension of the expected utility framework of Chapter 5. Within the expected-utility maximizing choice set, all lotteries must receive equal probability. In particular, all lotteries that are given positive probability must have equal probability. This violates the formulation of quantal response equilibria, according to which players should give higher probability to strategies with greater expected payoff, and lower but still positive probability to strategies with lower expected payoff. It also makes each player's response correspondence discontinuous, and so rules out existence of Nash equilibrium in many games.

6.8 Assessment

The quantal response equilibria of McKelvey and Palfrey (1995) may well have better predictive power than usual Nash equilibria, or than various refinements such as proper equilibria. For this reason, it might be very sensible to advise any one player in a game to use the quantal response idea, based on stochastic utility theory, in order to attach probabilities to other players' strategies, and then to maximize expected utility accordingly. But quantal response equilibria lack consequentialist foundations. So recommended behaviour based on such advice will depend on the extensive form of the game, in general.

More seriously, perhaps, it makes no sense to recommend randomization that attaches positive probability to inferior, even disastrous strategies. This makes the stochastic utility framework unsuitable when trying to construct a normative model of two or more players' behaviour simultaneously. So stochastic utility offers no satisfactory escape from the zero probability problem, at least for normative game theory.

7 Beyond Equilibrium

7.1 *Is Equilibrium Attainable?*

Early work on equilibrium in games addressed explicitly the issue of what players should believe of each other. In particular, Morgenstern (1928, 1935) had perceived the need to determine agents' expectations in order to make economic forecasts. This seems to have been what motivated his subsequent interest in von Neumann's (1928) pioneering mathematical work on "parlour games".[15] In fact, given a Nash equilibrium profile μ^I of mixed strategies $\mu_i \in \Delta(S_i)$ $(i \in I)$ which every player finds credible, it seems reasonable for each player $i \in I$ to believe that the joint distribution of other players' strategies is given by $\pi_i = \prod_{j \in I \setminus \{i\}} \mu_j \in \Delta(S_{-i})$. For games in which Nash equilibrium is unique, players' expectations are then determined uniquely. And for the two-person strictly competitive or "zero sum" games for which von Neumann and Morgenstern were able to find a generally agreed solution, at least both players' expected utilities are uniquely determined in equilibrium, even if their expectations are not. In fact, for such games, the set E of equilibria must be *exchangeable* or "interchangeable" (Luce and Raiffa, 1957, p. 106) in the sense of being the Cartesian product $E = E_1 \times E_2$ of the sets E_1 and E_2 of equilibrium mixed strategies for both players.

Even when there are only two players, if the set of Nash equilibria is not exchangeable, then reaching equilibrium requires, in effect, that each player i know the other player j's (mixed) strategy $\mu_j \in \Delta(S_j)$. Games as simple as Battle of the Sexes illustrate how restrictive this is. Similar difficulties arise in n-person pure coordination games like those Lewis (1969) used to model conventions. These take the form $\langle I, S^I, v^I \rangle$ where each $S_i = S$, independent of i, and also each $v_i(s^I) = \bar{v}$ if $s_i = s$ for all $i \in I$, independent of both i and s; otherwise $v_i(s^I) = w(s^I) < \bar{v}$, independent of i. Thus, all players have identical payoff functions, and also payoffs are equal to $\bar{v}$ in any possible Nash equilibrium. Yet reaching one of the multiple non-exchangeable equilibria with $s_i = s$ for all $i \in I$ still requires players to have correct beliefs about what the others will do. Much worse, Bernheim (1984, 1986) argues convincingly that such knowledge of the other player's strategy can be unduly restrictive even in games with a unique Nash equilibrium that happens to involve a pure strategy for each player.

When there are three or more players, Nash equilibrium generally requires any two players to have identical expectations concerning any third player's strategy. It also requires each player i to believe that other players' strategies

[15] For recent discussion of this issue, see Leonard (1995), especially pp. 745–51.

are uncorrelated, even if i has reason to believe that there may be some hidden common cause affecting the probabilities of these strategies — for example, two other players might be identical twins. For a much more careful and extensive discussion of what players must know about each other in order to reach Nash equilibrium, see Aumann and Brandenburger (1995). See also the recent interchange between Gul (1998) and Aumann (1998).

Harsanyi and Selten (1988) sought a way around this problem by devising a theory that would select a unique Nash equilibrium in each non-cooperative game — see also van Damme (1995). It seems fair to say, however, that this part of their work has been viewed as too implausible to become generally accepted. On the other hand, Fudenberg and Levine (1998) in particular summarize a large body of work which investigates how plausible is the hypothesis that players will learn to reach Nash equilibrium. Even in the most favourable case when the same game is repeated many times, it is by no means guaranteed that the learning process will converge, although if it does converge then typically it must be to a Nash equilibrium. For more recent work on this topic, see especially Foster and Young (2001, 2003), as well as Hart and Mas-Colell (2003).

7.2 The Zero Probability Problem

More devastating than the difficulty of attaining equilibrium, however, is the fact that zero probabilities must be avoided if consequentialist normal form invariance is to be maintained. As pointed out in Section 5, this contradicts the hypothesis that inferior responses are played with zero probability. That section considered the alternative proposal that players cannot avoid trembling, as a basis for the useful notions of perfect and proper equilibrium. But, even if all other players regard i as being likely to tremble, why should i hold those beliefs about himself? Alternatively, suppose one takes the justifiable view that it is only beliefs about other players that matter. Even so, when there are three or more players, one may ask what drives players other than i all to have the same probability belief that i will tremble — for example, 0.327×10^{-12}? Or, with trembles as a function of ϵ, what makes them believe the same relative probabilities of two different trembles, given ϵ?

One alternative, of course, is to abandon the hypothesis of consequentialist normal form invariance. If one does so, however, there is no obvious justification for the orthodox view that players should have preference orderings, or cardinal utility functions, or payoffs. It is very likely that empirical game theory would do better to abandon the invariance hypothesis and all of its implications. But for normative game theory, this hypothesis is about the only secure foundation we have. So it seems worth exploring quite a bit further to see if the invariance hypothesis can be maintained after all, in some form or other.

7.3 Beyond Objective Probability

Though the notion of equilibrium has played a fundamentally important role in non-cooperative game theory, there are many situations where it seems inapplicable. These are also situations in which it seems unreasonable to postulate that players' beliefs about each other are described by objective probabilities. The main alternative, of course, would appear to be a theory based on subjective probabilities. Before immersing ourselves completely in such a theory, however, it is important to see whether one can extend to non-cooperative game theory the axioms that justify the use of subjective probabilities in decision theory — as discussed in Chapter 6 of this *Handbook*.

8 Subjectively Expected Utility in Game Theory

8.1 The Mariotti Problem

Several recent works on game theory have simply asserted that, because of the axioms of subjective expected utility (SEU) theory, players should have subjective probabilistic beliefs about each other, and then choose from their respective best response correspondences induced by those beliefs.[16] Yet it is not immediately obvious how axioms like those discussed in Chapter 6 on SEU theory can be applied to non-cooperative games. Indeed, it is really quite a troublesome issue whether each player should attach subjective probabilities to strategies under the control of other players who have their own objectives. After all, orthodox single-person decision theory attaches them only to apparently capricious moves by nature.

Now, virtually every result in decision theory requires a broad range of different single-person decision problems to be considered. This includes the consequentialist normal form invariance hypothesis set out in Section 2, which implies consequentialist behaviour in each associated single-person decision tree. Yet the results concerning such behaviour rely on being able to consider, if not necessarily a completely unrestricted domain of decision trees with a fixed set of states of the world, then at least one that is rich enough. In particular, a player i's preference between the random consequences of two different strategies is revealed by forcing i to choose between just those two strategies. When such alterations in the options available to an agent occur in a single-person decision tree, there is no reason to believe that nature's exogenous "choice" will change. But in an n-person game, changes in the feasible set faced by any one player $i \in I$ will typically lead to changes in player i's behaviour within the game, as Mariotti (1996) and Battigalli (1996) have pointed out.

As an example, consider the 2-person game in normal form whose payoff matrix is displayed in Table 2.[17] Here, player P_2's strategy b_3 strictly dominates b_1, suggesting that P_2's preference ordering over the strategy set $S_2 = \{ b_1, b_2, b_3 \}$ should make b_3 strictly preferred to b_1. Yet if b_1 is removed from P_2's strategy set, then a_2 weakly dominates a_1 for player P_1. This suggests that P_2 should

[16] See, for example, the influential articles by Aumann (1987a) and by Tan and Werlang (1988), as well as authoritative textbooks such as Myerson (1991, p. 92) or Osborne and Rubinstein (1994, p. 5), and also the work by Harsanyi cited in Section 1.

[17] This game is adapted from Mariotti (1996, Figure 6.4).

regard a_2 as much more likely than a_1, and so that P_2 should strictly prefer b_2 to b_3. On the other hand, if b_3 is removed from P_2's strategy set, then a_1 weakly dominates a_2 for player P_1. This suggests that P_2 should regard a_1 as much more likely than a_2, and so that P_2 should strictly prefer b_1 to b_2. But then putting these three conclusions together implies that P_2 has a strict preference cycle over $S_2 = \{ b_1, b_2, b_3 \}$. There is accordingly no way in which any choice by player P_2 from S_2 can be explained as maximizing a preference relation based on choices in different games when only a pair of strategies is available.

P_1 \ P_2	b_1	b_2	b_3
a_1	1, 1	1, 0	0, 2
a_2	0, 0	1, 2	1, 1

Table 2 Example Adapted from Mariotti (1996)

Instead, P_2's ordering over S_2 has to be based on choices over pairs in different extensive form games which all have the common feature that P_1 believes all three strategies are available to P_2. Figure 4 illustrates an example of such an extensive form, whose normal form is that Table 2. Note that player P_2, when at node n_1, really does face the choice between only b_2 and b_3, though P_1 regards b_1 as also possible at the ensuing information set — the only one belonging to player P_1.

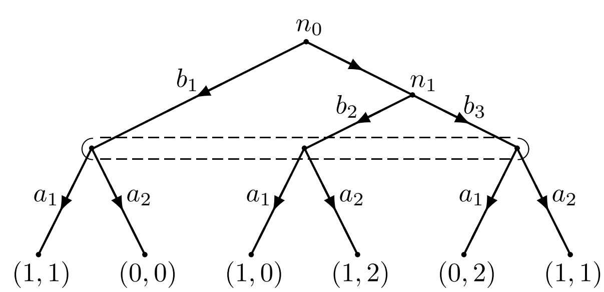

Figure 4 Extensive Game with Sequential Moves for Player P_2

Arguing as in Chapter 5, player P_2's choice from S_2 should be independent of the structure of the single-person decision tree whose terminal nodes constitute player P_1's information set. In particular, it should be independent of how many moves have to be made — one or two. This implies that there must be a preference ordering over S_2 explaining both the player's choice from S_2, as well as between two-element subsets in trees like that shown in Figure 2.

To conclude, it may be reasonable to treat nature as a passive but capricious bystander in a single-person decision problem or "game against nature". Indeed, this was the main theme in Chapter 6 of this *Handbook*. Yet in an n-person non-cooperative game, we still lack a justification for treating other active players in this way.

8.2 Battigalli's Construction

In order to surmount this difficulty, Battigalli's (1996) comment on Mariotti (1996) suggests introducing, for each player i whose subjective probabilities are to be determined, one extra player i^* who plays the role of an "external observer". In effect, this extra player is an exact copy or "behavioural clone" of player i. Following a somewhat similar idea in Nau and McCardle (1990), player i^* faces a variable opportunity to bet on how players other than i will play the original game, but is unable to affect the consequences available to all the other players, including i. Equivalently, one can ask how player i would bet if placed outside the game, being replaced by player i^* in the game.

With this useful and ingenious device, player i^* can be faced with each possible single-person decision tree in the unrestricted domain of trees where moves by nature amount to strategy choices by players other than i. This allows i^*'s subjective probabilities over strategy profiles for players other than i to be inferred. Moreover, they should apply to i^*'s behaviour when facing a single-person decision problem equivalent to that which i faces in the game itself. Because i^* is an exact copy of i, it follows that i's behaviour in the original game matches i^*'s in this equivalent single-person decision problem; in particular, i will maximize subjective expected utility using i^*'s subjective probabilities.

Using the notation defined in (2), let $i \in I$ be any player in the game form $\Gamma = \langle I, S^I, Y^I, \psi \rangle$. Following the analysis in Chapter 6, let $\mathcal{T}_i(S_{-i}, Y_i)$ denote the "almost unrestricted" domain of all allowable decision trees for player i, with every possible move at any chance node having positive probability, with $S(n_0) = S_{-i}$ as the set of states of the world which are possible at the initial node n_0, and with random consequences in $\Delta(Y_i)$ at every terminal node. In particular, each tree $T \in \mathcal{T}_i(S_{-i}, Y_i)$ should be regarded as separate from Γ, except that the possible states of nature happen to correspond exactly to other players' strategy profiles $s_{-i} \in S_{-i}$.

Given the consequentialist game form Γ, it will also be necessary to consider a family $\mathcal{G} = \{\Gamma\} \cup (\cup_{i \in I} \mathcal{G}_i)$ of game forms derived from Γ, where

$$\mathcal{G}_i = \{\, \Gamma(i, T) \mid T \in \mathcal{T}_i(S_{-i}, Y_i)\,\}$$

That is, for each player $i \in I$ and tree $T \in \mathcal{T}_i(S_{-i}, Y_i)$, there is a corresponding game form in $\mathcal{G}_i$ specified by

$$\Gamma(i, T) = \langle \{i^*\} \cup I, S^T \times S^I, Y_i \times Y^I, \tilde{\psi}_i^T \rangle$$

Because the extra player i^* is a copy of player i, player i^*'s consequence space, like i's, is Y_i. It is assumed that player i^*, as an external observer, effectively faces a single-person decision tree $T \in \mathcal{T}_i(S_{-i}, Y_i)$, in which the set of possible states of nature is S_{-i}. The set of i^*'s strategies in T is S^T, and the outcome function is denoted by $\psi^T : S^T \to \Delta(Y^{S_{-i}})$. In the game form $\Gamma(i, T)$, the value $\bar{\psi}_i^T(s^T, s^I)$ of the outcome function $\bar{\psi}_i^T : S^T \times S^I \to \Delta(Y_i \times Y^I)$ is assumed to be given by the product lottery $\psi^T(s^T) \times \psi(s^I)$ for all $(s^T, s^I) \in S^T \times S^I$. Note that, as far as all players $h \in I$ are concerned, including the particular player i of whom i^* is a copy, the outcome of $\Gamma(i, T)$ is the same as the outcome of Γ, independent of both T and also of i^*'s choice of strategy in T.

8.3 Players' Type Spaces

It may be useful to think of a game form as a book of rules, specifying what strategies players are allowed to choose, and what random consequence results from any allowable profile of strategic choices. So the family $\mathcal{G}$ of consequentialist game forms needs fleshing out with descriptions of players' preferences, beliefs, and behaviour. The Bayesian rationality hypothesis involves preferences represented by expected values of von Neumann–Morgenstern utility functions (NMUFs) attached to consequences. Also, beliefs take the the form of subjective probabilities attached jointly to combinations of other players' preferences, strategies, and beliefs. And behaviour should maximize subjectively expected utility. It has yet to be shown, however, that the consequentialist hypotheses imply such preferences, beliefs, and behaviour. To do so satisfactorily requires a framework for describing preferences, beliefs, and behaviour in game forms before the consequentialist hypotheses have been imposed. We shall postulate spaces of types similar to those considered by Harsanyi (1967–8) in his theory of games of incomplete information, as discussed in Section 4.3. However, here each player will have three separate type variables, corresponding to preferences, beliefs, and behaviour respectively.

Indeed, since one cannot directly assume that preferences exist, it is necessary to consider instead, for each player $i \in I$, a *decision type* $d_i \in D_i$ which determines what is normatively acceptable behaviour for i in any single-person finite decision tree $T \in \mathcal{T}(Y_i)$ without natural nodes that has random consequences in $\Delta(Y_i)$. Of course, consequentialist normal form invariance implies the consequentialist hypotheses for single-person decision theory. So if continuity of behaviour is added to these hypotheses, we know already that each player $i \in I$ will have a unique cardinal equivalence class of NMUFs $v_i(y_i; d_i)$ on Y parametrized by their decision type. The assumption that such a parameter d_i exists is without loss of generality because if necessary it could be one NMUF $v_i(y_i)$ in the equivalence class appropriate for i. Together, the list of all players' decision types forms a *decision type profile* $d^I \in D^I := \prod_{i \in I} D_i$.

As in orthodox equilibrium game theory, each player $i \in I$ is assumed next to have beliefs or an *epistemic type* $e_i \in E_i$, with $E^I := \prod_{i \in I} E_i$ as the set of all possible *epistemic type profiles.* It will be a result rather than an assumption of the theory that all such beliefs can be represented by subjective probabilities on an appropriately defined space. For the moment, each $e_i \in E_i$ is assumed to determine parametrically player i's *strategic behaviour* in the form of a non-empty set $\sigma_i(\Gamma', d_i, e_i) \subset S_i$ defined for every game form $\Gamma' \in \mathcal{G}$ and decision type d_i for player i. In orthodox game theory, $\sigma_i(\Gamma', d_i, e_i)$ is the set of i's "best responses" given the NMUF $v_i(y_i; d_i)$ and subjective probability beliefs over other players' strategies determined by e_i. The assumption that such a parameter e_i exists is without loss of generality because if necessary it could be the correspondence $(\Gamma', d_i) \mapsto\!\!\!\rightarrow \sigma_i$ itself. Finally, it is also necessary to define $\sigma_{i^*}(\Gamma', d_i, e_i)$ for the copy i^* of player i in every game $\Gamma' \in \mathcal{G}_i$. Note that, because i^* is a copy of i, player i^*'s behaviour depends on i's type pair (d_i, e_i), as the above notation reflects.

Maintaining the normative point of view throughout, each set $\sigma_i(\Gamma', d_i, e_i)$ already describes how i with decision type d_i and epistemic type e_i should play Γ'. However, in forming beliefs, it is not enough for player i (and also i^* if $\Gamma' \in \mathcal{G}_i$) to know the other players' sets $\sigma_j(\Gamma', d_j, e_j)$ $(j \in I \setminus \{i\})$; also relevant are the tie-breaking rules which the other players $j \in I \setminus \{i\}$ use to select one particular strategy s_j from the set $\sigma_j(\Gamma', d_j, e_j)$ whenever this set has more than one member. Accordingly, each player $i \in I$ is assumed to have in addition a *behaviour type* $b_i \in B_i$, with $B^I := \prod_{i \in I} B_i$ as the set of all possible *behaviour type profiles*. Each $b_i \in B_i$ is assumed to determine parametrically player i's *selection rule* yielding a single member $s_i(\Gamma', d_i, e_i, b_i) \in \sigma_i(\Gamma', d_i, e_i)$ of each strategic behaviour set. The assumption that b_i exists is without loss of generality because it could be the function $(\Gamma', d_i, e_i) \mapsto s_i$ itself. Note that player i^*'s behaviour type need not be specified because i^*'s behaviour has no effect on any other player.

To simplify notation in future, define for each player $i \in I$ a combined type space $\Theta_i := D_i \times E_i \times B_i$, whose members are triples $\theta_i := (d_i, e_i, b_i)$. Note that each player's selection rule can then be expressed as $s_i(\Gamma', \theta_i)$. Let $\Theta^I := D^I \times E^I \times B^I$ be the space of combined type profiles, with typical member $\theta^I := (d^I, e^I, b^I)$, and let $\Theta_{-i} := \prod_{j \in I \setminus \{i\}} \Theta_i$ denote the set of all possible types for players other than i. A complete epistemic type $e_i \in E_i$ should then describe in full player i's beliefs about the other players' types $\theta_{-i} \in \Theta_{-i}$, including their epistemic types e_{-i}. This creates a problem of circularity or infinite regress which is an inevitable and fundamental part of modern game theory. A possible resolution is the subject of Section 10.

8.4 Subjective Expectations

First, given any game $\Gamma' = \Gamma(i, T) \in \mathcal{G}_i$, consider an extensive form in which player i^* moves first, before any player $j \in I$. Later these players must move without knowing what i^* has chosen. In this extensive form, after i^* has moved, Γ is effectively a subgame of incomplete information. Now, given any player $j \in I$, and any combined type $\theta_j \in \Theta_j$ which player j may have, applying an obvious dynamic consistency hypothesis to the subgame Γ of $\Gamma' = \Gamma(i, T)$ yields the result that

$$\sigma_j(\Gamma', d_j, e_j) = \sigma_j(\Gamma, d_j, e_j) \quad \text{and} \quad s_j(\Gamma', \theta_j) = s_j(\Gamma, \theta_j)$$

In particular, for each $j \in I$, both $\sigma_j(\Gamma', d_j, e_j)$ and $s_j(\Gamma', \theta_j)$ are effectively independent of whatever player $i \in I$ is copied and of whatever tree $T \in \mathcal{T}_i(S_{-i}, Y_i)$ is given to the copy i^* of player i. So variations in i^*'s decision tree within the domain $\mathcal{T}_i(S_{-i}, Y_i)$ are possible without inducing changes in the behaviour of other players $j \in I$. This justifies applying the consequentialist and continuous behaviour hypotheses to the whole domain $\mathcal{T}_i(S_{-i}, Y_i)$ of single-person decision trees that player i^* may face, while treating each $s_{-i} \in S_{-i}$ as a state of nature determined entirely outside the tree. So the usual arguments imply the existence of unique and strictly positive subjective probabilities $P_i(s_{-i})$ $(s_{-i} \in S_{-i})$ such that behaviour in trees $T \in \mathcal{T}_i(S_{-i}, Y_i)$ maximizes the subjectively expected value of a von Neumann–Morgenstern utility function $v_i(y_i; d_i)$ parametrized by i's decision type $d_i \in D_i$.

It remains to consider player i's behaviour in the game form Γ itself. To do so, consider the special decision tree $T_i^\Gamma \in \mathcal{T}_i(S_{-i}, Y_i)$ in which the set of i^*'s strategies is S_i, equal to i's strategy set in Γ, and the outcome function ψ_i^Γ from S_i to the set $\Delta(Y_i^{S_{-i}})$ of lotteries over $Y_i^{S_{-i}}$, the Cartesian product of $\#S_i$ copies of player i's consequence domain Y_i, is given by the product lottery

$$\psi_i^\Gamma(s_i)(y^{S_{-i}}) = \prod_{s_{-i} \in S_{-i}} \psi(s_i, s_{-i})(y_{s_{-i}})$$

for all $y^{S_{-i}} = \langle y_{s_{-i}} \rangle_{s_{-i} \in S_{-i}} \in Y_i^{S_{-i}}$. Then, at least under Anscombe and Aumann's reversal of order axiom which was discussed in Chapter 6, both the strategy set S_i and the outcome function ψ_i^Γ are exactly the same as in Γ itself. In this case T_i^Γ and Γ are consequentially equivalent from i's (or i^*'s) point of view, so consequentialism requires i's behaviour in Γ to match that of i^* in T_i^Γ or $\Gamma(i, T_i^\Gamma)$. This implies that $\sigma_{i^*}(\Gamma(i, T_i^\Gamma), d_i, e_i) = \sigma_i(\Gamma, d_i, e_i)$. Therefore player i should choose $s_i \in S_i$ to maximize subjectively expected utility based on the subjective probabilities $P_i(s_{-i})$ $(s_{-i} \in S_{-i})$ that are appropriate for all decision trees in $\mathcal{T}_i(S_{-i}, Y_i)$. Really, one should write these probabilities as $P_i(s_{-i}, e_i)$ to indicate that they represent player i's epistemic type and so characterize i's acceptable behaviour sets $\sigma_i(\Gamma', d_i, e_i)$ on the domain $\mathcal{G}$ of game forms Γ', including Γ itself.

So, by applying them in a suitable context, the axioms presented earlier in Chapter 6 *can* be used to justify the claim that each player's behaviour should conform with the SEU model, just as most game theorists have always asserted. The same axioms also justify the specification (1) of a game in normal form, with "payoff" functions $v^I : S^I \to \mathbb{R}^I$ that are really NMUFs. There is the obvious qualification that each player's payoff function or NMUF is only determined up to a unique cardinal equivalence class. Clearly this is unimportant, because transformations within this class have no effect on the different players' best response correspondences $B_i(\pi_i)$ — or, in games with incomplete information, on their type-dependent best response correspondences $B_i(\pi_i; t_i)$. Note, however, that the framework used here differs from that of Börgers (1993), whose assumptions do not allow decision problems with objective probabilities to be considered, and so yield only a unique ordinal equivalence class of utility functions.

Another requirement is that each player i's SEU model include a unique specification of appropriate subjective probabilities $P_i(s_{-i})$ over other players' strategy profiles in the game form Γ. Failure to specify these probabilities leaves the description of the players' decision models fundamentally incomplete. Yet specifying them arbitrarily ignores the fact that, in the end, other players' strategies are really not like states of nature, because other players face their own individual decision problems in the game, which they try to resolve rationally, at least to some extent. This tension is precisely the challenge that non-cooperative game theory must meet.

A more serious worry is that, as explained in the chapter on SEU theory, the consequentialist hypothesis actually implies the SEU* model, with null events excluded. In particular, this implies that each player i's subjective probability attached to *any* strategy profile $s_{-i} \in S_{-i}$ must be strictly positive. This contradicts much of orthodox game theory, where each player is required to attach zero probability to the event that one or more other players choose strategies which are not best responses. In particular, the probability of any player choosing a strictly dominated strategy must be zero. This topic will be taken up again in Section 11.

Given any game in extensive form, any player i in that game effectively faces a one-person game in which some moves are made, not by nature, but by other players. Also, unlike the decision trees analysed in the two earlier chapters on expected utility theory, there may be non-trivial information sets rather than decision nodes. But then it is fairly easy to extend the dynamic programming arguments in these previous chapters to cover such one-person games. So part (2) of Theorem 5 in Chapter 6 remains valid, implying that subjective expected utility maximizing behaviour does satisfy the consequentialist and other axioms set out in that chapter.

8.5 Arbitrage Choice Theory

This section concludes with a brief discussion of the very different "arbitrage choice theory" due to Nau and McCardle (1990, 1991) as well as Nau (1999).

To quote from the 1990 paper (p. 445): "The central idea is an extension to the multi-player setting of de Finetti's operational criterion of rationality, namely that choices under uncertainty should be *coherent* in the sense of not presenting opportunities for arbitrage ('Dutch books') to an outside observer who serves as betting opponent. That is, a rational individual should not let himself be used as a money pump."

Further (p. 446): "Players who subscribe to the standard of joint coherence are those who do not let themselves be used *collectively* as a money pump. Our result is that a strategy is jointly coherent if and only if it occurs with positive probability in some correlated equilibrium."

So, like Battigalli's construction, some outsider to the game is making gambles on the side. But the concept of rationality is quite different from consequentialism and, in a multi-person context, seems to require the players to coordinate in order to avoid exploitation by an outsider. In Battigalli's construction, on the other hand, the outsider is a behavioural clone of any one of the players. Also, that construction presumes no coordination whatsoever, and reaches only the weaker conclusion that all players should have subjective probability beliefs and maximize subjective expected utility.

9 Rationalizable Expectations

9.1 Rationalizable Strategies

As discussed in Section 5.5, the probabilities that lie behind the usual Nash equilibrium concept of non-cooperative game theory must be objective, at least implicitly. Similarly for the refinements of Nash equilibria, or for correlated equilibria. Then Section 7 argued that such apparent objectivity lacks intuitive appeal in many non-cooperative games. With the apparent intention of escaping from these somewhat implausible Nash constraints on expectations, Bernheim (1984) and Pearce (1984) independently proposed an entirely novel approach to non-cooperative game theory. To do so, they defined sets $\mathrm{Rat}_i \subset S_i$ of *rationalizable strategies* as the largest family of sets $Z_i \subset S_i$ which together satisfy

$$Z_i = B_i(\Delta(Z_{-i})) \tag{34}$$

for all players $i \in I$.[18]

Let $\mathrm{Rat}_{-i} \subset S_{-i}$ denote the Cartesian product set $\prod_{h \in I \setminus \{i\}} \mathrm{Rat}_h$. Then the associated sets $\Delta(\mathrm{Rat}_{-i})$ are *rationalizable expectations.* In other words, each set Rat_i consists of the entire range of best responses to rationalizable expectations which attach probability one to the event that all players other than i choose a rationalizable strategy profile $s_{-i} \in \mathrm{Rat}_{-i}$; moreover, Rat_i is the largest set with this property.

The sets Rat_i $(i \in I)$ are well defined, non-empty, and can be constructed iteratively, starting with $Z_i^0 := S_i$, then letting $Z_i^k := B_i(\Delta(Z_{-i}^{k-1}))$ $(k = 1, 2, \ldots)$. Indeed, because $Z_i^1 \subset Z_i^0$ (all $i \in I$), it is easy to prove by induction that $\emptyset \neq Z_i^k \subset Z_i^{k-1}$ $(k = 1, 2, \ldots)$. So one can finally define the limit set $\mathrm{Rat}_i := \cap_{k=1}^{\infty} Z_i^k$. In fact, because each player i's strategy set S_i is assumed to be finite, the construction converges after a finite number of iterations — i.e., there exists a finite k, independent of i, such that $\emptyset \neq \mathrm{Rat}_i = Z_i^k \subset Z_i^{k+1}$ (all $i \in I$). From this it is easy to see that $\emptyset \neq \mathrm{Rat}_i = B_i(\Delta(\mathrm{Rat}_{-i}))$ and also that, whenever $Z_i \subset \mathrm{Rat}_i$ (all $i \in I$), then $B_i(\Delta(Z_{-i})) \subset \mathrm{Rat}_i$. Therefore the

[18]Strictly speaking, in games with more than two players, this extends the original definition of rationalizability by allowing each player $i \in I$ to have correlated expectations regarding the other players' choice of strategy profile $s_{-i} \in S_{-i}$. This extension seems on the way to becoming generally accepted. For example, in their textbook Osborne and Rubinstein (1994, ch. 4) mention only in the notes at the end that their definition involves such an extension.

sets Rat_i $(i \in I)$ do indeed form the largest family satisfying (34), and are non-empty.

In fact, it is not difficult to show that similar non-empty sets of rationalizable strategies exist even when the sets S_i are not all finite, provided they are compact in a topology that makes all the payoff functions v_i continuous. This is just an implication of the finite intersection property of compact sets.

Note also that each set Rat_i is large enough to include all "Nash strategies" — i.e., all strategies which i plays with positive probability in some Nash equilibrium. But Rat_i may well be larger than i's set of Nash strategies, as Bernheim's (1984, pp. 1024–5) Cournot oligopoly example clearly shows.

Indeed, each set Rat_i is large enough to include all strategies which i plays with positive probability in some correlated equilibrium. To prove this, let $\bar{\mu} \in \Delta(S^I)$ be any correlated equilibrium, and for each $i \in I$, define

$$\bar{S}_i := \{\, s_i \in S_i \mid \bar{\mu}_i(\{s_i\} \times S_{-i}) > 0 \,\}$$

as the set of strategies which i plays with positive probability in equilibrium. By definition of correlated equilibrium, if $s_i \in \bar{S}_i$, it must be because $s_i \in B_i(\pi_i)$ where $\pi_i = \bar{\mu}(\cdot|s_i)$ is the induced conditional distribution on S_{-i} given s_i. But $\bar{\mu}(s_{-i}|s_i) > 0$ only if $\bar{\mu}(s_i, s_{-i}) > 0$, which implies that $s_h \in \bar{S}_h$ for all $h \in I \setminus \{i\}$. So π_i is in the set $\Delta(\bar{S}_{-i})$ of distributions attaching probability one to the set $\bar{S}_{-i} := \prod_{h \in I \setminus \{i\}} \bar{S}_h$. It follows that $s_i \in B_i(\Delta(\bar{S}_{-i}))$ and so, because s_i was an arbitrary strategy in $\bar{S}_i$, that $\bar{S}_i \subset B_i(\Delta(\bar{S}_{-i}))$. The final step in the argument is an induction proof, beginning with the trivial observation that $\bar{S}_i \subset S_i = Z_i^0$ for each $i \in I$. As the induction hypothesis, suppose that $\bar{S}_i \subset Z_i^{k-1}$ (all $i \in I$) for some $k = 1, 2, \ldots$. Then $\bar{S}_{-i} \subset Z_{-i}^{k-1}$, so

$$\bar{S}_i \subset B_i\left(\Delta(\bar{S}_{-i})\right) \subset B_i\left(\Delta(Z_{-i}^{k-1})\right) = Z_i^k$$

Thus $\bar{S}_i \subset Z_i^k$ for $k = 1, 2, \ldots$, by induction on k, and so $\bar{S}_i \subset \cap_{k=0}^{\infty} Z_i^k = \mathrm{Rat}_i$, as required.

9.2 Iterated Removal of Strictly Dominated Strategies

Given collections of strategies $Z_i \subset S_i$ $(i \in I)$, say that i's strategy $s_i \in Z_i$ is *strictly dominated relative to* $Z^I = \prod_{i \in I} Z_i$ if there exists a mixed strategy $\mu_i \in \Delta(Z_i)$ such that

$$\sum_{s_i' \in Z_i} \mu_i(s_i')\, v_i(s_i', s_{-i}) > v_i(s_i, s_{-i})$$

for all other players' strategy profiles $s_{-i} \in Z_{-i}$. And say that i's strategy $s_i \in Z_i$ is *weakly dominated relative to* Z^I if there exists a mixed strategy $\mu_i \in \Delta(Z_i)$ such that

$$\sum_{s_i' \in Z_i} \mu_i(s_i')\, v_i(s_i', s_{-i}) \geq v_i(s_i, s_{-i})$$

for all other players' strategy profiles $s_{-i} \in Z_{-i}$, with strict inequality for at least one such profile.

Now the results of Sections 3.4 and 3.5 can be applied to the game in which each player $i \in I$ is restricted to choosing a strategy s_i from Z_i instead of S_i. The first implication is that $s_i \in Z_i$ is a best response among strategies in the set Z_i to some expectations $\pi_i \in \Delta(Z_{-i})$ if and only if s_i is not strictly dominated relative to Z^I. The second implication is that $s_i \in Z_i$ is a cautious best response among strategies in the set Z_i to some expectations $\pi_i \in \Delta^0(Z_{-i})$ if and only if s_i is not weakly dominated relative to Z^I.

The first of these two results implies that constructing the sets Rat_i is equivalent to iteratively removing strictly dominated strategies for each player. This property makes it relatively easy to compute the relevant sets.

As a necessary condition for rationality, rationalizability seems quite appealing. It completely resolves some well known games such as finitely repeated Prisoner's Dilemma, but places no restrictions at all on players' behaviour or beliefs in others such as Battle of the Sexes or pure coordination games.

9.3 Strictly Rationalizable Strategies

In Section 8 it was claimed that each player i should attach subjective probabilistic beliefs π_i to other players' strategy profiles s_{-i}, and then maximize expected utility accordingly. But a major complication arises: the consequentialist hypotheses require that all subjective probabilities be strictly positive. At first, therefore, it seems that i's beliefs π_i must belong to the set $\Delta^0(S_{-i})$ consisting of probability distributions π_i that are in the interior of $\Delta(S_{-i})$ because they satisfy $\pi_i(s_{-i}) > 0$ for all $s_{-i} \in S_{-i}$. In games where some strictly dominated strategies can be eliminated, this contradicts the previous suggestion that expectations be rationalizable — i.e., that $\pi_i \in \Delta(\mathrm{Rat}_{-i})$.

At this stage, following Börgers and Samuelson (1992, p. 20), it might be tempting to look for "strictly rationalizable" strategies in the largest sets $Z_i \subset S_i$ $(i \in I)$ which satisfy

$$Z_i = B_i(\Delta^0(Z_{-i})) \tag{35}$$

instead of (34). Hence, every player must attach positive probability to every profile of the other players' strictly rationalizable strategies. This suggests that one should adapt the previous iterative procedure for finding the sets of rationalizable strategies. Rather than taking $Z_i^k = B_i(\Delta(Z_{-i}^{k-1}))$ at each step $k = 1, 2, \ldots$, try $\hat{Z}_i^k = B_i(\Delta^0(\hat{Z}_{-i}^{k-1}))$ instead. However, an immediate problem arises from the awkward fact that $\Delta^0(\hat{Z}_{-i}^k)$ and $\Delta^0(\hat{Z}_{-i}^{k-1})$ will be disjoint subsets of $\Delta(S_{-i})$ whenever $\hat{Z}_{-i}^k$ is a proper subset of $\hat{Z}_{-i}^{k-1}$. Therefore, it may not be true that $\hat{Z}_i^{k+1} \subset \hat{Z}_i^k$. To ensure that the constructed sets really are nested, so that each player's rationalizable beliefs and strategies become more restricted at each stage, the construction really needs amending to

$$\bar{Z}_i^k = B_i(\Delta^0(\bar{Z}_{-i}^{k-1})) \cap \bar{Z}_i^{k-1} \tag{36}$$

Now let us apply the result of Section 3.5 to a game in which each player $i \in I$ is restricted to the strategy set $\bar{Z}_i^{k-1}$. The implication is that $s_i \in B_i(\Delta^0(\bar{Z}_{-i}^{k-1}))$ if and only if s_i is not weakly dominated when other players are restricted to playing a strategy profile $s_{-i} \in \bar{Z}_{-i}^{k-1}$. Also, it turns out that (36) makes $\bar{Z}_i^k$ equal to the set of strategies s_i which are best responses to expectations $\pi_i \in \Delta^0(\bar{Z}_{-i}^{k-1})$ subject to the constraint that $s_i \in \bar{Z}_i^{k-1}$ — see Hammond (1993, pp. 286–8). So each limit set $Z_i^* := \cap_{k=1}^\infty \bar{Z}_i^k$ $(i \in I)$ is well-defined and equal to the set of strategies that remain after iteratively deleting all weakly dominated strategies for all players — a procedure that extends to general normal form games an idea going back to the work of Gale (1953), followed by Luce and Raiffa (1957, pp. 108–9), and also Farquharson (1969, pp. 74–75).[19] However, (36) implies only that these limit sets satisfy $Z_i^* = B_i(\Delta^0(Z_{-i}^*)) \cap Z_{-i}^*$. In particular, these sets need not satisfy (35). Furthermore, the precise decision-theoretic foundations of this iterative procedure remain far from clear, despite the recent work by Stahl (1995), Veronesi (1997), and Brandenburger and Keisler (2002), amongst others.

9.4 The Centipede Game

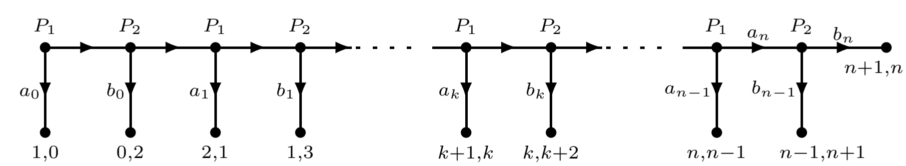

Figure 5 Rosenthal's Centipede Game, Modified

Iterative deletion of weakly dominated strategies can lead to controversial, almost paradoxical, conclusions. One famous example arises in a variation of what Binmore (1987) calls the "centipede game," which is due to Rosenthal (1981, Example 3). This is illustrated in Figure 5 (see also Osborne and Rubinstein, 1994). There are two players labelled P_1 and P_2, with respective strategy sets

$$S_1 = \{ a_i \mid i = 0, 1, \ldots, n \}, \quad S_2 = \{ b_j \mid j = 0, 1, \ldots, n \}$$

If P_1 chooses $a_i \in S_1$, then unless P_2 ends the game beforehand, P_1 moves across in the tree exactly i successive times. Similarly, if P_2 chooses $b_j \in S_2$, then unless P_1 ends the game beforehand, P_2 moves across exactly j successive times. The game ends immediately once either player has moved down. If $i = j = n$, it ends anyway after P_2 has chosen b_n. Outside this case, if the two

[19] As in Section 12, Farquharson considers iterative deletion only of strategies that are weakly dominated by other pure strategies, thus retaining strategies that are weakly (or even strictly) dominated by mixed strategies only.

players choose (a_i, b_j) where $i = k \leq j$, then the game ends when P_1 moves down after they have both moved across k times. But if $i > k = j$, then P_1 moves across $k + 1$ times but P_2 only k times before moving down to end the game. Hence, the two players' respective payoffs are

$$v_1(a_i, b_j) = \begin{cases} i+1 & \text{if } i \leq j \\ j & \text{if } i > j \end{cases} \quad \text{and} \quad v_2(a_i, b_j) = \begin{cases} i & \text{if } i \leq j \\ j+2 & \text{if } i > j \end{cases}$$

It is now easy to see how iterative deletion of weakly dominated strategies proceeds. First P_2's strategy b_n is removed because it is weakly dominated by b_{n-1}. Then P_1's strategy a_n is removed because it is weakly dominated by a_{n-1} once b_n has been eliminated. But then, by backward induction on k, for $k = n-1, \ldots, 1$, each successive b_k is weakly dominated by b_{k-1} once $a_n, \ldots, a_{k+1}$ have all been eliminated. Similarly, for $k = n-1, \ldots, 1$, each successive a_k is weakly dominated by a_{k-1} once $b_n, \ldots, b_k$ have all been eliminated. Therefore, by backward induction, the iterative procedure successively deletes all strategies except P_2's strategy b_0 and then P_1's strategy a_0. The only remaining strategy profile (a_0, b_0) is a Nash equilibrium, of course, though there are other Nash equilibria as well.

Starting with Rosenthal (1981) and Binmore (1987), several game theorists have found this backward induction argument unconvincing, for the following reason. Suppose P_1 were unexpectedly faced with the opportunity to play a_k after all, because neither player has yet played down, and in fact each player has played across k times already. Backward induction applied to the remaining subtree leads to the conclusion that P_2, if given the move, will play b_k next time, so P_1 should play a_k. Yet P_2 has already played across k times, whereas backward induction implies that P_2 should move across whenever there is a move to make. So, as Binmore in particular argues most persuasively, if k is large enough, P_1 has every reason to doubt whether the backward induction argument applies to P_2's behaviour after all. Furthermore, if $n-k$ is also large, there may be much to gain, and at most 1 unit of payoff to lose, from allowing the game to continue by moving across instead of playing a_k.

Of course, P_2 can then apply a similar argument when faced with the choice between b_k and continuing the game. Also, P_1 should understand how moving across once more instead of playing a_k will reinforce P_2's doubt about whether the backward induction argument applies to P_1, and so make it more likely that P_2 will decline to play b_k. This strengthens P_1's reasons for not playing a_k. Similar reasoning then suggests that P_2 should not play b_{k-1}, that P_1 should not play a_{k-1}, etc. It may be sensible in the end for P_1 not to play a_0, for P_2 not to play b_0, etc. Indeed, there are some obvious close parallels between this idea and the reputation arguments of Kreps *et al.* (1982) for players first to cooperate and then play tit-for-tat in the early stages of a finitely repeated Prisoner's Dilemma. Or for a chain-store to play "tough" in the paradox due to Selten (1978). These parallels are thoroughly discussed in Kreps (1990, pp. 537–542). See also Fudenberg and Tirole (1991, Ch. 9) for a general discussion of reputation in games.

This argument strongly suggests that a theory of individual behaviour in n-person games is too restrictive if it eliminates all weakly dominated strategies iteratively without considering most of the details lying behind players' rationalizable expectations. This view is supported by theoretical work such as Dekel and Fudenberg (1990), Ben-Porath and Dekel (1992), and Ben-Porath (1997).[20]

10 Hierarchies of Beliefs

10.1 Rationalizable Types

In Sections 8 and 9, the formulation of each player i's decision problem treated other players' strategy profiles s_{-i} as states of nature. Yet these strategy profiles, if they consist of rationalizable strategies for each player, must be those other players' respective best responses, given their individual beliefs. And really all these other players' beliefs $\pi_h \in \Delta(S_{-h})$ $(h \in I \setminus \{i\})$ are also unknown to player i. As are their beliefs about other players' beliefs, etc. In fact, unless the players all know that a particular equilibrium will be played, and know that the others know it, ... , they are effectively in a game of incomplete information, as described in Section 4.3.

Accordingly, given the game Γ as in (1), there are corresponding *type spaces* T_i $(i \in I)$. Each type is a pair $t_i = (s_i, \theta_i) \in S_i \times \Theta_i$ consisting of a strategy $s_i \in S_i$ combined with a belief in the form of a probability distribution or measure θ_i over the profile $t_{-i} \in T_{-i} = \prod_{h \in I \setminus \{i\}} T_h$ of other players' types. There is a corresponding marginal distribution $\pi_i = \text{marg}_{S_{-i}} \theta_i$ over other players' strategy profiles $s_{-i} \in S_{-i}$. A type (s_i, θ_i) is said to be *rationalizable* if $s_i \in B_i(\text{marg}_{S_{-i}} \theta_i)$ and θ_i attaches probability one to the event that all other players' types are rationalizable.

The above definitions of type space and of rationalizable type, like the complete epistemic types described in Section 8.3, are circular. This is because types include descriptions of beliefs about other players' types, and rationalizable types of each player are almost sure that other all players' types are rationalizable. Of course, a very similar circularity arises in Harsanyi's (1967–8) definition (13) of a game of incomplete information. Some time after Harsanyi, several game theorists set out to show how such circularities can be resolved without any logical inconsistency. They typically did so by constructing for each player an infinite hierarchy of successively richer type spaces, whose respective limits have the desired circularity property. See for instance Armbruster and

[20] For some recent lively and enlightening discussions, especially of backward induction, see Aumann (1995, 1996a, b), Binmore (1996), Binmore and Samuelson (1996), and Nagel (1995), as well as Feinberg (2002).

Finally, Vasin (1999) presents some interesting results on versions of the folk theorem for repeated games with perturbed payoffs, using rationalizability. It remains to be seen how many similar results hold for more general games that are usually solved by backward induction.

Böge (1979), Böge and Eisele (1979), Mertens and Zamir (1985), and Brandenburger and Dekel (1993).

10.2 Mathematical Preliminaries

The standard construction which follows relies on some mathematical concepts concerning Polish spaces, the topology of weak convergence of measures, and the Prohorov metric corresponding to this topology. These concepts are briefly reviewed below. Chapter 5 contains some discussion of the relevant concepts in measure theory.

A *metric space* (X, d) consists of a set X together with a *metric* $d : X \times X \to \mathbb{R}_+$ satisfying the conditions: (i) $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0 \iff x = y$; (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$; (iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (the *triangle inequality*). The space is said to be *complete* if the sequence $(x_n)_{n=1}^\infty$ in X converges whenever it is a *Cauchy sequence* — i.e., whenever for each $\epsilon > 0$ there exists N_ϵ such that $d(x_m, x_n) < \epsilon$ for all $m, n > N$. A metric space (X, d) is said to be *separable* if there is a countable set $\{ x_n \mid n = 1, 2, \ldots \}$ in X whose closure is the whole of X.

A *Polish space* is defined to be a complete separable metric space equipped with its *Borel σ-algebra* — i.e., the smallest σ-algebra generated by the sets which are open in its metric topology. It should be noted that any finite or even a countably infinite Cartesian product of Polish spaces is also a Polish space — see, for example, Bertsekas and Shreve (1978).

Two Polish spaces X_1, X_2, together with their associated metrics, are said to be *isomorphic* if they are homeomorphic as topological spaces — i.e., if there exists a continuous mapping ψ from X_1 onto X_2 which has a continuous inverse $\psi^{-1} : X_2 \to X_1$. This mapping is called an isomorphism rather than a homeomorphism because continuous mappings are measurable w.r.t. the Borel σ-algebras — i.e., the inverse image of any measurable subset of the range is a measurable subset of the domain — so ψ actually makes the Borel σ-algebras of the two sets isomorphic, as well as their topologies.

Given any Polish space X with Borel σ-algebra $\mathcal{B}$, the set $\Delta(X, \mathcal{B})$ of all Borel probability measures on the measurable space $(X, \mathcal{B})$ is also a Polish space provided that it is given the *topology of weak convergence of probability measures*. This topology corresponds to the *Prohorov metric* ρ, according to which the distance between any pair of probability measures $\mu, \nu \in \Delta(X, \mathcal{B})$ is

$$\begin{aligned} \rho(\mu, \nu) := \inf_\epsilon \big\{ \epsilon > 0 \mid \forall E \in \mathcal{B}(X) : \mu(E) \quad &\leq \quad \nu(N_\epsilon(E)) + \epsilon \\ \text{and} \quad \nu(E) \quad &\leq \quad \mu(N_\epsilon(E)) + \epsilon \big\}, \end{aligned}$$

where $N_\epsilon(E)$ denotes the set of all points in X which are within a distance $\epsilon > 0$ of points in E. This topology of weak convergence of probability measures derives its name from the property that a sequence of measures $(\mu_n)_{n=1}^\infty$ in $\Delta(X, \mathcal{B})$ converges to the limit $\mu \in \Delta(X, \mathcal{B})$ if and only if, for every bounded continuous function $f : X \to \mathbb{R}$, the expected value $\int_X f(x)\, \mu_n(dx)$ of f with respect to the probability measure μ_n converges in $\mathbb{R}$ to the expected value $\int_X f(x)\,\mu(dx)$ of f with respect to the probability measure μ.

A result of fundamental importance is that $\Delta(X, \mathcal{B})$ is compact and Polish whenever X is a compact Polish space. See, for example, Parthasarathy (1967) or Aliprantis and Border (1999).

10.3 A Sequence of Type Spaces

The standard construction of the hierarchy whose limit is the space of rationalizable types begins with the finite sets $T_i^0 := S_i$ $(i \in I)$. Hence each player's type of order zero $t_i^0 \in T_i^0$ is equivalent to a possible choice of strategy. The construction continues at the next step with the first-order type spaces

$$T_i^1 := \{\, (s_i, \theta_i^0) \in S_i \times \Delta(S_{-i}) \mid s_i \in B_i(\theta_i^0) \,\} \subset S_i \times \Delta(T_{-i}^0)$$

where $\Delta(S_{-i})$ denotes the set of probability distributions θ_i^0 on the finite set $S_{-i} = T_{-i}^0$. Thus, T_i^1 is the graph of player i's best response correspondence. Note that each T_i^1 is the Cartesian product of a finite set S_i with a space of probability measures on a finite set S_{-i}. By choosing the Prohorov metric on $\Delta(S_{-i})$, which induces the topology of weak convergence, this product space can be made compact and Polish. The same is true of the Cartesian product $T_{-i}^1 = \prod_{j \in I \setminus \{i\}} T_j^1$. It follows that there is a well-defined space $\Delta(T_{-i}^1)$ of Borel probability measures on T_{-i}^1 which is also a compact Polish space.

Next in the hierarchy comes the second-order type space

$$T_i^2 := \{\, (s_i, \theta_i^0, \theta_i^1) \in T_i^1 \times \Delta(T_{-i}^1) \mid \theta_i^0 = \operatorname{marg}_{T_{-i}^0} \theta_i^1 \,\} \tag{37}$$

Thus, each member of T_i^2 consists of a probabilistic belief $\theta_i^1 \in \Delta(T_{-i}^1)$ concerning the profile $t_{-i}^1 \in T_{-i}^1$ of other players' first-order types, together with the induced marginal belief $\theta_i^0 = \operatorname{marg}_{S_{-i}} \theta_i^1 \in \Delta(S_{-i})$ concerning other players' zero-order types, and a best response $s_i \in B_i(\theta_i^0)$ to this marginal belief. Because $\theta_i^1 \in \Delta(T_{-i}^1)$, this belief attaches probability one to the event that all other players $h \in I \setminus \{i\}$ choose best responses $s_h \in B_h(\theta_h^0)$ to what i expects their marginal beliefs $\theta_h^0 \in \Delta(S_{-h})$ to be. As pointed out at the end of Section 10.2, the space of probability measures on any compact Polish space is compact and Polish when given the topology generated by the Prohorov metric. It follows that each set T_i^2 is a compact Polish space, as a closed subset of the product of two compact Polish spaces.

For each subsequent $k > 2$, define the kth order type space

$$T_i^k := \{\, (s_i, \theta_i^0, \theta_i^1, \ldots, \theta_i^{k-1}) \in T_i^1 \times \prod_{r=1}^{k-1} \Delta(T_{-i}^r) \mid \theta_i^{r-1} = \operatorname{marg}_{T_{-i}^{r-1}} \theta_i^r \ (r = 1, 2, \ldots, k-1) \,\}$$

as the natural extension of (37). Each $t_i^k \in T_i^k$ therefore consists of a belief $\theta_i^{k-1} \in \Delta(T_{-i}^{k-1})$ over other players' types of order $k-1$, together with the induced hierarchy $(\theta_i^0, \theta_i^1, \ldots, \theta_i^{k-2})$ of marginal lower order beliefs which satisfy $\theta_i^r = \operatorname{marg}_{T_{-i}^r} \theta_i^{k-1}$ $(r = 0, 1, \ldots, k-2)$, and finally a best response $s_i \in B_i(\theta_i^0)$ to the induced belief $\theta_i^0 = \operatorname{marg}_{S_{-i}} \theta_i^{k-1}$. Once again, each T_i^k is a compact Polish space, as a closed subset of the Cartesian product of a finite collection of compact Polish spaces.

There is an important relationship between the construction of the type spaces T_i^k and that of the sets $Z_i^k = B_i(\Delta(Z_{-i}^{k-1}))$ whose intersection $\cap_{k=1}^{\infty} Z_i^k$ forms the set Rat_i of i's rationalizable strategies. Indeed, for $k = 0, 1, 2, \ldots$, the projection of T_i^{k+1} onto S_i satisfies

$$\begin{aligned} \operatorname{proj}_{S_i} T_i^{k+1} &= \{ s_i \in S_i \mid \exists \theta_i^k \in \Delta(T_{-i}^k) : s_i \in B_i(\operatorname{marg}_{S_{-i}} \theta_i^k) \} \\ &= B_i(\operatorname{marg}_{S_{-i}} \Delta(T_{-i}^k)) \quad = \quad B_i(\Delta(\operatorname{proj}_{S_i} T_{-i}^k)) \end{aligned}$$

It follows by induction on k that each Z_i^{k+1} is the projection $\operatorname{proj}_{S_i} T_i^{k+1}$, and that $\operatorname{marg}_{S_{-i}} \Delta(T_{-i}^k) = \Delta(Z_{-i}^k)$.

10.4 The Limit Space

Finally, there is a well defined limit space

$$T_i := \{ (s_i, \theta_i^0, \theta_i^1, \theta_i^2, \ldots) \in T_i^1 \times \prod_{r=1}^{\infty} \Delta(T_{-i}^r) \mid \theta_i^{r-1} = \operatorname{marg}_{T_{-i}^{r-1}} \theta_i^r \ (r = 1, 2, \ldots) \}$$

It is not too difficult to prove that each limit space T_i is non-empty, compact and Polish when given the obvious product topology. Most important, one can construct a natural bijection h_i between T_i and the space $S_i \times \Delta(T_{-i})$ of strategies combined with beliefs concerning other players' types, including their strategies and their hierarchies of beliefs concerning other players' types, etc.

Indeed, this construction begins by applying the well-known existence or consistency theorem of Kolomogov regarding stochastic processes. This result implies that, for each sequence $\langle \theta_i^r \rangle_{r=0}^{\infty} \in \prod_{r=0}^{\infty} \Delta(T_{-i}^r)$ which is *consistent* in the sense that $\theta_i^r = \operatorname{marg}_{T_{-i}^r} \theta_i^{r+1}$ for $r = 0, 1, 2, \ldots$, there exists a unique limit probability measure $\theta_i = \xi_i(\langle \theta_i^r \rangle_{r=0}^{\infty}) \in \Delta(T_{-i})$ satisfying $\theta_i^r = \operatorname{marg}_{T_{-i}^r} \theta_i$ for $r = 0, 1, 2, \ldots$. The basic idea here is that each Borel set G in the infinite product space $T_{-i} = \prod_{h \in I \setminus \{i\}} [T_h^1 \times \prod_{r=1}^{\infty} \Delta(T_{-h}^r)]$ can be expressed as the limiting infinite intersection $G = \cap_{k=1}^{\infty} G^k$ of the shrinking sequence of cylindrical sets

$$G^k := \operatorname{proj}_{T_{-i}^k} G \times \prod_{h \in I \setminus \{i\}} \prod_{r=k}^{\infty} \Delta(T_{-h}^r)$$

Moreover, because the sequence of sets G^k is shrinking, the sequence of measures $\theta_i^k \left(\operatorname{proj}_{T_{-i}^k} G \right)$ is non-increasing and bounded below. So there is a well-defined limit given by

$$\theta_i(G) := \inf{}_{k=0}^{\infty} \theta_i^k \left(\operatorname{proj}_{T_{-i}^k} G \right) = \lim_{k \to \infty} \theta_i^k \left(\operatorname{proj}_{T_{-i}^k} G \right)$$

which turns out to be the required probability attached to the set G.

Furthermore, the mapping ξ_i constructed this way is evidently a bijection between $\prod_{r=0}^{\infty} \Delta(T^r_{-i})$ and $\Delta(T_{-i})$, with an inverse given by $\xi_i^{-1}(\theta_i) = \langle \text{marg}_{T^r_{-i}} \theta_i \rangle_{r=0}^{\infty}$ for all $\theta_i \in \Delta(T_{-i})$. So one can define $h_i : T_i \to S_i \times \Delta(T_{-i})$ for each $i \in I$ by $h_i(t_i^1, \langle \theta_i^r \rangle_{r=0}^{\infty}) := (t_i^1, \xi_i(\langle \theta_i^r \rangle_{r=0}^{\infty}))$, which then gives the promised bijection.

This construction of T_i as the set of player i's rationalizable types resolves the circularity problem. For more details, see especially Brandenburger and Dekel (1993).

The construction requires that the spaces Θ_i consist of countably additive probability measures. Otherwise, if for example one attempts to represent knowledge either instead of or in addition to beliefs, no countable construction is entirely adequate. In fact, there may even be no family of "universal" type spaces large enough to represent all players' possible knowledge and beliefs about each others' types. See, for example, Heifetz and Samet (1998), as well as Fagin *et al.* (1999). Aumann (1999a, b) proposes circumventing this difficulty by using a syntactic approach. But even this has its theoretical limitations, as Heifetz (1999) points out.

11 Trembling Best Responses

11.1 The Zero Probability Problem

The argument for iterative deletion of weakly dominated strategies, as well as the implied backward induction paradox, arose from the desire to attach zero subjective probability to strategies that are not rationalizable, in combination with the need to attach positive subjective probabilities to all strategies that have not already been eliminated as weakly dominated. Recall that positive probabilities are required so that Bayesian updating is always possible; except in trivial cases, they are implied by the hypothesis that behaviour in decision trees should be consequentialist. The paradoxes created by iterative deletion of all weakly dominated strategies might be avoided, therefore, if positive infinitesimal probabilities were allowed, and could be attached to all strategies that are not rationalizable. Such infinitesimal probabilities are positive but smaller than $1/n$ for any positive integer n; they are not real numbers, but belong to the space of "hyperreals" which are the basis of non-standard analysis.[21] In game theory it usually does no harm to think of an infinitesimal as an infinite sequence of real numbers that converges to zero.

Infinitesimal probabilities have already appeared in game theory in various guises. As discussed in Section 5, Selten (1975) and Myerson (1978) considered arbitrarily small "trembles" in players' optimal mixed strategies. Following earlier work by Rényi (1955, 1956) and other mathematicians, Myerson (1986) rediscovered "complete conditional probability systems" that allow condition-

[21] See Anderson (1991) for an economist's brief introduction and useful general bibliography.

ing on events whose probability is zero. Kreps and Wilson (1982), followed by Blume, Brandenberger and Dekel (1991a), introduced lexicographic hierarchies of probabilities in game theory. In Hammond (1994a) I discussed the history of such ideas in greater detail, and showed the close relationship between several different concepts. The papers by McLennan (1989a, b), Blume, Brandenberger and Dekel (1991b), Brandenberger (1992), Battigalli (1994a, b, 1998), Battigalli and Siniscalchi (1999), and Rajan (1998) all explore the usefulness of infinitesimal probabilities in game theory; in Hammond (1999a, b) I explore their use in consequentialist single-person decision theory, for the cases of objective and subjective probabilities respectively.

Infinitesimal probabilities may also help to avoid the kind of paradox that backward induction can create. In the centipede game of Section 9.4, for example, they allow the probability of reaching any node of the tree to remain positive. Moreover, if the game is modelled as having incomplete information because the players lack common knowledge of each other's theories of how the game will be played, then successive moves by the same player may be correlated in a way that makes it rational for each player to continue the game for quite a while. See Binmore (1987) for further elaboration, as well as the discussion at the end of Section 9.4.[22]

11.2 B-Perfect Rationalizability

Instead of introducing infinitesimal probabilities, a simpler alternative in the same spirit may be to allow small positive probabilities which tend to zero. This is what was done in Section 5, when discussing trembles and perfect and proper equilibria. Here the same idea can be applied to rationalizable strategies.

In this subsection, the definition of Bernheim (1984, p. 1021) will be modified to allow each player to have correlated beliefs regarding other players' pure strategies, as discussed in Section 9. This modification will lead to a concept to be called "B-perfect rationalizability". It is stronger than the newer and more convenient concept of "W-perfect rationalizability" due to Herings and Vannetelbosch (1999, 2000), which will be discussed in the next subsection.

The first definition will be of η-constrained rationalizable strategies, for each $i \in I$ and each small strictly positive vector $\eta = \langle\langle \eta_i(s_i)\rangle_{s_i \in S_i}\rangle_{i\in I} \in \prod_{i\in I} \mathbb{R}^{S_i}_{++}$ of trembles $\eta_i(s_i)$ attached to each strategy $s_i \in S_i$ of each player $i \in I$. The iterative definition is based on strategy sets $Z_i^k(\eta) \subset S_i$, together with sets $P_i^k(\eta) \subset \Delta(S_i)$ of other players' permissible beliefs about what i will play, and associated sets $E_i^k(\eta) \subset \Delta(S_{-i})$ of player i's permissible beliefs about other players' strategy profiles.

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[22]Positive infinitesimal probabilities help avoid the contradictions involved in normal form game theory that were noticed by Abreu and Pearce (1984). Considering such probabilities casts doubt on two of their axioms — namely, (A3) (Dominance) and (A4) (Subgame Replacement). In fact, the contradictions noted in their Propositions 1 and 2 can be avoided by attaching positive but infinitesimal probabilities to dominated or other inferior responses.

The construction starts with $Z_i^0(\eta) := S_i$. Then, for $k = 0, 1, 2, \ldots$ in turn, having already constructed the strategy set $Z_i^k(\eta) \subset S_i$, define the associated sets

$$\begin{aligned} P_i^k(\eta) &:= \{ \mu_i \in \Delta^\eta(S_i) \mid \forall s_i \in S_i \setminus Z_i^k(\eta) : \mu_i(s_i) = \eta_i(s_i) \} \\ E_i^k(\eta) &:= \text{co}\left[\prod\nolimits_{h \in I \setminus \{i\}} P_i^k(\eta) \right], \end{aligned} \tag{38}$$

where $\Delta^\eta(S_i)$ is defined as in (15) of Section 5.2. That is, $\mu_i \in \Delta^\eta(S_i)$ iff $\mu_i \in \Delta(S_i)$ with $\mu_i(s_i) \geq \eta_i(s_i)$ for all $s_i \in S_i$. Note that $P_i^k(\eta)$ is the set of η-perfect "trembling beliefs" about i's strategy, which attach the minimum allowable probability $\eta_i(s_i)$ to each strategy $s_i \notin Z_i^k(\eta)$. Then $E_i^k(\eta)$ is the set of i's beliefs about all other players' strategy choices. These take the form of joint distributions on S_{-i} in the convex set whose extreme points arise when, for each $h \in I \setminus \{i\}$, player i attaches the maximum allowable probability $1 - \sum_{s'_h \in S_h \setminus \{s_h\}} \eta_h(s'_h)$ to one particular strategy $s_h \in Z_h^k(\eta)$, and the minimum allowable probability $\eta_h(s'_h)$ to each strategy $s'_h \in S_h \setminus \{s_h\}$. In effect, each $\pi_i \in \Delta(Z_{-i}^k(\eta))$ becomes replaced by the new lottery

$$\sum_{s_{-i} \in Z_{-i}^k(\eta)} \pi_i(s_{-i})\, e_i(s_{-i})$$

where $e_i(s_{-i})$ denotes the product $\prod_{h \in I \setminus \{i\}} e_{ih}(s_h)$ of independent lotteries defined for all $h \in I \setminus \{i\}$ by

$$e_{ih}(s_h)(s'_h) = \begin{cases} \eta_h(s'_h) & \text{if } s'_h \neq s_h; \\ 1 - \sum_{\tilde{s}_h \in S_h \setminus \{s_h\}} \eta_h(\tilde{s}_h) & \text{if } s'_h = s_h. \end{cases}$$

For games with two players 1 and 2, because $P_i^k(\eta)$ is already convex, one will have $E_1^k(\eta) = P_2^k(\eta)$ and $E_2^k(\eta) = P_1^k(\eta)$ for all k and η.

At the next stage of the iteration, define

$$Z_i^{k+1}(\eta) := B_i(E_i^k(\eta)) = \bigcup_{\pi_i \in E_i^k(\eta)} B_i(\pi_i) \tag{39}$$

as the set of unconstrained best responses to expectations in $E_i^k(\eta)$. For $k = 1, 2, \ldots$ it is then easy to see that $P_i^k(\eta) = B_i^\eta(E_i^{k-1}(\eta))$, the set of η-constrained best responses to expectations in $E_i^{k-1}(\eta)$, as defined in (16) of Section 5.2.

Because $Z_i^0(\eta) = S_i$ and so $Z_i^1(\eta) \subset Z_i^0(\eta)$ (all $i \in I$), it is easy to prove by induction that $Z_i^{k+1}(\eta) \subset Z_i^k(\eta)$, $P_i^{k+1}(\eta) \subset P_i^k(\eta)$, and $E_i^{k+1}(\eta) \subset E_i^k(\eta)$ ($k = 1, 2, \ldots$). So there is a limit set

$$Z_i^\infty(\eta) := \cap_{k=0}^\infty Z_i^k(\eta) \tag{40}$$

of *η-perfectly rationalizable strategies.* As when rationalizable strategy sets were being constructed in Section 9.1, because each S_i is finite, the iteration must end in finitely many rounds — i.e., there exists a finite r such that $Z_i^k(\eta) = Z_i^r(\eta)$, $P_i^k(\eta) = P_i^r(\eta)$, and $E_i^k(\eta) = E_i^r(\eta)$ for all $i \in I$ and all $k \geq r$. Of course, this implies that there are associated limit sets satisfying

$$P_i^\infty(\eta) := \cap_{k=0}^\infty P_i^k(\eta) \quad \text{and} \quad E_i^\infty(\eta) := \cap_{k=0}^\infty E_i^k(\eta)$$

as well as $Z_i^\infty(\eta) = B_i(E_i^\infty(\eta))$, and $P_i^\infty(\eta) = B_i^\eta(E_i^\infty(\eta))$. In addition, $E_i^\infty(\eta)$ must be the convex hull of $\prod_{h \in I \setminus \{i\}} P_h^\infty(\eta)\,\}$.

Finally, for each player $i \in I$, say that the strategy $s_i \in S_i$ is *B-perfectly rationalizable* if and only if there exists a vanishing decreasing sequence $\eta_n \downarrow 0$ of allowable vectors in $\prod_{i \in I} \mathbb{R}_{++}^{S_i}$ such that $s_i \in Z_i^\infty(\eta_n)$ for all large n. Because $Z_i^\infty(\eta)$ is a non-empty subset of the finite set S_i for all $\eta \gg 0$, each player's limit set BPerf Rat$_i$ is non-empty.

Let $\mu^I \in \prod_{i \in I} \Delta(S_i)$ be any trembling-hand perfect equilibrium, as defined in Section 5.2. Then μ^I must be the limit as $n \to \infty$ of an infinite sequence μ_n^I of η_n-perfect equilibria, for some vanishing sequence η_n $(n = 1, 2, \dots)$ in $\prod_{i \in I} \mathbb{R}_{++}^{S_i}$. Suppose $s_i \in S_i$ is a strategy for which $\mu_i(s_i) > 0$. Then $\mu_{in}(s_i) > \eta_{in}(s_i)$ for all large n, implying that $s_i \in B_i(\pi_{in})$ where $\pi_{in} = \prod_{h \in I \setminus \{i\}} \mu_{hn}$. But now an easy induction argument shows that for $k = 0, 1, 2, \dots$ one has $s_i \in Z_i^k(\eta_n)$, $\mu_{in} \in P_i^k(\eta_n)$, and $\pi_{in} \in E_i^k(\eta_n)$. So in the limit as $k \to \infty$, one has $s_i \in Z_i^\infty(\eta_n)$, $\mu_{in} \in P_i^\infty(\eta_n)$, and $\pi_{in} \in E_i^\infty(\eta_n)$. Taking the limit as $n \to \infty$ and so $\eta_n \downarrow 0$, it follows that $s_i \in$ BPerf Rat$_i$. Thus, any strategy which appears with positive probability in some trembling-hand perfect equilibrium must be B-perfectly rationalizable.

It is instructive to consider how the concepts of rationalizability and B-perfect rationalizability perform in the centipede game of Section 9.4. In the first place, because neither player has a strictly dominated strategy, every strategy is rationalizable for each player.

In the "biped" case when $n = 1$, only the backward instruction strategies a_0 and b_0 are B-perfectly rationalizable. In the more interesting case when $n > 1$, only player P_2's strategy b_n is not B-perfectly rationalizable; all the others are. The reason is that player P_1's strategy a_i is a best response to $\pi_1 \in \Delta^\eta(S_2)$ provided that $\pi_1(b_i)$ is close to 1 and $\pi_1(b_{i+1}) = \eta_2(b_{i+1})$ is close to 0. Under B-perfect rationalizability, this is possible whenever $i \leq n - 1$. On the other hand, player P_2's strategy b_j is a best response to $\pi_2 \in \Delta^\eta(S_1)$ provided that $\pi_2(a_{j+1})$ is close to 1 and $\pi_2(a_{j+2}) = \eta_1(a_{j+2})$ is close to 0. Under B-perfect rationalizability, this is possible whenever $j \leq n - 1$.

11.3 Börgers' Example

Table 3 describes an important example taken from Börgers (1994), but with the row and column players interchanged to save space. The two players P_1 and P_2 have strategy sets $S_1 = \{\, a_1, a_2, a_3 \,\}$ and $S_2 = \{\, b_1, b_2, b_3, b_4, b_5 \,\}$ respectively.

For player P_2, strategies b_4 and b_5 are both weakly dominated. Thus, neither can be best responses when a_1, a_2 and a_3 are all expected to occur with positive probability, even if this is merely the result of trembling by player P_1. Hence,

P_2

P_1		b_1	b_2	b_3	b_4	b_5
	a_1	0, 3	0, 0	0, 2	3, 0	0, 0
P_1	a_2	0, 1	0, 1	0, 0	2, 0	2, 0
	a_3	0, 0	0, 3	0, 2	0, 0	3, 0

Table 3 Börgers' Example

any η-constrained best response for player P_2 satisfies $\mu_2(b_4) = \eta_2(b_4)$ and $\mu_2(b_5) = \eta_2(b_5)$.

Note that player P_1's payoff is 0 whenever P_2 chooses b_1, b_2 or b_3, regardless of what strategy P_1 chooses. In fact, P_1's expected payoff from a_1 is $3\eta_2(b_4)$, whereas from a_2 it is $2[\eta_2(b_4) + \eta_2(b_5)]$, and from a_3 it is $3\eta_2(b_5)$. Accordingly, player P_1's best response is entirely determined by the likelihood ratio $\rho := \eta_2(b_4)/\eta_2(b_5)$ of the two strategies b_4 and b_5 when P_2 trembles. In fact, there are five different cases:

> *Case 1:* $\rho > 2$. Here a_1 is P_1's unique best response to P_2's tremble, and (a_1, b_1) is the unique pair of η-perfectly rationalizable strategies.
>
> *Case 2:* $\frac{1}{2} < \rho < 2$. Here a_2 is P_1's unique best response to P_2's tremble, and, except in the boundary case when $\eta_1(a_1) = \eta_1(a_3)$, either (a_2, b_1) or (a_2, b_2) is the unique pair of η-perfectly rationalizable strategies, depending upon the likelihood ratio $\eta_1(a_1)/\eta_1(a_3)$ of the two strategies a_1 and a_3 when P_1 trembles. In no case is b_3 an η-perfectly rationalizable strategy for player P_2.
>
> *Case 3:* $\rho < \frac{1}{2}$. Here a_3 is P_1's unique best response to P_2's tremble, and (a_3, b_2) is the unique pair of η-perfectly rationalizable strategies.
>
> *Case 4:* $\rho = 2$. Here a_1 and a_2 are both best responses for player P_1 to P_2's tremble. Note that b_1 is P_2's unique best response whenever a_1, a_2 and a_3 all occur with positive probability, and the probability of a_3 is sufficiently small. So the only η-perfectly rationalizable strategies are a_1 or a_2 for player P_1, and b_1 for player P_2.
>
> *Case 5:* $\rho = \frac{1}{2}$. Here a_2 and a_3 are both best responses for player P_1 to P_2's tremble. Note that b_2 is P_2's unique best response whenever a_1, a_2 and a_3 all occur with positive probability, and the probability of a_1 is sufficiently small. So the only η-perfectly rationalizable strategies are a_2 or a_3 for player P_1, and b_2 for player P_2.

From this analysis, after considering all the different possible limits as $\eta \to 0$, it follows that the sets of B-perfectly rationalizable strategies are the whole of

S_1 for player P_1, and $\{ b_1, b_2 \}$ for player P_2. The fact that b_3 in particular is not B-perfectly rationalizable for player P_2 will be used later on to argue that B-perfect rationalizability may be too stringent a requirement.

11.4 W-Perfect Rationalizability

As remarked at the beginning of Section 11.2, Herings and Vannetelbosch (1999, 2000) have introduced a weaker concept which they call "weakly perfect rationalizability". Here the term "W-perfect rationalizability" is used instead. This new concept has some advantages over Bernheim's, even in two-person games for which the issue of whether to allow each player to have correlated beliefs regarding other players' pure strategies does not arise.

Consider any fixed small strictly positive vector $\xi = \langle\langle \xi_i(s_{-i})\rangle_{s_i \in S_i}\rangle_{i\in I} \in \prod_{i\in I} \mathbb{R}^{S_{-i}}_{++}$ of minimum possible probabilities $\xi_i(s_{-i})$ that each player $i \in I$ is allowed to attach to each strategy profile $s_{-i} \in S_{-i}$. The iterative construction is based on strategy sets $\bar{Z}^k_i(\xi) \subset S_i$, together with associated sets $\bar{E}^k_i(\xi) \subset \Delta(S_{-i})$ of permissible beliefs about other players' strategy profiles.

The construction starts with $\bar{Z}^0_i(\xi) := S_i$. Then, for $k = 0, 1, 2, \ldots$ in turn, given the already constructed strategy set $\bar{Z}^k_i(\xi) \subset S_i$, define the set

$$\bar{E}^k_i(\xi) := \{ \pi_i \in \Delta^0(S_{-i}) \mid \pi_i(s_{-i}) > \xi_i(s_{-i}) \Longrightarrow s_{-i} \in \bar{Z}^k_{-i}(\xi) \} \tag{41}$$

of permissible beliefs, where $\bar{Z}^k_{-i}(\xi) := \prod_{h\in I\setminus\{i\}} \bar{Z}^k_h(\xi)$. At the next stage of the iteration, define

$$\bar{Z}^{k+1}_i(\xi) := B_i(\bar{E}^k_i(\xi)) = \bigcup_{\pi_i \in \bar{E}^k_i(\xi)} B_i(\pi_i) \tag{42}$$

as the set of unconstrained best responses to expectations in $\bar{E}^k_i(\xi)$.

Because $\bar{Z}^0_i(\xi) = S_i$ and so $\bar{Z}^1_i(\xi) \subset \bar{Z}^0_i(\xi)$ for all $i \in I$, it is easy to prove by induction on k that $\bar{Z}^{k+1}_i(\xi) \subset \bar{Z}^k_i(\xi)$, and $\bar{E}^{k+1}_i(\xi) \subset \bar{E}^k_i(\xi)$ $(k = 1, 2, \ldots)$. So one can define the limit sets

$$\bar{Z}^\infty_i(\xi) := \cap^\infty_{k=0} \bar{Z}^k_i(\xi) \quad \text{and} \quad \bar{E}^\infty_i(\xi) := \cap^\infty_{k=0} \bar{E}^k_i(\xi) \tag{43}$$

The latter consists of *ξ-perfect trembling beliefs*. As with rationalizable strategies, because each S_i is finite, the iteration must end in finitely many rounds. Of course, the associated limit sets satisfy

$$\begin{aligned} \bar{Z}^\infty_i(\xi) &= B_i(\bar{E}^\infty_i(\xi)) \\ \bar{E}^\infty_i(\xi) &= \{ \pi_i \in \Delta^0(S_{-i}) \mid \pi_i(s_{-i}) > \xi_i(s_{-i}) \Longrightarrow s_{-i} \in \bar{Z}^\infty_{-i}(\xi) \} \end{aligned}$$

Finally, for each player $i \in I$, say that the strategy $s_i \in S_i$ is *W-perfectly rationalizable* if and only if there exists a vanishing decreasing sequence $\xi_n \downarrow 0$ of allowable vectors in $\prod_{i\in I} \mathbb{R}^{S_{-i}}_{++}$ such that $s_i \in \bar{Z}^\infty_i(\xi_n)$ for all large n. Because $\bar{Z}^\infty_i(\xi)$ is a non-empty subset of the finite set S_i for all $\xi \gg 0$, each player's limit set Perf Rat_i is non-empty.

Note that W-perfect rationalizability is indeed a weaker concept than B-perfect rationalizability, in the sense that BPerf $\text{Rat}_i \subset$ Perf Rat_i for all $i \in I$. To show this, it is enough to prove by induction on k that, when $\xi_i(s_{-i}) = \max_h\{\, \eta_h(s_h) \mid h \in I \setminus \{i\}\,\}$ for all $i \in I$ and all $s_{-i} \in S_{-i}$, then $Z_i^k(\eta) \subset \bar{Z}_i^k(\xi)$ and $E_i^k(\eta) \subset \bar{E}_i^k(\xi)$ for all $i \in I$ and for $k = 0, 1, 2, \ldots$. Of course, when $k = 0$, then $Z_i^0(\eta) \subset \bar{Z}_i^0(\xi) = S_i$ and

$$E_i^0(\eta) = \text{co} \prod_{h \in I \setminus \{i\}} \Delta^\eta(S_h) \subset \Delta^0(S_{-i}) = \bar{E}_i^0(\xi)$$

As the induction hypothesis, suppose that $Z_i^k(\eta) \subset \bar{Z}_i^k(\xi)$ for all $i \in I$. Suppose that $\pi_i \in E_i^k(\eta)$. Let s_{-i} be any strategy profile in S_{-i} for which $\pi_i(s_{-i}) > \xi_i(s_{-i})$. By definition of $E_i^k(\eta)$, there exist m and, for $j = 1, 2, \ldots, m$, mixed strategies $\mu_h^j \in P_h^k(\eta)$ for all $h \in I$ and convex weights $\lambda^j \geq 0$ such that $\sum_{j=1}^m \lambda^j = 1$ and $\pi_i = \sum_{j=1}^m \lambda^j \pi_i^j$ where $\pi_i^j(s_{-i}) = \prod_{h \in I \setminus \{i\}} \mu_h^j(s_h)$. Because $\pi_i(s_{-i}) > \xi_i(s_{-i})$, there exists at least one $j \in \{\, 1, 2, \ldots, m \,\}$ such that $\pi_i^j(s_{-i}) > \xi_i(s_{-i})$. In fact, for at least one j it must be true that

$$\pi_i^j(s_{-i}) = \prod_{h \in I \setminus \{i\}} \mu_h^j(s_h) > \xi_i(s_{-i}) = \max_h\{\, \eta_h(s_h) \mid h \in I \setminus \{i\} \,\}$$

Because $\mu_h^j(s_h) \leq 1$ for each $h \in I \setminus \{i\}$, it follows that

$$\mu_{h'}^j(s_{h'}) \geq \prod_{h \in I \setminus \{i\}} \mu_h^j(s_h) > \max_h\{\, \eta_h(s_h) \mid h \in I \setminus \{i\} \,\}$$

for each $h' \in I \setminus \{i\}$. This implies that for all $h \in I \setminus \{i\}$ one has $\mu_h^j(s_h) > \eta_h(s_h)$ and so $s_h \in Z_h^k(\eta)$. Hence $\pi_i(s_{-i}) > \xi_i(s_{-i})$ implies that $s_{-i} \in Z_{-i}^k(\eta)$. But $Z_{-i}^k(\eta) \subset \bar{Z}_{-i}^k(\xi)$ by the induction hypothesis.

So it has been proved that $\pi_i \in E_i^k(\eta)$ implies that $\pi_i \in \bar{E}_i^k(\xi)$. Hence $E_i^k(\eta) \subset \bar{E}_i^k(\xi)$, from which it is obvious from the definitions that $Z_{-i}^{k+1}(\eta) \subset \bar{Z}_{-i}^{k+1}(\xi)$. This completes the induction step of the proof.

The fact that W-perfect rationalizability is strictly weaker than B-perfect rationalizability will be shown at the end of Section 11.5, after introducing a useful characterization of W-perfect rationalizability.

11.5 The Dekel–Fudenberg Procedure

In Section 9.2 it was shown that rationalizable strategies are precisely those which survive iterated deletion of strictly dominated strategies. On the other hand, we have also discussed the iterative procedure of eliminating on each round every strategy that is weakly dominated. In their discussion of games with payoff uncertainty, Dekel and Fudenberg (1990) were led to investigate a new iterative procedure for eliminating dominated strategies. This procedure is stronger than iterated deletion of strictly dominated strategies, but weaker than iterated deletion of all weakly dominated strategies. In fact, the first step of the *Dekel–Fudenberg* (or DF) *procedure* does eliminate all weakly dominated strategies from each player's strategy set S_i. On each later round, however, only strictly dominated strategies are removed from those that remain; strategies that are merely weakly dominated are all retained.

Formally, the DF procedure constructs a sequence of strategy sets D_i^k ($k = 0, 1, 2, \dots$) for each player $i \in I$. As usual, $D_i^0 = S_i$. But then D_i^1 is the set of strategies in S_i that are not weakly dominated, and so $D_i^1 = B_i(\Delta^0(S_i))$. At each later stage of the construction, D_i^{k+1} is the set of strategies in D_i^k that are not strictly dominated when the other players' strategy profile s_{-i} is restricted to $D_{-i}^k := \prod_{h \in I \setminus \{i\}} D_h^k$. As with previous constructions, the result is a nested sequence of non-empty sets satisfying $\dots D_i^{k+1} \subset D_i^k \subset \dots \subset D_i^1 \subset D_i^0$ for all $i \in I$ and for $k = 0, 1, 2, \dots$. Once again, because each S_i is finite, the procedure converges in finitely many steps to a family of non-empty sets D_i^∞ ($i \in I$).

Recall that a strategy is not strictly dominated if and only if it is a best response in the strategy set S_i to some probability distribution $\pi \in \Delta(S_{-i})$. Similarly, D_i^{k+1} must consist of those *constrained* best responses within the set D_i^k to some probability distribution $\pi \in \Delta(D_{-i}^k)$.

Apart from its intrinsic interest, the DF procedure also happens to yield exactly the same set of strategies for each player as the W-perfect rationalizability criterion discussed in the previous subsection. In the first place, note that $Z_i^1(\xi) \subset B_i(\Delta^0(S_{-i}))$, so $s_i \in Z_i^1(\xi)$ only if s_i is not weakly dominated by any mixed strategy in $\Delta(S_i)$. That is, the first step of the iteration eliminates weakly dominated strategies from S_i. On each later round, strictly dominated strategies are removed from those that remain. Hence, a strategy is W-perfectly rationalizable only if it survives the DF procedure. The converse is also true, as Herings and Vannetelbosch (2000) have recently demonstrated.

This equivalence can also be used to illuminate the difference between B- and W-perfect rationalizability. Indeed, consider once again Börgers' example which was the subject of Section 11.3. Player P_2's strategies b_4 and b_5 are obviously weakly dominated; none of the others are. None of player P_1's strategies are even weakly dominated. The same is obviously true once b_4 and b_5 have been removed. Thus, the DF procedure leads to the respective strategy sets S_1 for player P_1 and $\{ b_1, b_2, b_3 \}$ for player P_2. Of course, these are the W-perfectly rationalizable strategy sets. Because player P_2's set of B-perfectly rationalizable strategies was shown to be only $\{ b_1, b_2 \}$, this confirms that B-perfect rationalizability is a strict refinement of W-perfect rationalizability.

11.6 Proper Rationalizability

Perfectly rationalizable strategies clearly suffer from exactly the same defect as perfect equilibria: they rely on all inferior responses being treated equally, even though some inferior responses may be much worse than others. Just as Myerson's concept of proper equilibrium is able to deal with this difficulty, so here we can consider properly rationalizable strategies.

The first definition will be of ϵ-proper rationalizable strategies, following Schuhmacher (1999) — see also Herings and Vannetelbosch (1999, 2000) and Asheim (2002). These are based on sets of best responses $\hat{Z}_i^k(\epsilon) \subset S_i$, sets of completely mixed strategies $\hat{P}_i^k(\epsilon) \subset \Delta^0(S_i)$, and associated sets of expectations $\hat{E}_i^k(\epsilon) \subset \Delta^0(S_{-i})$ $(k = 0, 1, 2, \dots)$. These sets are constructed iteratively for each $i \in I$ and small $\epsilon > 0$, starting with $\hat{Z}_i^0(\epsilon) := S_i$, $\hat{P}_i^0(\epsilon) := \Delta^0(S_i)$ and $\hat{E}_i^0(\epsilon) := \Delta^0(S_{-i})$. Then, for each $k = 1, 2, \dots$ in turn, let

$$\begin{aligned}
\hat{Z}_i^k(\epsilon) &:= B_i(\hat{E}_i^{k-1}(\epsilon)) \\
\hat{P}_i^k(\epsilon) &:= \{ \mu_i \in \Delta^0(S_i) \mid \exists \pi_i \in \hat{E}_i^{k-1}(\epsilon) : \forall s_i, s_i' \in S_i : \\
&\qquad V(s_i, \pi_i) > V(s_i', \pi_i) \Longrightarrow \mu_i(s_i') \leq \epsilon\, \mu_i(s_i) \} \\
\hat{E}_i^k(\epsilon) &:= \operatorname{co} \hat{P}_{-i}^k(\epsilon) \qquad (44)
\end{aligned}$$

where $\hat{P}_{-i}^k(\epsilon)$ denotes $\prod_{h \in I \setminus \{i\}} \hat{P}_h^k(\epsilon)$. Thus, $\hat{Z}_i^k(\epsilon)$ is the set of all strategies that are best responses for some expectations $\pi_i \in \hat{E}_i^{k-1}(\epsilon)$, whereas $\hat{P}_i^k(\epsilon)$ is the set $\cup_{\pi_i \in \hat{E}_i^{k-1}(\epsilon)} \hat{P}_i^\epsilon(\pi_i)$ of all ϵ-proper reponses to some expectations $\pi_i \in \hat{E}_i^{k-1}(\epsilon)$, as defined by (19) in Section 5.4. The definition of $\hat{E}_i^k(\epsilon)$ is like that of $E_i^k(\epsilon)$ in the construction of B-perfectly rationalizable strategies — see Section 11.2.

Given any $\epsilon \in (0, 1)$, let $\bar{\mu}^I = \prod_{i \in I} \bar{\mu}_i$ be any ϵ-proper equilibrium. From (19) and (44) it is clear that $\bar{\mu}_i \in \hat{P}_i^\epsilon(\bar{\pi}_i) \subset \hat{P}_i^1(\epsilon) \subset \hat{P}_i^0(\epsilon)$ for all $i \in I$, where $\bar{\pi}_i = \prod_{h \in I \setminus \{i\}} \bar{\mu}_h$. Then it is easy to prove by induction that $\bar{\pi}_i \in \hat{E}_i^k(\epsilon) \subset \hat{E}_i^{k-1}(\epsilon)$, $\hat{Z}_i^k(\epsilon) \subset \hat{Z}_i^{k-1}(\epsilon)$, and $\bar{\mu}_i \in \hat{P}_i^k(\epsilon) \subset \hat{P}_i^{k-1}(\epsilon)$ for $k = 1, 2, \dots$. It follows that one can define

$$\hat{Z}_i^\infty(\epsilon) := \cap_{k=0}^\infty \hat{Z}_i^k(\epsilon), \quad \hat{P}_i^\infty(\epsilon) := \cap_{k=0}^\infty \hat{P}_i^k(\epsilon) \quad \text{and} \quad \hat{E}_i^\infty(\epsilon) := \cap_{k=0}^\infty \hat{E}_i^k(\epsilon)$$

as the limit sets of *ϵ-properly rationalizable* strategies, responses, and expectations respectively. Moreover $\bar{\mu}_i \in \hat{P}_i^\infty(\epsilon)$ and $\bar{\pi}_i \in \hat{E}_i^\infty(\epsilon)$. This shows that any ϵ-proper equilibrium $\bar{\mu}^I$ is made up of ϵ-properly rationalizable responses $\bar{\mu}_i$ to $\bar{\pi}_i$ for each player $i \in I$. The existence theorem for ϵ-proper equilibrium presented in Section 5.4 implies that the three sets $\hat{Z}_i^\infty(\epsilon)$, $\hat{P}_i^\infty(\epsilon)$, and $\hat{E}_i^\infty(\epsilon)$ must all be non-empty. Also, because the correspondence $\pi_i \mapsto\!\!\!\to B_i(\pi_i)$ has a closed graph, $\hat{Z}_i^\infty(\epsilon) = B_i(\hat{E}_i^\infty(\epsilon))$, $\hat{P}_i^\infty(\epsilon) = \hat{P}_i^\epsilon(\hat{E}_i^\infty(\epsilon))$ and $\hat{E}_i^\infty(\epsilon) = \operatorname{co} \hat{P}_{-i}^\infty(\epsilon)$.

After these preliminaries, define each player i's set $\hat{E}_i$ of *properly rationalizable* expectations so that $\pi_i \in \hat{E}_i$ if and only if there exists a decreasing vanishing sequence $\epsilon_n \downarrow 0$ and a sequence $\pi_{in} \in \hat{E}_i^\infty(\epsilon_n)$ such that $\pi_{in} \to \pi_i$ as $n \to \infty$. Finally, define each player i's set $\text{Prop Rat}_i := B_i(\hat{E}_i) := \cup_{\pi_i \in E_i} B_i(\pi_i)$ of *properly rationalizable strategies* as those which lie within the range of possible best responses, given the set $\hat{E}_i$ of properly rationalizable expectations.

11.7 Properties

Let $\bar{\mu}^I \in \prod_{i \in I} \Delta(S_i)$ be any proper equilibrium. Then $\bar{\mu}^I$ must be the limit as $n \to \infty$ of an infinite sequence $\bar{\mu}_n^I$ of ϵ_n-proper equilibria, for some vanishing sequence ϵ_n $(n = 1, 2, \dots)$. As shown above, for all $i \in I$ these ϵ_n-proper equilibria must satisfy $\bar{\mu}_{in} \in \hat{P}_i^\infty(\epsilon_n)$ and $\bar{\pi}_{in} \in \hat{E}_i^\infty(\epsilon_n)$ for $n = 1, 2, \dots$, where $\bar{\pi}_{in} = \prod_{h \in I \setminus \{i\}} \bar{\mu}_{hn}$ and so $\bar{\pi}_{in} \to \bar{\pi}_i = \prod_{h \in I \setminus \{i\}} \bar{\mu}_h$ as $n \to \infty$. It follows that $\bar{\pi}_i \in \hat{E}_i$.

Suppose that $\bar{\mu}_i(s_i) > 0$. Then $\bar{\mu}_{in}(s_i) > \epsilon_n$ for all large n, implying that $s_i \in B_i(\bar{\pi}_{in})$. Taking the limit as $n \to \infty$ and so $\epsilon_n \to 0$, it follows that $s_i \in B_i(\bar{\pi}_i) \subset B_i(\hat{E}_i) =$ Prop Rat$_i$. Thus, any strategy which appears with positive probability in some proper equilibrium must be properly rationalizable.

Next, one can show that any properly rationalizable strategy must be W-perfectly rationalizable. Indeed, as was argued in Section 5.4, given any player $i \in I$ with expectations $\pi_i \in \Delta(S_{-i})$, any ϵ-proper response $\mu_i \in \hat{P}_i^\epsilon(\pi_i)$ is an ϵ-perfect response in $P_i^\epsilon(\pi_i)$ — that is, $\hat{P}_i^\epsilon(\pi_i) \subset P_i^\epsilon(\pi_i)$. It follows by induction on k that $\hat{Z}_i^k(\epsilon) \subset Z_i^k(\epsilon)$, $\hat{E}_i^k(\epsilon) \subset E_i^k(\epsilon)$ and $\hat{P}_i^k(\epsilon) \subset P_i^k(\epsilon)$ for $k = 0, 1, 2, \dots$.

Consider any extensive form game of perfect information which is *generic* in the sense that no player is indifferent between any pair of consequences at terminal nodes. Then backward induction selects the unique subgame perfect equilibrium. As shown by Schuhmacher (1999), this is the unique profile of properly rationalizable strategies. To see why, first note that a proper equilibrium exists and must be subgame perfect. So the unique backward induction outcome is also the unique proper equilibrium. But any strategies appearing with positive probability in a proper equilibrium must be properly rationalizable, implying that backward induction determines a profile of properly rationalizable strategies.

Conversely, the only properly rationalizable strategies in a generic extensive form game of perfect information must be those resulting from backward induction. The reason is that each backward induction step eliminates strategies which are inferior in each successive subgame, given that the relevant subgame is reached with positive probability. Proper responses involve playing such inferior strategies with a probability relative to other strategies in the subgame that converges to 0 as $\epsilon \to 0$. Thus, backward induction selects for each player the unique properly rationalizable strategy in each subgame, and so a unique profile of properly rationalizable strategies in the game as a whole.

Finally, proper rationalizability is a strict refinement of W-perfect rationalizability, but there is no logical relationship between B-perfect and proper rationalizability. Showing this requires two examples.

The first example is the centipede game of Section 9.4, with $n > 1$. By our earlier argument, the unique properly rationalizable strategies are a_0 for player P_1 and b_0 for player P_2. Yet, as discussed at the end of Section 11.2, all strategies except b_n for player P_2 are B-perfectly rationalizable, so also W-perfectly rationalizable.

The second example is once again the game due to Börgers analysed in Section 11.3. There it was shown that the B-perfectly rationalizable strategy sets are all of S_1 for player P_1, and $\{ b_1, b_2 \}$ for player P_2. Yet b_3 is a properly rationalizable strategy for player P_2. To show this, note that given any $\epsilon \in$

$(0,1)$, an ϵ-proper equilibrium takes the form

$$\mu_1(a_1) = \mu_1(a_3) = \frac{1}{2+\epsilon}; \qquad \mu_1(a_2) = \frac{\epsilon}{2+\epsilon};$$
$$\mu_2(b_1) = \mu_2(b_2) = \frac{\epsilon}{1+2\epsilon+2\epsilon^2}; \qquad \mu_2(b_3) = \frac{1}{1+2\epsilon+2\epsilon^2};$$
$$\mu_2(b_4) = \mu_2(b_5) = \frac{\epsilon^2}{1+2\epsilon+2\epsilon^2}.$$

Taking the limit as $\epsilon \to 0$, one obtains a proper equilibrium with $\mu_1(a_1) = \mu_1(a_3) = \frac{1}{2}$ and $\mu_2(b_3) = 1$. In particular, b_3 must be a properly rationalizable strategy for player P_2, even though it is not B-perfectly rationalizable.

12 Rationalizable Preferences over Pure Strategies

12.1 Quasi-Orderings as Dominance Relations

Except in Section 6, all the previous discussion in this chapter has been of solution concepts in which players are assumed to choose strategies that maximize expected utility. This is true whether probabilities are effectively objective, as in the case of Nash equilibrium and its refinements discussed in Sections 3–5, or whether probabilities are explicitly recognized to be subjective, as in the case of rationalizability and its refinements discussed in Sections 9 and 11. In particular, the notion of utility is essentially cardinal, insofar as solutions are invariant whenever each player's utility function is replaced by one that is cardinally equivalent — i.e., related by an increasing affine transformation.

Inspired by the fact that rationalizable strategies are precisely those which survive iterative deletion of strictly dominated strategies, as well as the corresponding relationship between W-perfect rationalizability and the Dekel–Fudenberg procedure, this section constructs a binary dominance relation over pure strategies for each player. These dominance relations will depend only on the profile of players' preference orderings over the consequences of different pure strategy profiles. In this sense, the relevant concept of utility is entirely ordinal, insofar as the relations are invariant whenever each player's utility function is replaced by one that is ordinally equivalent — i.e., related by *any* increasing transformation, not necessarily affine.[23]

As a preliminary, recall that a *quasi-ordering* on any set X is a binary relation Q that is irreflexive (there is no $x \in X$ satisfying $x\ Q\ x$) and transitive (for all $x, y, z \in X$, if $x\ Q\ y$ and $y\ Q\ z$, then $x\ Q\ z$). Evidently, a quasi-ordering is also asymmetric — i.e., no pair $x, y \in X$ can satisfy both $x\ Q\ y$ and $y\ Q\ x$, otherwise there would be a violation either of irreflexivity or transitivity. Unlike an ordering, a quasi-ordering may be incomplete — i.e., there may be pairs $x, y \in X$ such that neither $x\ Q\ y$ nor $y\ Q\ x$,

[23] As remarked in the introduction, Farqhuarson's notion of iterated deletion of strategies that are weakly dominated by other pure strategies shares this ordinality property, as does Börgers' (1993) different concept of pure strategy dominance.

In what follows, a *dominance relation* D on the set X will be any quasi-ordering. We shall often identify any dominance relation D with its *graph*, defined as the subset of pairs $(x, y) \in X \times X$ satisfying $x \; D \; y$.

12.2 A Recursive Construction

Consider any game $\langle I, S^I, v^I \rangle$ in normal form. The aim of this section will be to construct a profile $D^I = \langle D_i \rangle_{i \in I}$ of "rationalizable" dominance relations for each player, where each D_i is a quasi-ordering on that player's strategy set S_i. There will be associated sets $\mathcal{R}(D_i)$ of "rationalizable" preference orderings for each player, consisting of all orderings R_i on S_i whose corresponding strict preference relations satisfy $D_i \subset P_i$ (when each relation is interpreted as its graph).

The recursive construction that follows begins with each D_i^0 as the *null relation* whose graph is the empty set. But each $D_i^1 \subset S_i \times S_i$ will be the graph of the weak dominance relation for pure strategies. Also, succeeding relations D_i^k $(k = 2, 3, \dots)$ will have graphs that are supersets of preceding relations. Thus, the dominance relations gradually extend the usual weak dominance relation (which extends the usual strict dominance relation). The corresponding set of undominated strategies therefore becomes more and more refined. As does the corresponding set $\mathcal{R}(D_i^k)$ of rationalizable preference orderings, and the set of (undominated) strategies which maximize at least one rationalizable preference ordering.

Let $i \in I$ be any player, and let $D_{-i} = \langle D_h \rangle_{h \in I \setminus \{i\}}$ denote any profile of dominance relations for the other players. Define the associated binary relation $\succ_{-i}(D_{-i})$ on the set S_{-i} of other players' strategy profiles so that

$$
\begin{aligned}
s_{-i} \succ_{-i}(D_{-i}) \; s'_{-i} \quad \iff \quad & \exists h \in I \setminus \{i\} : s_h \; D_h \; s'_h \\
& \text{and} \quad \forall h \in I \setminus \{i\} : [s_h \; D_h \; s'_h \text{ or } s_h = s'_h]
\end{aligned}
\tag{45}
$$

Then $\succ_{-i}(D_{-i})$ is obviously both irreflexive and transitive, so a quasi-ordering or dominance relation in its own right.

In the special case of two players, this definition implies that $\succ_{-i}(D_{-i}) = D_h$, where $h \neq i$ is the only other player. More generally, note that if $s_h \neq s'_h$ for any $h \in I \setminus \{i\}$ and $s_{-i} \succ_{-i}(D_{-i}) \; s'_{-i}$, this can only be because $s_h \; D_h \; s'_h$. That is, for any player $h \in I \setminus \{i\}$ whose strategy s_h in the dominant profile s_{-i} differs from s'_h in the profile s'_{-i}, it must be true that s_h dominates s'_h. Accordingly, an informal interpretation of $s_{-i} \succ_{-i}(D_{-i}) \; s'_{-i}$ could be that s_{-i} is much more likely than s'_{-i} precisely because the two profiles differ, and moreover every player with a different strategy in the two profiles has switched to a dominating strategy.

An obvious implication of the above definition is that whenever $D'_{-i} = \langle D'_h \rangle_{h \in I \setminus \{i\}}$ is an alternative profile of *strengthened* dominance relations whose graphs satisfy $D_h \subset D'_h$ for all $h \in I \setminus \{i\}$, then $\succ_{-i}(D'_{-i})$ must strengthen $\succ_{-i}(D_{-i})$ in the same sense — i.e., their two graphs must satisfy $\succ_{-i}(D_{-i}) \subset \succ_{-i}(D'_{-i})$.

The complementary part of the construction starts with any quasi-ordering $\succ_{-i}$ on S_{-i}, and uses it to generate an associated dominance relation $D_i(\succ_{-i})$ for player i. Specifically, $D_i(\succ_{-i})$ is defined by

$$\begin{aligned} s_i \; D_i(\succ_{-i}) \; s_i' \quad \iff \quad & \exists \bar{s}_{-i} \in S_{-i} : v_i(s_i, \bar{s}_{-i}) > v_i(s_i', \bar{s}_{-i}) \\ & \text{and} \quad \big\{ \forall s_{-i} \in S_{-i} : v_i(s_i, s_{-i}) < v_i(s_i', s_{-i}) \\ & \qquad \Longrightarrow [\exists \tilde{s}_{-i} \succ_{-i} s_{-i} : v_i(s_i, \tilde{s}_{-i}) > v_i(s_i', \tilde{s}_{-i})] \big\} \end{aligned} \tag{46}$$

In fact, $s_i \; D_i(\succ_{-i}) \; s_i'$ if and only if s_i is better than s_i' for at least one $\bar{s}_{-i} \in S_{-i}$; moreover, if s_i is worse than s_i' for any $s_{-i} \in S_{-i}$, then it must be better for some other $\tilde{s}_{-i} \succ_{-i} s_{-i}$ which is much more likely than s_{-i}.

Suppose one strengthens $\succ_{-i}$ in the sense of replacing it with a different relation $\succ'_{-i}$ whose graph is a superset of the graph of $\succ_{-i}$. Then, as with $\succ_{-i} (D_{-i})$, the relation $D_i(\succ_{-i})$ becomes strengthened in the sense that its graph is a subset of that of the new relation $D_i(\succ'_{-i})$.

Clearly $D_i(\succ_{-i})$ always extends weak dominance, or is equal to it in case $\succ_{-i}$ is the null relation. It is also evident that $D_i(\succ_{-i})$ is irreflexive; demonstrating transitivity, however, is a little intricate.

Indeed, suppose that $s_i \; D_i(\succ_{-i}) \; s_i'$ and $s_i' \; D_i(\succ_{-i}) \; s_i''$, with $v_i(s_i, \bar{s}_{-i}) > v_i(s_i', \bar{s}_{-i})$ in particular. If $v_i(s_i', \bar{s}_{-i}) \geq v_i(s_i'', \bar{s}_{-i})$, then obviously $v_i(s_i, \bar{s}_{-i}) > v_i(s_i'', \bar{s}_{-i})$. Otherwise, if $v_i(s_i', \bar{s}_{-i}) < v_i(s_i'', \bar{s}_{-i})$, then $s_i' \; D_i(\succ_{-i}) \; s_i''$ implies that there exists $\bar{s}^1_{-i} \succ_{-i} \bar{s}_{-i}$ such that $v_i(s_i', \bar{s}^1_{-i}) > v_i(s_i'', \bar{s}^1_{-i})$. In which case, either $v_i(s_i, \bar{s}^1_{-i}) \geq v_i(s_i', \bar{s}^1_{-i})$, so $v_i(s_i, \bar{s}^1_{-i}) > v_i(s_i'', \bar{s}^1_{-i})$, or alternatively $v_i(s_i, \bar{s}^1_{-i}) < v_i(s_i', \bar{s}^1_{-i})$. But in the latter case $s_i \; D_i(\succ_{-i}) \; s_i'$ implies that there exists $\bar{s}^2_{-i} \succ_{-i} \bar{s}^1_{-i}$ such that $v_i(s_i, \bar{s}^2_{-i}) > v_i(s_i', \bar{s}^2_{-i})$.

Because $\succ_{-i}$ is transitive and S_{-i} is finite, this process of constructing successive $\bar{s}^r_{-i} \in S_{-i}$ with

$$\bar{s}^r_{-i} \succ_{-i} \bar{s}^{r-1}_{-i} \succ_{-i} \ldots \succ_{-i} \bar{s}^2_{-i} \succ_{-i} \bar{s}^1_{-i} \succ_{-i} \bar{s}_{-i}$$

must terminate after finitely many steps. So in the end there must exist $\bar{s}^r_{-i} \in S_{-i}$ such that $v_i(s_i, \bar{s}^r_{-i}) \geq v_i(s_i', \bar{s}^r_{-i}) \geq v_i(s_i'', \bar{s}^r_{-i})$, with at least one strict inequality, implying in particular that $v_i(s_i, \bar{s}^r_{-i}) > v_i(s_i'', \bar{s}^r_{-i})$.

The same argument shows that, if $v_i(s_i, s_{-i}) < v_i(s_i'', s_{-i})$ for any $s_{-i} \in S_{-i}$, then because either $v_i(s_i, s_{-i}) < v_i(s_i', s_{-i})$ or $v_i(s_i', s_{-i}) < v_i(s_i'', s_{-i})$, there must exist $\tilde{s}_{-i} \succ_{-i} s_{-i}$ such that $v_i(s_i, \tilde{s}_{-i}) \geq v_i(s_i', \tilde{s}_{-i}) \geq v_i(s_i'', \tilde{s}_{-i})$, with at least one strict inequality, implying in particular that $v_i(s_i, \tilde{s}_{-i}) > v_i(s_i'', \tilde{s}_{-i})$. Hence, $D_i(\succ_{-i})$ is transitive, as well as irreflexive, and so is a quasi-ordering that can serve as a dominance relation.

After these essential preliminary definitions and results, each player i's rationalizable dominance relation D_i^∞ can be constructed recursively. The recursion starts with D_i^0 equal to the null relation, whose graph is the empty subset of $S_i \times S_i$. Then, for each successive $k = 0, 1, 2, \ldots$, define

$$\succ^k_{-i} := \succ_{-i} (D^k_{-i}) \quad \text{and} \quad D_i^{k+1} := D_i(\succ^k_{-i}) \tag{47}$$

Clearly $\succ_{-i}^0$ is also equal to the null relation, but on S_{-i} instead of S_i. Then, as already remarked, D_i^1 is the usual weak dominance relation between pure strategies, defined on S_i. In particular, $D_i^0 \subset D_i^1$. Successive application of the "strengthening property" satisfied by both mappings $\succ_{-i}(D_{-i})$ and $D_i(\succ_{-i})$ then implies that the relations must have graphs satisfying the inclusions

$$\succ_{-i}^k \subset \succ_{-i}^{k+1} \quad \text{and} \quad D_i^k \subset D_i^{k+1} \quad (k = 0, 1, 2, \dots) \tag{48}$$

for each player $i \in I$. Because both S_i and S_{-i} are finite sets, this recursion terminates in finitely many stages. So for each $i \in I$ there exist both a limiting *rationalizable dominance relation* D_i^∞ on S_i, and also an associated quasi-ordering $\succ_{-i}^\infty$ on S_{-i}, whose respective graphs are defined by

$$D_i^\infty := \bigcup_{k=0}^\infty D_i^k \quad \text{and} \quad \succ_{-i}^\infty := \bigcup_{k=0}^\infty \succ_{-i}^k \tag{49}$$

Moreover, these relations satisfy $D_i^\infty = D_i^r$ and $\succ_{-i}^\infty = \succ_{-i}^r$ for some finite r. From which it follows that D_i^∞ is the dominance relation $D_i(\succ_{-i}^\infty)$ associated with the limit quasi-ordering $\succ_{-i}^\infty$, whereas $\succ_{-i}^\infty$ is the quasi-ordering associated with the profile of other players' limit dominance relations D_h^∞ $(h \in I \setminus \{i\})$.

12.3 Assessment

So far only a few very preliminary implications of this definition have been explored. It has already been remarked that D_i^1 is the usual weak dominance relation among pure strategies, so any strategies which are weakly dominated by other pure strategies will be eliminated from S_i^1. Thereafter, each profile s_{-i} of other players' strategies involving a dominated strategy is $\succ_{-i}$-dominated by some other profile s'_{-i}. So it is rather obvious that strategies which are strictly dominated by other pure strategies when other players are restricted to profiles in S_{-i}^k must be eliminated from S_i^{k+1}, for $k = 1, 2, \dots$ and all $i \in I$. The conclusion is that any strategy in S_i^∞ must have survived one round of eliminating all strategies which are weakly dominated by other pure strategies, followed by iteratively eliminating strategies which are strictly dominated by other pure strategies. In other words, the strategies that are undominated in the limit must have survived an obvious modification of the Dekel–Fudenberg procedure described in Section 11.5. The key difference is that, like Farqhuarson (1969), the new procedure only eliminates strategies which are dominated by other *pure* strategies; strategies which are dominated by mixed strategies, but not by other pure strategies, may be retained, unlike with orthodox rationalizability. This is only natural in a framework which deliberately avoids using cardinal utilities.

The rationalizable dominance criterion is finer than this, however. Indeed, consider any generic extensive form game of perfect information with the property that no player is indifferent between any pair of terminal nodes (or rather the consequences attached to those terminal nodes). In this case, backward induction reaches a unique outcome, as in the centipede game discussed in

Section 9.4. Rationalizable dominance then leads to the same outcome, which coincides with what remains after iterative deletion of all strategies that are weakly dominated by other pure strategies. This is because it is easy to prove that strategies which are eliminated at stage k of the usual backward induction argument based on best responses must be D_i^k-dominated for whatever player i has the move at each relevant node of the game tree. So any profile s_{-i} involving strategies which are eliminated in this way will be $\succ^k_{-i}$-dominated. On the other hand, as is well known, it takes as many rounds as necessary of iterative removal of weakly dominated strategies to achieve the same result — it is not enough to remove weakly dominated strategies just once, followed by removing only strictly dominated strategies iteratively.

A second example described by Table 4 is due to Dekel and Fudenberg (1990, Fig. 7.1, pp. 265–6). It can be interpreted as a game of Battle of the Sexes in which each player is presented with an additional outside option that avoids the "battle" entirely. Specifically, by choosing a_1, player P_1 guarantees each player's third-best outcome. But if player P_1 avoids a_1, then player P_2 can choose b_1, which guarantees each player's second-best outcome. On the other hand, if both players avoid these respective "outside options", then the game reduces to Battle of the Sexes.

		P_2		
		b_1	b_2	b_3
P_1	a_1	2, 2	2, 2	2, 2
	a_2	3, 3	4, 1	0, 0
	a_3	3, 3	0, 0	1, 4

Table 4 Battle of the Sexes with Two Outside Options

In this game, initially the only dominance relation is that strategy b_1 for player P_2 weakly dominates b_2. All strategies are therefore rationalizable. When b_2 is removed, still no strategy strictly dominates any other. Hence, the pure strategy version of the DF-procedure terminates after b_2 has been removed. So does the construction discussed in this section.

Once b_2 has been removed, however, player P_1's strategy a_3 weakly dominates a_2. But after a_2 has been removed, player P_2's strategy b_3 weakly dominates b_1, leaving only strategy b_3 for player P_2. But when this is P_2's only remaining strategy, player P_1's strategy a_1 strictly dominates a_3. So (a_1, b_3) is the only strategy profile that survives iterative deletion of all weakly dominated strategies. It follows that this iterative procedure refines the construction described in this section.

13 Conclusion: Insecure Foundations?

This chapter has shown how the results of Chapter 6 of Volume I could be used in order to justify applying the subjective expected utility (SEU) model to n-person games, with each player attaching subjective probabilities to the different strategy profiles that can be chosen by the other $n-1$ players. Some remaining difficulties in applying the SEU model are then discussed.

Only games such as finitely repeated Prisoner's Dilemma, in which a unique strategy profile can be found by deleting strictly dominated strategies, iteratively if necessary, seem to have a clear and generally accepted solution. In other games, it seems reasonable to restrict beliefs to rationalizable expectations of the kind described in Sections 9–11. But often rationalizability is not much of a restriction. For example, in Matching Pennies or Battle of the Sexes, it is no restriction at all. Then psychological or other influences that are extraneous to the game are likely to determine players' beliefs about how the game will be played, as well as about what players should believe about each other. In most decision problems or games, not even normative decision theory suffices in general to determine an agent's beliefs, just as it generally fails to determine tastes or attitudes to risk. Utility theory, of course, is unable to do any better.

Most worrying, however, may be the artificiality of the Battigalli construction used in Section 8.2 to justify the existence of subjective probabilities. That construction was based on each player having a "behavioural clone" who functions as an external observer able to choose among bets whose outcome depends on the strategy profile chosen by the other players. In the end, an approach such as that of Section 12, which does not rely on expected utilities at all, may be more suitable. But if players have only rationalizable preference orderings over each of their own strategy sets, we are back to using only ordinal utility. With a pair of utility functions, moreover, of which the first is defined on the consequences of strategy profiles, whereas the second is defined on the player's own pure strategies. Indeed, even the domain of this second utility function must be different in each game. That seems to be a radical departure from traditional game theory, and it may even remove most of the usefulness of utility as a tool.

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